(12-1)

§8. Hecke operators.

Recall we've defined, for $f: \mathfrak{H} \to \mathbb{C}$, $\gamma \in GL_2(\mathbb{R})^+$

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2} j(\gamma,z)^{-k} f(\gamma(z)) \quad [j(\gamma,z) = cz+d]$$

$M_k(\Gamma(1)) = \{f : \mathfrak{H} \to \mathbb{C}$ holomorphic $\mid \forall \gamma \in \Gamma(1),\ f|_k\gamma = f$ and f hol. at $\infty\}$.

Will turn out that spaces M_k, S_k have extra structure – they are representations of a certain algebra associated to $GL_2(\mathbb{Q})$.

$f \in M_k(\Gamma(1))$. Well, $\Gamma(1) \subset GL_2(\mathbb{R})^+$ so we could look at $g = f|_k\gamma$, $\gamma \in GL_2(\mathbb{R})^+$. Not a modular form of level 1 in general: if $\delta \in \Gamma(1)$

$$g|_k\delta = f|_k\gamma\delta = f|_k\gamma\delta\gamma^{-1}|_k\gamma, \text{ so } g|_k\delta = g \iff f|_k\gamma\delta\gamma^{-1} = f$$

and $\Gamma(1)$ is not normal in $GL_2(\mathbb{R})^+$, so usually $\gamma\delta\gamma^{-1} \notin \Gamma(1)$.

[in fact, normaliser $= \langle \Gamma(1), \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \rangle$.]

However not all is lost:–

Lemma. Suppose $\gamma_1, \dots, \gamma_n \in GL_2(\mathbb{R})^+ \supset \Gamma, \Gamma'$ such that the family of cosets $\Gamma\gamma_i$ $(1 \le i \le n)$ is invariant under right mult. by Γ'. Let $f: \mathfrak{H} \to \mathbb{C}$ s.t. $f|_k\gamma = f$ $\forall \gamma \in \Gamma$.

Then if $g = \sum_i f|_k\gamma_i$, also $g|_k\gamma = g$ $\forall \gamma \in \Gamma'$

Proof. Let $\delta \in \Gamma'$. By hypothesis, $\exists \pi \in S_n$ and elements $\varepsilon_i \in \Gamma$ such that $\gamma_i\delta = \varepsilon_i\gamma_{\pi(i)}$

$$\Rightarrow g|_k\delta = \sum_i f|_k\gamma_i\delta = \sum_i f|_k\varepsilon_i\gamma_{\pi(i)} = \sum_i f|_k\gamma_{\pi(i)} = g \qquad \square$$

Let's look at this in a more abstract setting.

G group, $\Gamma \subset G$ a subgroup. Consider $\Gamma\backslash G = \{$cosets Γg, $g \in G\}$ on which G (hence also Γ) acts by right multiplication.

Fact. The stabiliser in Γ of $\Gamma g \in \Gamma\backslash G$ is $g^{-1}\Gamma g \cap \Gamma$. (Proof obvious.)

Hypothesis (H): for every $g \in G$, $(\Gamma : g^{-1}\Gamma g \cap \Gamma) < \infty$

So (G, Γ) satisfies (H) $\iff$ $\Gamma\backslash G$ is a union of finite orbits under action of Γ.

Thm $G = GL_2(\mathbb{Q})$, $\Gamma \subset SL_2(\mathbb{Z})$ finite index. Then (G, Γ) satisfies (H).

Proof (i) $\Gamma = SL_2(\mathbb{Z})$. First assume $g \in Mat_2(\mathbb{Z})$, $|\det g| = N \neq 0$.

Suppose $\gamma \in \Gamma$, $\gamma \equiv I \bmod N$ i.e. $\gamma \in \Gamma(N) = \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}))$. Then:

$$N.\, g\gamma g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\gamma\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv N.I \pmod N$$

$\Rightarrow g\gamma g^{-1} \in \Gamma$; so $\Gamma(N) \subset g^{-1}\Gamma g \cap \Gamma$.

In general if $g \in GL_2(\mathbb{Q})$, $\exists M$ s.t. $g' = Mg \in Mat_2(\mathbb{Z})$ and $g'\gamma g'^{-1} = g\gamma g^{-1}$.

So (G, Γ) satisfies (H)

(ii) $\Gamma \subset \Gamma(1)$ finite index.

Recall that if $(G:H), (G:H') < \infty$, then $(G : H \cap H') < \infty$ (consider action of G on $G/H \times G/H'$)
Apply this and (i) to the "butterfly" diagram, in which each subgroup is the intersection of the two subgroups above:-

$$
\begin{tikzcd}[column sep=small]
 & \Gamma(1) \arrow[dl, dash, "\text{finite}"'] \arrow[dr, dash, "\text{finite by (i)}"] & & g^{-1}\Gamma(1)g \arrow[dl, dash, "\text{finite by (i)}"'] \arrow[dr, dash, "\text{finite}"] & \\
\Gamma \arrow[dr, dash] & & \Gamma(1) \cap g^{-1}\Gamma(1) g \arrow[dl, dash] \arrow[dr, dash] & & g^{-1}\Gamma g \arrow[dl, dash] \\
 & \Gamma \cap g^{-1}\Gamma(1) g \arrow[dr, dash] & & \Gamma(1) \cap g^{-1}\Gamma g \arrow[dl, dash] & \\
 & & \Gamma \cap g^{-1}\Gamma g & &
\end{tikzcd}
$$

□

From now on, assume (H). Then if $\{\Gamma g_i\}_{1 \le i \le n}$ is an orbit of Γ,
$\coprod_{i=1}^{n} \Gamma g_i = \Gamma g_j \Gamma$ for any j; conversely, for any $g \in \Gamma$, its orbit under Γ $\{\Gamma g_1 = \Gamma g, \Gamma g_2, \dots \Gamma g_n\}$ is finite, of order $(\Gamma : \Gamma \cap g^{-1}\Gamma g)$, and $\coprod \Gamma g_i = \Gamma g \Gamma$. This is called a <u>double coset</u>.

Double cosets are just orbits of $\Gamma \times \Gamma$ acting on G by $(\gamma, \delta) : g \mapsto \gamma g \delta^{-1}$ (left action) so G is a disj. union of double cosets $\Gamma g \Gamma$.

<u>NB.</u> For a single coset Γg, the elements are in bijection with Γ ($\gamma g \longleftrightarrow \gamma$).
For a double coset, the elements of $\Gamma g \Gamma$ need not be in bijection with $\Gamma \times \Gamma$
eg. take $g = 1$. (If G is finite, the double cosets generally have different sizes.)

<u>Aim:</u> define a ring $\mathcal{H}(G, \Gamma)$ from double cosets.
As an abelian group, $\mathcal{H}(G, \Gamma) =$ free abelian group on symbols $[\Gamma g \Gamma]$ one for each double coset $\Gamma g \Gamma$

First define action of double cosets on G-modules.

Let M be any right G-module: <u>ie</u> an abelian group with an action of G on the right which is $\mathbb{Z}$-linear.

Let $M^\Gamma = \{m \in M \mid m\gamma = m \ \forall \gamma \in \Gamma\}$. For $g \in G$, $m \in M^\Gamma$ define

$$m | [\Gamma g \Gamma] = \sum_{i=1}^{n} m g_i \quad \text{if} \quad \Gamma g \Gamma = \coprod_{i=1}^{n} \Gamma g_i$$

<u>Properties</u> (i) $m|[\Gamma g \Gamma]$ depends only on the double coset $\Gamma g \Gamma$, not on coset reps. g_i
(if $g_i' = \gamma_i g_i$, $\gamma_i \in \Gamma$ then $m g_i' = (m\gamma_i) g_i = m g_i$).

(ii) $m|[\Gamma g \Gamma] \in M^\Gamma$ (same as Lemma above, since $\{\Gamma g_i\}$ is invariant under Γ).

<u>Last time.</u> (G,Γ) satisfying (H) ($\Leftrightarrow$ orbits of Γ on $\Gamma\backslash G$ are all finite)

$G = \coprod$ of double cosets $\Gamma g \Gamma$, each $\Gamma g \Gamma = \coprod_{\text{finite}} \Gamma g_i$

M a right G-module. Defined, for $\Gamma g \Gamma = \coprod_{i=1}^{r} \Gamma g_i$ and $m \in M^\Gamma$,

$$m \,|\, [\Gamma g \Gamma] \stackrel{\text{def}}{=} \sum_{i=1}^{r} m g_i \qquad (*)$$

& showed

(i) $(*)$ doesn't depend on choice of coset reps. g_i

(ii) $m \,|\, [\Gamma g \Gamma] \in M$

Defined $\mathcal{H}(G,\Gamma)$ = free abel. gp. on $[\Gamma g \Gamma]$.

<u>Thm.</u> $\exists$ product on on $\mathcal{H}(G,\Gamma)$ making it an associative ring, unit $[\Gamma e \Gamma] = [\Gamma]$, such that $\forall$ G-module M, M^Γ is a right $\mathcal{H}$-module under $(*)$.

In the proof and later, will use following: let $\mathbb{Z}[\Gamma\backslash G]$ = free abel. gp. on cosets $[\Gamma g]$, G acting by right multiplication. Then have iso$^{\underline{\text{m}}}$

$$\Theta : \mathcal{H}(G,\Gamma) \xrightarrow{\sim} \mathbb{Z}[\Gamma\backslash G]^\Gamma$$
$$[\Gamma g \Gamma] \longmapsto \sum [\Gamma g_i] \qquad \left(\Gamma g \Gamma = \coprod \Gamma g_i\right)$$

<u>Proof.</u> Take $M = \mathbb{Z}[\Gamma\backslash G]$. Then if $[\Gamma g \Gamma] = \coprod \Gamma g_i$, $[\Gamma h \Gamma] = \coprod \Gamma h_j$ have by (ii)

$\sum_i [\Gamma g_i] \,\Big|\, [\Gamma h \Gamma] = \sum_{i,j} [\Gamma g_i h_j] \in M^\Gamma$. By (i) this gives a well-defined

product $[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \Theta^{-1}\left(\sum_{i,j} [\Gamma h_i g_j]\right)$. Obviously associative, and $[\Gamma]$ is unit.

If M is <u>any</u> G-module, $m \in M^\Gamma$ then

$$m \,\big|\, [\Gamma g \Gamma] \,\big|\, [\Gamma h \Gamma] = \sum_{i,j} m g_i h_j = m \,\big|\, [\Gamma g \Gamma] \cdot [\Gamma h \Gamma].$$

so M^Γ is a right $\mathcal{H}(G,\Gamma)$-module. $\square$

Compute in terms of double cosets. Pick $S \subset G$ with $G = \coprod_{g \in S} \Gamma g \Gamma$.

<u>Prop$^{\underline{\text{n}}}$.</u> Let $\Gamma g \Gamma = \coprod_{i=1} \Gamma g_i$, $\Gamma h \Gamma = \coprod_{j=1} \Gamma h_j$. Then

$$[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \sum_{k \in S} \sigma(k) [\Gamma k \Gamma], \quad \text{where } \sigma(k) = \#\{(i,j) \mid \Gamma g_i h_j = \Gamma k\}.$$

<u>Proof.</u> Consider the family of cosets $\Gamma g_i h_j$ $(1 \le i \le r,\ 1 \le j \le s)$ (invariant under Γ by (ii)).

So $\exists\, k_\ell \in S$ $(1 \le \ell \le t)$ (not nec. distinct) and a partition $\{(i,j) : \substack{1 \le i \le r \\ 1 \le j \le s}\} = \coprod_{\ell=1}^{t} T_\ell$

s.t. $\Gamma k_\ell \Gamma = \coprod_{(i,j) \in T_\ell} \Gamma g_i h_j$; and for each ℓ, $\Gamma k_\ell = \Gamma g_i h_j$ for exactly one $(i,j) \in T_\ell$.

So $\sigma(k) = \#\{\ell \mid k = k_\ell\}$ and so $[\Gamma g \Gamma] \cdot [\Gamma h \Gamma] = \sum_{\ell=1}^{t} [\Gamma k_\ell \Gamma] = \sum_{k \in S} \sigma(k) [\Gamma k \Gamma]$. $\square$

$\boxed{\text{From now on, } \Gamma = \Gamma(1) \subset G = GL_2(\mathbb{Q})^+}$

What are cosets, double cosets?

- Let $\gamma \in Mat_2(\mathbb{Z})$, $\det \gamma = n > 0$. Then rows of γ generate a sgp. $\Lambda = \gamma \mathbb{Z}^2 \subset \mathbb{Z}^2$ of index n. If the rows of γ' also generate Λ, then $\exists \delta \in GL_2(\mathbb{Z})$ s.t. $\gamma' = \delta\gamma$, so $\det(\gamma') = \pm n$; if $\det \gamma' = n$ then $\delta \in SL_2(\mathbb{Z})$. This gives a bijection

$$\{\text{cosets } \Gamma\gamma,\ \gamma \in Mat_2(\mathbb{Z}),\ \det\gamma = n > 0\} \xleftrightarrow{\sim} \{\text{sgps } \Lambda \subset \mathbb{Z}^2 \text{ of index } n\}.$$

- $\Lambda \subset \mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let $\Lambda \cap \mathbb{Z}e_2 = \mathbb{Z}.de_2$, $d > 0$.

Then $\Lambda = \langle ae_1 + be_2, de_2 \rangle$ for some $a > 0$, $b \in \mathbb{Z}$.
If we require $0 \le b < d$, then b is uniquely determined. So if we define

$$\Pi_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Mat_2(\mathbb{Z}) \;\middle|\; \begin{matrix} a, d \ge 1, \ ad = n \\ 0 \le b < d \end{matrix} \right\}.$$

then

$$\{\gamma \in Mat_2(\mathbb{Z}) \mid \det\gamma = n\} = \coprod_{\gamma \in \Pi_n} \Gamma\gamma$$

LHS is Γ-biinvariant. What is its decomposition into double cosets?

<u>Prop</u> (i) Let $\gamma \in Mat_2(\mathbb{Z})$, $\det \gamma = n \ge 1$. Then

$$\Gamma\gamma\Gamma = \Gamma \begin{pmatrix} n_1 & \\ & n_2 \end{pmatrix} \Gamma$$

for unique $n_1, n_2 \ge 1$ with $n_2 | n_1$, $n_1 n_2 = n$.

<u>Proof</u> (i) Let $\Lambda \subset \mathbb{Z}^2$ be sgp. generated by rows of γ. $\exists$ basis e_1, e_2 for $\mathbb{Z}^2$ such that $\Lambda = \langle n_1 e_1, n_2 e_2 \rangle$, $1 \le n_2 | n_1 = n/n_2$ (by elementary divisors / Smith Normal Form).
Replacing e_1 by $-e_1$, may assume $\det\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = +1$.

Then $\gamma = \delta \begin{pmatrix} n_1 e_1 \\ n_2 e_2 \end{pmatrix}$ for some $\delta \in GL_2(\mathbb{Z})$; $\det\gamma = n = n_1 n_2$
$\Rightarrow \delta \in SL_2(\mathbb{Z})$.

As n_1, n_2 are elementary divisors of $\mathbb{Z}^2/\Lambda$ they are unique. $\square$

Recall: (i) Let $\gamma \in Mat_2(\mathbb{Z})$, $\det \gamma = n \geq 1$. Then

$$\Gamma \gamma \Gamma = \Gamma \begin{pmatrix} n_1 & \\ & n_2 \end{pmatrix} \Gamma \quad \text{for unique } n_1, n_2 \geq 1 \text{ with } n_2 | n_1,\ n_1 n_2 = n.$$

Prop (cont'd) (ii) $$\{\gamma \in Mat_2(\mathbb{Z}) \mid \det \gamma = n\} = \coprod_{\substack{1 \leq n_2 | n_1 \\ n_1 n_2 = n}} \Gamma \begin{pmatrix} n_1 & \\ & n_2 \end{pmatrix} \Gamma$$

(iii) Notation as in (i), if $d \geq 1$ then $\Gamma (d^{-1}\gamma) \Gamma = \Gamma \begin{pmatrix} n_1/d & \\ & n_2/d \end{pmatrix} \Gamma$

Both follow from (i). □

Corollary. $\left\{ \left[\Gamma \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Gamma \right] : r_1, r_2 \in \mathbb{Q}_{>0},\ r_1/r_2 \in \mathbb{Z} \right\}$ is a basis for $\mathcal{H}(G, \Gamma)$.

□

Now define following elements of $\mathcal{H}(G, \Gamma)$:-

- for $1 \leq n_2 | n_1$, $T(n_1, n_2) = \left[\Gamma \begin{pmatrix} n_1 & \\ & n_2 \end{pmatrix} \Gamma \right]$
- for $n \geq 1$, $R(n) = \left[\Gamma \begin{pmatrix} n & \\ & n \end{pmatrix} \right] = T(n, n)$ (so $T(1,1) = T(1) = R(1) = 1$;

$$T(n) = \sum_{\substack{1 \leq n_2 | n_1 \\ n_1 n_2 = n}} T(n_1, n_2).$$ $T(n) = T(n,1)$ for n □-free)

Thm. (i) $R(mn) = R(m) R(n)$, $R(m) T(n) = T(n) R(m)$ $\forall m, n$.

(ii) $T(m) T(n) = T(mn)$ if $(m,n) = 1$.

(iii) $T(p) T(p^r) = T(p^{r+1}) + p R(p) T(p^{r-1})$ if $r \geq 1$

Coroll. $\mathcal{H}(G, \Gamma)$ is commutative, generated by $\{T(p), R(p), R(p)^{-1} \mid p \text{ prime}\}$.

Proof (i) is easy. For (ii), we have $T(n) \overset{\Theta}{\longmapsto} \sum_{\gamma \in \Pi_n} [\Gamma \gamma]$

and $\{\gamma . \mathbb{Z}^2 \mid \gamma \in \Pi_n\} = \{$subgroups of $\mathbb{Z}^2$ of index $n\}$.

So $T(m) T(n) \overset{\Theta}{\longmapsto} \sum_{\substack{\delta \in \Pi_m \\ \gamma \in \Pi_n}} [\Gamma \delta \gamma]$ and $\{\delta \gamma \mathbb{Z}^2 \mid \delta \in \Pi_m\} = \{$sgps of $\gamma \mathbb{Z}^2$ of index $m\}$.

As $(m,n) = 1$, every sgp. $\Lambda \subset \mathbb{Z}^2$ of index mn has a unique sgp. of index n.

$\therefore T(m) T(n) \overset{\Theta}{\longmapsto} \sum_{\varepsilon \in \Pi_{mn}} [\Gamma \varepsilon] \overset{\Theta^{-1}}{\longmapsto} T(mn)$.

(iii). Have $T(p^r) T(p) \overset{\Theta}{\longmapsto} \sum_{\delta \in \Pi_{p^r},\ \gamma \in \Pi_p} [\Gamma \delta \gamma]$,

and for fixed γ, $\{\delta \gamma \mathbb{Z}^2\} = \{$sgps. of $\gamma \mathbb{Z}^2$ of index $p^r\}$

$T(p^{r+1}) \overset{\Theta}{\longmapsto} \sum_{\varepsilon \in \Pi_{p^{r+1}}} [\Gamma \varepsilon]$, and $\{\varepsilon \mathbb{Z}^2\} = \{$sgps of $\mathbb{Z}^2$ of index $p^{r+1}\}$

Every $\Lambda = \varepsilon \mathbb{Z}^2$ of index p^{r+1} is a sgp. of index p^r of some sgp. $\Lambda' = \gamma \mathbb{Z}^2$ of index p.

 (14–2)

If $\Lambda \not\subset p\mathbb{Z}^2$ then Λ' is unique; $\Lambda' = \Lambda + p\mathbb{Z}^2$.

If $\Lambda \subset p\mathbb{Z}^2$ (ie. $\varepsilon = \begin{pmatrix} p & \\ & p \end{pmatrix}\varepsilon'$ for some $\varepsilon' \in \mathrm{Mat}_2(\mathbb{Z})$) then there are $(p+1)$ such Λ' (corresponding to the $(p+1)$ subgps of order p of $\mathbb{Z}^2/p\mathbb{Z}^2$)

$$\text{So } \Theta(T(p^r)T(p)) = \sum_{\varepsilon \in \Pi_{p^{r+1}} \setminus \begin{pmatrix} p & \\ & p\end{pmatrix}\Pi_{p^{r-1}}} [\Gamma\varepsilon] + (p+1)\sum_{\varepsilon' \in \Pi_{p^{r-1}}} \left[\Gamma\begin{pmatrix} p & \\ & p\end{pmatrix}\varepsilon'\right]$$

$$= \sum_{\varepsilon \in \Pi_{p^{r+1}}} [\Gamma\varepsilon] + p\sum_{\varepsilon' \in \Pi_{p^{r-1}}} \left[\Gamma\begin{pmatrix} p & \\ & p\end{pmatrix}\varepsilon'\right] = \Theta\left(T(p^{r+1}) + pR(p)T(p^{r-1})\right) \quad \square$$

ex. sheet — **Remark.** A more direct proof of commutativity:- consider the linear automorphism t of $\mathcal{H}(G,\Gamma)$, $t : [\Gamma g\Gamma] \longmapsto [{}^t(\Gamma g \Gamma)] = [\Gamma\, {}^tg\, \Gamma]$. As ${}^t(gh) = {}^th\,{}^tg$ it is an anti-isomorphism: $t(XY) = t(Y).t(X)$. But $\mathcal{H}$ is spanned by $\left[\Gamma\begin{pmatrix} r_1 & \\ & r_2\end{pmatrix}\Gamma\right]$, so t is the identity on $\mathcal{H}$. $\Rightarrow$ $\mathcal{H}$ commutative.

<u>Back to modular forms.</u> Let $V_k = \{\text{all holo. fns. } f.\ \mathfrak{h} \longrightarrow \mathbb{C}\}$ with $G = GL_2(\mathbb{Q})^+$ acting by $g : f \longmapsto f|_k g$. Then $M_k \subset V_k^\Gamma$.

Then for $f \in V_K^\Gamma$, $f|_k[\Gamma g\Gamma] := \sum f|_k g_i \in V_K^\Gamma$ if $\Gamma g\Gamma = \coprod \Gamma g_i$.

Note that $f|_k R(n) = f \quad \forall n \geq 1$.

Define $T_n = T_n^k : V_k \to V_k$, $T_n f = n^{k/2-1} f|_k T(n)$. (Since $\mathcal{H}(G,\Gamma)$ is commutative we may put on left instead of right.)

<u>Prop$^{\underline{n}}$.</u> (i) $T^k_{mn} = T^k_m T^k_n$ if $(m,n)=1$; $T^k_{p^{r+1}} = T^k_p T^k_{p^r} - p^{k-1}T^k_{p^{r-1}}$.

(ii) $f \in \begin{cases} M_k \\ S_k \end{cases} \Rightarrow T_n f \in \begin{cases} M_k \\ S_k\end{cases}$; so $T^k_n \in \mathrm{End}\, M_k$.

(iii) q-expansions: $$a_n(T_m f) = \sum_{1 \leq d | (m,n)} d^{k-1} a_{mn/d^2}(f)$$ and $a_0(T_m f) = \sigma_{k-1}(m)\, a_0(f)$.

<u>Pf.</u> (i) follows from the corresponding identity for $\{T(n)\}$.

(ii) f holomorphic $\Rightarrow T_n f$ is holomorphic. So sufficient to check holomorphy/vanishing at ∞, which will follow from (iii).

If $r \in \mathbb{Z}$, $$q^r|_k T(m) = m^{k/2} \sum_{\substack{e|m \\ 0 \leq b < e}} e^{-k} \exp\left(2\pi i r\frac{mz}{e^2} + 2\pi i \frac{br}{e}\right) \qquad \left(\Pi_m = \left\{\begin{pmatrix} m/e & b \\ 0 & e\end{pmatrix}\right\}\right)$$

$$= m^{k/2}\sum_{e|(n,r)} e^{1-k} q^{mr/e^2} \quad \text{as} \quad \sum_b \exp\left(2\pi i \frac{br}{e}\right) = \begin{cases} 0 & \text{if } r/e \notin \mathbb{Z} \\ e & \text{if } r/e \in \mathbb{Z}\end{cases}$$

$$\therefore\ T_m f = \sum_{r \geq 0}\sum_{e|(m,r)} (m/e)^{k-1} a_r(f)\, q^{mr/e^2} = \sum_{1 \leq d | m} d^{k-1} \sum_{s \geq 0} a_{\frac{ms}{d}}(f)\, q^{ds} \qquad \begin{pmatrix} d = \frac{m}{e} \\ r = es = \frac{ms}{d}\end{pmatrix}$$

$$= \sum_{n \geq 0}\sum_{d|(m,n)} d^{k-1} a_{mn/d^2}(f)\, q^n. \qquad (n = ds)$$

$\square$

Eigenforms

15-1

Corollary Suppose $f \in M_k$ and $T_m f = \lambda_m f$, some $m \geq 1$, $\lambda_m \in \mathbb{C}$.

(i) $\forall\, n \geq 1, (m,n) = 1$: $a_{mn}(f) = \lambda_m \cdot a_n(f)$

(ii) If $a_0(f) \neq 0$, then $\lambda_m = \sigma_{k-1}(m)$

Proof $a_n(T_m f) = \lambda_m a_n(f)$; then apply (iii) $\square$.

Of special interest is the case when f is an eigenfunction for all T_m, $m \geq 1$.

Coroll. Suppose $0 \neq f \in M_k$ $(k \geq 4)$ and $T_m f = \lambda_m f$ $\forall m \geq 1$. Then:-

i) $f \in S_k \Rightarrow a_1(f) \neq 0$ and $f = a_1(f) \cdot \sum_{n \geq 1} \lambda_n \cdot q^n$.

ii) $f \notin S_k \Rightarrow f = a_0(f) \cdot E_k$.

Proof i) $a_n(f) = \lambda_n a_1(f)$ $\forall n$.

ii) $a_0(f) \neq 0 \Rightarrow a_n(f) = \sigma_{k-1}(n) \cdot a_1(f)$

so $f = a_0(f) + a_1(f) \cdot \sum \sigma_{k-1}(n) q^n = A + BE_k$ for some $A, B \in \mathbb{C}$

$f - BE_k \in M_k \Rightarrow A = 0$. $\square$

We say f is a <u>Hecke eigenform</u>. If $a_1(f) = 1$ it is said to be <u>normalized</u>.

Thm. $\exists$ basis of S_k consisting of Hecke eigenforms.

Proof (mostly later) $\exists$ inner product on S_k, w.r.t. which the $\{T_m\}$ are symmetric operators. By a thm in linear algebra, they can be simultaneously diagonalised.

Ex. $\dim S_{12} = 1$, so $\Delta = \sum_{n \geq 1} \tau(n) q^n$ is a normalised Hecke eigenform (as $\tau(1) = 1$).

So $\tau(n) = \lambda_n$, and we deduce:-

* $\tau(m)\tau(n) = \tau(mn)$ if $(m,n) = 1$

* $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$.

Ex. $\dim S_{24} = 2$, spanned by $F = E_4^3 \Delta$ and $G = \Delta^2$

It is a simple matter to compute matrix of T_2 and find that it has distinct eigenvalues, & thus to compute the Hecke eigenforms

§ 9. L-functions of eigenforms

$f = \sum_{n\geq 1} a_n q^n \in S_k$. Can form Dirichlet series $L(f,s) = \sum_{n\geq 1} a_n . n^{-s}$

Prop: $L(f,s)$ converges absolutely for $\mathrm{Re}(s) > k/2 + 1$.

Proof. This follows from:

Prop^n: $|a_n| \leq c . n^{k/2}$

Proof Recall (Ex. sheet) $y^{k/2}|f|$ is bounded on $\mathfrak{H}$. Now

$$|a_n(f)| = \left| \frac{1}{2\pi} \int_{|q|=r} q^{-n} . \tilde{f}(q) \frac{dq}{q} \right| \quad \text{for any } r<1$$

$$= \left| \int_0^1 e^{-2\pi i n(x+iy)} f(x+iy)\, dx \right| \quad \text{for any } y>0$$

$$\leq e^{-2\pi n y} \sup_{x\in[0,1]} |f(x+iy)|$$

$$\leq C_1 . y^{-k/2} e^{-2\pi n y} \leq C_2 . n^{k/2} \quad \text{taking } y = 1/n \qquad \square$$

Euler product. Suppose f is a normalised eigenform. Then

$$L(f,s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}.$$

Proof We have:

$$(1 - a_p p^{-s} + p^{k-1-2s})(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots\cdots)$$

$$= 1 + \sum_{r\geq 2} (a_{p^r} - a_p a_{p^{r-1}} + p^{k-1} a_{p^{r-2}}) p^{-rs} = 1$$

so $\mathrm{RHS} = \prod_p (1 + a_p p^{-s} + a_{p^2} p^{-2s} + \cdots) = L(f,s)$ as $a_{mn} = a_m a_n$ for $(m,n)=1$.

AC + FE. Thm. If $f \in S_k(\Gamma(1))$, then $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s) L(f,s) = M(f,s)$ is an entire function of s, and $\Lambda(f, k-s) = (-1)^{k/2} \Lambda(f,s)$.

This will follow from:-

Thm. Suppose $0 \neq f(z) = \sum_{n\geq 1} a_n q^n$ holo. on $\mathfrak{H}$, $|a_n| = O(n^R)$ for some R

and $\exists N \in \mathbb{R}_{>0}$ s.t. $f\Big|_k \begin{pmatrix} & -1 \\ N & \end{pmatrix}(z) = c . f(z)$ $\quad (k \in \mathbb{Z}_{>0})$.

Then $L(s) = \sum a_n n^{-s}$ is entire; $c^2 = (-1)^k$; and if $\Lambda(s) = (2\pi)^{-s}\Gamma(s) L(s)$ then

$$\Lambda(k-s) = \varepsilon N^{s-k/2} \Lambda(s), \qquad \varepsilon = c . i^k \in \{\pm 1\}$$

(If $f \in S_k$ then take $N=1$; $f\Big|_k \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = f$ so $c=1$, $\varepsilon = (-1)^{k/2}$ and get previous result.)

Proof $f|_k\begin{pmatrix} & -1 \\ N & \end{pmatrix}(z) = N^{-k/2} z^{-k} f\left(\frac{-1}{Nz}\right)$, $f|_k\begin{pmatrix} & -1 \\ N & \end{pmatrix}\Big|_k\begin{pmatrix} & -1 \\ N & \end{pmatrix} = f|_k\begin{pmatrix} -N & \\ & -N \end{pmatrix} = (-1)^k f = c^2 f$.

$$\Lambda(f,s) = M(f(iy),s) = \int_{1/\sqrt{N}}^{\infty} + \int_0^{1/\sqrt{N}} f(iy)\, y^s \frac{dy}{y}. \qquad (\mathrm{Re}\, s > \tfrac{k}{2}+1) \qquad 16\text{-}1$$

and $$\int_0^{1/\sqrt{N}} = \int_{1/\sqrt{N}}^{\infty} f(i/Ny)\,(Ny)^{-s}\frac{dy}{y} = \int_{1/\sqrt{N}}^{\infty} c\, i^k N^{k/2-s} f(iy)\, y^{k-s}\frac{dy}{y} \qquad (y \mapsto 1/Ny)$$

$$\therefore \quad \Lambda(f,s) = \int_{N^{-1/2}}^{\infty} f(iy)\left(y^s + \varepsilon N^{\frac{k}{2}-s} y^{k-s}\right)\frac{dy}{y}$$

analytic $\forall s \in \mathbb{C}$, and satisfies FE. $\square$

(NB Some people define $\Lambda(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(s)$, in which case FE is just $\Lambda(k-s) = \varepsilon\Lambda(s)$.)

Inverse Mellin transform

Thm $f : (0,\infty) \to \mathbb{C}$ C^∞-function s.t.

i) $\forall N, n \geq 0$, $y^N f^{(n)}(y)$ is bounded as $y \to \infty$

ii) $\exists k \in \mathbb{Z}$ s.t. $\forall n \geq 0$, $y^{n+k} f^{(n)}(y)$ is bounded as $y \to 0$.

(These hypotheses can be considerably weakened).

Let $\Phi(s) = M(f,s)$, analytic for $\mathrm{Re}(s) > k$. Then $\forall \sigma > k$,

$$f(y) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \Phi(s)\, y^{-s}\, ds.$$

Proof Let $g(x) = g_\sigma(x) = e^{2\pi\sigma x} f(e^{2\pi x}) \in C^\infty(\mathbb{R})$

Then $\forall N, n \geq 0$, $e^{Nx} g^{(n)}(x)$ is bounded as $x \to +\infty$; and as $x \to -\infty$,

$$g^{(n)}(x) \ll \sum_{j=0}^{n} e^{2\pi(\sigma+j)x} f^{(j)}(e^{2\pi x}) \ll \sum e^{2\pi(\sigma+j)x} \cdot e^{-2\pi(j+k)x} \ll e^{2\pi(\sigma-k)x}$$

$$\Rightarrow g \in \mathcal{S}(\mathbb{R}), \text{ and } \hat{g}(-t) = \int_{-\infty}^{\infty} e^{2\pi\sigma x} f(e^{2\pi x})\, e^{2\pi i x t}\, dx = \frac{1}{2\pi}\int_0^\infty y^{\sigma+it} f(y)\frac{dy}{y} \quad (y = e^{2\pi x})$$
$$= \frac{1}{2\pi}\Phi(\sigma+it)$$

So by Fourier inversion,

$$f(y) = y^{-\sigma} g\left(\frac{\log y}{2\pi}\right) = \frac{y^{-\sigma}}{2\pi}\int_{-\infty}^{\infty} e^{-2\pi i t(\log y/2\pi)}\,\Phi(\sigma+it)\, dt = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} y^{-s}\,\Phi(s)\, ds. \quad \square$$

Applications:-

1. The (simplest) converse theorem. Let $L(s) = \sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series, $a_n \ll n^R$ for some R. Assume:-

- $L(s)$ can be analytically continued to $\{\mathrm{Re}(s) > \frac{k}{2} - \varepsilon\}$ (for some $\varepsilon > 0$) and that $|L(s)|$ is bounded in vertical strips $\{\sigma_0 \leq \mathrm{Re}(s) \leq \sigma_1\}$, $k/2 \leq \sigma_0 < \sigma_1$
- $\Lambda(s) = (2\pi)^{-s}\Gamma(s)L(s) = (-1)^{k/2}\Lambda(k-s)$.

Then $f = \sum a_n q^n \in S_k(\Gamma(1))$.

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Proof. STP $f(i/y) = (iy)^k f(iy)$. We have

$$f(iy) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Lambda(s)\, y^{-s}\, ds = \frac{1}{2\pi i}\int_{k/2-i\infty}^{k/2+i\infty} \Lambda(s)\, y^{-s}\, ds \quad \text{by Cauchy's thm.}$$

$$= \frac{(-1)^{k/2}}{2\pi i}\int_{k/2-i\infty}^{k/2+i\infty} \Lambda(k-s)\, y^{-s}\, ds \underset{s\mapsto k-s}{=} \frac{(-1)^{k/2}}{2\pi i}\int_{k/2-i\infty}^{k/2+i\infty} \Lambda(s)\, y^{s-k}\, ds = (-1)^{k/2}\cdot y^{-k} f(i/y).$$

$\square$

Remark. In the proof, need to know that $\displaystyle\int_{k/2\pm iT}^{\sigma\pm iT} \Lambda(s)\, y^{-s}\, ds \to 0$ as $T\to\infty$

Uses fact that $\Gamma(\sigma+iT) \to 0$ rapidly as $T\to\pm\infty$, uniformly for $\sigma\in[\sigma_0,\sigma_1]$.

2. Transformation formula for $E_2(z) = 1 - 24\sum_{n\geq 1}\sigma_1(n)\, q^n$.

Know $E_2(z+1) = E_2(z)$. What about $E_2(-1/z)$?

Let $f(y) = \dfrac{1-E_2(iy)}{24} = \sum_{m\geq 1}\sigma_1(m)\, e^{-2\pi m y}$

Claim $M(f,s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1)$

$$M(f,s) = (2\pi)^{-s}\Gamma(s)\sum_{m\geq 1}\sigma_1(m)\, m^{-s} = (2\pi)^{-s}\Gamma(s)\prod_p\left(1+(p+1)p^{-s}+(p^2+p+1)p^{-2s}+\dots\right)$$
$$= (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1)$$

as $\sigma_1(mn) = \sigma_1(m)\sigma_1(n)$ for $(m,n)=1$

and $(1-p^{-s}).(1+(p+1)p^{-s}+(p^2+p+1)p^{-2s}+\dots) = 1+p^{1-s}+p^{2-2s}+\dots = 1/(1-p^{1-s})$ $\square$

$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ D.F. – $\pi^{1/2}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2)$

$= 2(2\pi)^{-s}.\pi^{-1/2}.2^{s-1}\Gamma(s/2)\Gamma(\frac{s+1}{2})$

$= \pi^{-s/2}.\pi^{-(s+1)/2}.\Gamma(s/2).\Gamma(\frac{s+1}{2})$

Previously: transformation formula for $f \Rightarrow$ F.E. for $M(f,s)$ 17-1

Here: $\Leftarrow$.

Duplication formula $\Rightarrow$ $(2\pi)^{-s}\Gamma(s) = \frac{s-1}{4\pi}\Gamma_{\mathbb{C}}(s-1) = \frac{s-1}{4\pi}\cdot\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s-1)$

$\therefore\ M(f,s) = \frac{s-1}{4\pi} Z(s) Z(s-1) = -M(f,2-s)$. Entire apart from simple poles at $s=0,1,2$.
(FE for Z).

$\mathrm{Res}_{s=2} M = \frac{1}{4\pi}\cdot Z(2)\cdot \mathrm{Res}_{s=1} Z(s) = \frac{1}{4\pi}\cdot\frac{\pi}{6}\cdot 1 = \frac{1}{24} = -\mathrm{Res}_{s=0}$ (by FE)

$\mathrm{Res}_{s=1} M = \frac{1}{4\pi}\cdot \mathrm{Res}_{s=1} Z\ \mathrm{Res}_{s=0} Z = -\frac{1}{4\pi}$. Have $\sigma_1(m) \le \sum_{d\le m} d \le m^2$.

$$\therefore\ f^{(n)}(y) \ll \sum_{m\ge1} m^{n+2}\cdot e^{-2\pi m y} \begin{cases} \ll y^{-N} & \text{for any } N \text{ as } y\to\infty \\ \ll \dfrac{1}{(1-e^{2\pi y})^{n+3}} \ll y^{-n-3} & \text{as } y\to 0. \end{cases}$$

So f satisfies conditions of Thm with $k=3$.

So if $\sigma>3$ then
$$f(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} M(f,s)\, y^{-s}\, ds.$$

$$\Rightarrow f(1/y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} -M(f,2-s)\,y^{s}\,ds = -\frac{1}{2\pi i}\int_{2-\sigma-i\infty}^{2-\sigma+i\infty} M(f,s)\,y^{2-s}\,ds$$

$$\Rightarrow f(y) + y^{-2} f(1/y) = \frac{1}{2\pi i}\left[\int_{\sigma-i\infty}^{\sigma+i\infty} - \int_{2-\sigma-i\infty}^{2-\sigma+i\infty}\right] M(f,s)\, y^{-s}\, ds$$

$$= \sum_{s_0=0,1,2} \mathrm{Res}_{s=s_0}\left[M(f,s)\,y^{-s}\right]$$
(assuming $\int$ along horizontal lines $\to 0$ as $|\mathrm{Im}\, s|\to\pm\infty$ – using asymptotics for $\Gamma(s)$ and $\zeta(s)$.)

$$= \frac{1}{24} - \frac{1}{4\pi}\cdot y^{-1} + \frac{1}{24} y^{-2}$$

So $E_2(iy) = 1-24f(y) = -y^{-2}E_2(i/y) + 24\cdot\frac{1}{4\pi} y^{-1}$. So by AC:

Theorem $E_2(-1/z) = z^2 E_2(z) + 12z/2\pi i$.

Corollary $\Delta(z) = q\prod_{m\ge1}(1-q^m)^{24}$.

Proof. Let $D(z) = $ RHS; holo. on $\mathbb{C}$, $D(z+1) = D(z)$, cusp form.

If $D|_{12}\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) = D$ then $D\in S_{12} \Rightarrow D = \Delta$. So STP $D(-1/z) = z^{12}D(z)$.

Now $D'/D = 2\pi i - 24\sum_{m\ge1}\frac{2\pi i m q}{1-q^m} = 2\pi i\left(1-24\sum_{m\ge1, d\ge1} m q^{md}\right) = 2\pi i\, E_2(z)$.
as $dq/dz = 2\pi i q$

$$\therefore\ \frac{d}{dz}\left[\log D(-1/z)\right] = \frac{1}{z^2}\left(\frac{D'}{D}\right)(-1/z) = D'/D(z) + 12/z = \frac{d}{dz}\log D(z) + 12\frac{d}{dz}\log z$$

$$\Rightarrow \log D(-\tfrac{1}{z}) = \log D(z) + 12\log z + c.$$

$$\Rightarrow D(-1/z) = D(z)\cdot z^{12} C\ ;\ z=i \Rightarrow C=1. \qquad \square$$

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§10. Modular forms for subgroups of $SL_2(\mathbb{Z})$

$\Gamma \subset \Gamma(1)$ sgp. of finite index. Want to define modular forms for Γ.

Trivially: if $\overline{\Gamma(1)} = \coprod_{i=1}^{d} \overline{\Gamma}\gamma_i$ then $\coprod \gamma_i \mathcal{D}$ is a fund. domain for Γ.

[Recall $\overline{\Gamma}$ = image$(\Gamma \to PSL_2(\mathbb{Z}))$] Ex $\Gamma^0(p) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z}) \;\middle|\; b\equiv 0 \ (p)\right\}$

$= \left\{\gamma\in SL_2(\mathbb{Z}) \;\middle|\; \gamma \ (\mathrm{mod}\ p) \in \begin{pmatrix}*&0\\ *&*\end{pmatrix}\subset SL_2(\mathbb{F}_p)\right\}$

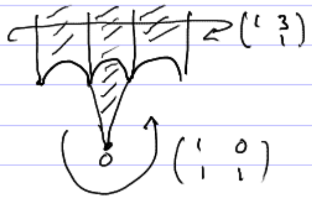
So $(\Gamma(1):\Gamma^0(p)) = p+1 = (\overline{\Gamma(1)}:\overline{\Gamma^0(p)})$

and coset reps are $\left\{\begin{pmatrix}1&b\\0&1\end{pmatrix} \;\middle|\; b \bmod p\right\} \cup \left\{\begin{pmatrix}&-1\\1&\end{pmatrix}\right\}$

and $SL_2(F) = \coprod_{b\in F}\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}*&0\\ *&*\end{pmatrix} \amalg \begin{pmatrix}*&*\\ *&0\end{pmatrix}$

$\begin{pmatrix}*&*\\ *&0\end{pmatrix} = \begin{pmatrix}&-1\\1&\end{pmatrix}\begin{pmatrix}*&0\\ *&*\end{pmatrix}$

p=3



So we need to control f as $\mathrm{Im}(z)\to\infty$ but also as $z\to 0$ somehow.

Defn. The cusps of Γ (or $\overline{\Gamma}$) are the orbits of Γ acting on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}\cup\{\infty\} \subset \mathbb{P}^1(\mathbb{C})$.

Prop. $\Gamma(1)$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$; only cusp is (orbit of) ∞.

Frequent abuse of notation: identify orbit with one of its representatives.

Proof $-\frac{d}{c}\in\mathbb{Q}$, $(c,d)=1 \Rightarrow \exists \begin{pmatrix}a&b\\c&d\end{pmatrix}=\gamma\in SL_2(\mathbb{Z})$ and then $\gamma(-\frac{d}{c})=\infty$. □

<u>Recall</u> $\Gamma \subset \Gamma(1)$ sgp. of finite index.

The <u>cusps</u> of Γ (or $\overline{\Gamma}$) are the orbits of Γ acting on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$

Since $(\Gamma(1) : \Gamma) < \infty$, $\Gamma_\infty = \Gamma \cap \begin{pmatrix} * & * \\ & * \end{pmatrix}$ is of finite index in $\Gamma(1)_\infty = \{\pm\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b \in \mathbb{Z}\}$.

$\Rightarrow$ $\overline{\Gamma}_\infty = \langle \pm\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \rangle$ for some $m \geq 1$. By conjugation, can do same at other cusps:-

<u>Def^n.</u> Let $\alpha \in \mathbb{Q} \cup \{\infty\}$ be (a representative of) a cusp of Γ. Let $g \in \Gamma(1)$ with $g(\infty) = \alpha$. Then $\gamma(\alpha) = \alpha \Leftrightarrow g^{-1}\gamma g(\infty) = \infty$, so

$$g^{-1}\overline{\Gamma}_\alpha g = (g^{-1}\overline{\Gamma} g)_\infty = \langle \pm\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \rangle \text{ for some integer } m \geq 1. \quad (\overline{\Gamma}_\alpha = \text{stabilizer of } \alpha \text{ in } \overline{\Gamma}.)$$

– m is the <u>width</u> of the cusp α.

<u>Note</u>: g as above not unique, but if g' is another then $g' = g\begin{pmatrix} \pm 1 & n \\ & \pm 1 \end{pmatrix}$ for some $n \in \mathbb{Z}$ and m is therefore indep. of choice of g.

Also, if $-I \notin \Gamma$, $g^{-1}\Gamma_\alpha g$ need not contain $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$ (it might only contain $-\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$)

<u>Prop^n.</u> Let Γ have ν cusps of widths $m_1, \dots, m_\nu$. Then $\sum_{i=1}^{\nu} m_i = (\overline{\Gamma(1)} : \overline{\Gamma})$.

<u>Proof</u>. Consider the surjective map
$$\overline{\Gamma}\backslash\overline{\Gamma(1)} \xrightarrow{\pi} \{\text{cusps of } \Gamma\}$$
$$\overline{\Gamma}.\gamma \longmapsto \overline{\Gamma}.\gamma(\infty)$$

Let $\alpha = g(\infty) \in \mathbb{P}^1(\mathbb{Q})$, $g \in \Gamma(1)$.

Then $\pi(\overline{\Gamma}.\gamma g) = \overline{\Gamma}.\alpha \Leftrightarrow \overline{\Gamma}\gamma g(\infty) = \overline{\Gamma}\alpha \Leftrightarrow \gamma(\alpha) \in \overline{\Gamma}\alpha \Leftrightarrow \gamma \in \overline{\Gamma}.\overline{\Gamma(1)}_\alpha$.

Now $\overline{\Gamma}\backslash\overline{\Gamma}.\overline{\Gamma(1)}_\alpha \simeq \overline{\Gamma}_\alpha\backslash\overline{\Gamma(1)}_\alpha$. So $\overline{\Gamma}.\overline{\Gamma(1)}_\alpha$ is a disjoint union of $(\overline{\Gamma(1)}_\alpha : \overline{\Gamma}_\alpha) = m_\alpha$

$\overline{\Gamma}.\gamma \longleftarrow\!\shortmid \overline{\Gamma}_\alpha.\gamma$

cosets of $\overline{\Gamma}$: i.e. $\pi^{-1}(\overline{\Gamma}\alpha)$ consists of m_α cosets $\overline{\Gamma}\gamma$. $\therefore \sum m_\alpha = (\overline{\Gamma(1)} : \overline{\Gamma})$. $\square$

<u>Ex.</u> $\Gamma = \Gamma_0(p)$: $\Gamma.\infty$ is a cusp of width 1 and ~~that's~~ all.
$\Gamma.0$ ——————— p

<u>Proof</u>. $\overline{\Gamma}_\infty = \langle \pm\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \rangle$, $\overline{\Gamma}_0 = \langle \pm\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \rangle$.

Geometrically: cusp of width m is on the boundary of m copies of $\mathcal{D}$.

<u>Group-theoretic.</u> Let $G_\infty = \begin{pmatrix} * & * \\ & * \end{pmatrix} \subset G = GL_2(\mathbb{Q})^+$.

Any conjugate P of G_∞ is called a <u>parabolic sgp</u> of G. ($\Leftrightarrow G/P$ is a projective algebraic variety)

If $P = g\,G_\infty\,g^{-1}$ then $P = G_\alpha$ where $g(\infty) = \alpha \in \mathbb{P}^1(\mathbb{Q})$

So $\mathbb{P}^1(\mathbb{Q}) \simeq \{\text{parabolic sgps. of } G\}$, and $\{\text{cusps of } \Gamma\} \simeq \{\Gamma\text{-conjugacy classes of parabolic sgps. of } G\}$

$\alpha \longleftrightarrow G_\alpha$.

$\Gamma\alpha \longleftrightarrow \Gamma\text{-conjugacy class of } G_\alpha$
$= \{\gamma G_\alpha \gamma^{-1} \mid \gamma \in \Gamma\}$
$= \{G_{\gamma(\alpha)} \mid \gamma \in \Gamma\}$

<u>Defn.</u> $\Gamma \subset SL_2(\mathbb{Z})$, $k \in \mathbb{Z}$. A <u>modular form of wt k on Γ</u> is a holomorphic $f : \mathfrak{h} \longrightarrow \mathbb{C}$ st.

i) $f|_k \gamma = f \quad \forall \gamma \in \Gamma$

ii) f is holomorphic at the cusps of Γ.

Say f is a <u>cusp form</u> if moreover iii) f vanishes at the cusps of Γ.

<u>Meaning of (ii), (iii):</u> let $\alpha = g(\infty) \in \mathbb{P}^1(\mathbb{Q})$, $g \in \Gamma(1)$. Then $\bar{\Gamma}_\alpha = g \langle \pm \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \rangle g^{-1}$, m = width of cusp Γ_α.

So $g^{-1}\Gamma_\alpha g \ni \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$ or $-\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$, so contains $\begin{pmatrix} 1 & 2m \\ & 1 \end{pmatrix} = \left[\pm \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}\right]^2$.

$\therefore \; f|_k g = f|_k g |_k \begin{pmatrix} 1 & 2m \\ & 1 \end{pmatrix} = f|_k g(z+2m)$, and therefore $\exists$ Fourier expansion

$$f|_k g = \tilde{f}_g(q) = \sum_{n \in \mathbb{Z}} (\text{constant}) \times q^{n/2m} = \sum_{\substack{n \in \mathbb{Q} \\ 2mn \in \mathbb{Z}}} a_{g,n}(f) . q^n \quad \left(\text{here } q^n := e^{2\pi i n z} \text{ for } n \in \mathbb{Q}\right)$$

f <u>holomorphic</u> at α if $a_{g,n}(f) = 0 \; \forall n < 0$ (ie iff $f|_k g$ is holomorphic at ∞).
<u>vanishes</u> at α if moreover $a_{g,0}(f) = 0$.

<u>Dependence on g</u>: if $\alpha = g(\infty) = g'(\infty)$, $g \in GL_2(\mathbb{Q})^+$, then $g' = gh$, $h(\infty) = \infty$

$\Rightarrow h = \pm \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ for some $a, b, d \in \mathbb{Q}$ with $a, d > 0$.

$\Rightarrow f|_k g' = (f|_k g)|_k h = \sum a_{g,n}(f) q^n \,\Big|_k \pm \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (\pm 1)^k \sum_n e^{2\pi i \frac{b}{nd}} a_{g,n}(f) q^{an/d}$.

So condition (ii) is equivalent to:

(ii)' for all $g \in GL_2(\mathbb{Q})^+$, $f|_k g$ is holomorphic at ∞

(ie $f|_k g = \sum_{n \geq 0} c_n . q^{rn}$ for some $r \in \mathbb{Q}_{>0}$, $c_n \in \mathbb{C}$.)

<u>Note</u> This condition does not involve Γ.

Likewise, (iii) $\Leftrightarrow$ (iii'): $f|_k g$ vanishes at ∞ for all $g \in GL_2(\mathbb{Q})^+$.

Notation: $M_k(\Gamma)$, $S_k(\Gamma)$: spaces of modular/cusp forms on Γ of weight k.

for $\Gamma = \Gamma(1)$, $M_k = 0$ for k odd, since $f|_k(-I) = (-1)^k f$. More generally

‖ Suppose $-I \in \Gamma$. Then for odd k, $M_k(\Gamma) = 0$.

<u>Propn.</u> $\Gamma \underset{f.i.}{\subset} \Gamma(1)$, $g \in G = GL_2(\mathbb{Q})^+$. Then $g^{-1}\Gamma g \cap \Gamma(1) = \Gamma'$ has finite index in $\Gamma(1)$,
and $f \in \begin{Bmatrix} M_k(\Gamma) \\ S_k(\Gamma) \end{Bmatrix} \Rightarrow f|_k g \in \begin{Bmatrix} M_k(\Gamma') \\ S_k(\Gamma') \end{Bmatrix}$.

<u>Proof.</u> 1st part since (G, Γ) has property (H) [see beginning of §8]
$\gamma \in \Gamma' \Rightarrow g\gamma g^{-1} \in \Gamma \Rightarrow f|_k g|_k \gamma = f|_k g$; and condition (ii)' holds for $f \Leftrightarrow$ it holds for $f|_k g$. $\square$

<u>Ex.</u> $\Gamma = \Gamma(1)$, $g = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$. Then $g^{-1}\Gamma(1) g = \left\{ \begin{pmatrix} a & N^{-1}b \\ Nc & d \end{pmatrix} \,\Big|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \right\}$

$\Rightarrow g^{-1}\Gamma(1) g \cap \Gamma(1) = \Gamma_0(N)$

So $f(z) \in M_k(\Gamma(1)) \Rightarrow f(Nz) \in M_k(\Gamma_0(N)) \quad \forall N \geq 1$.

Thm. $M_k(\Gamma) = \begin{cases} 0 & \text{if } k<0 \\ \mathbb{C} & \text{if } k=0 \end{cases}$ & $\dim M_k(\Gamma) \le 1 + \frac{k}{12}(\Gamma(1):\Gamma)$ in general.

(Cheap) Proof. Let $\Gamma(1) = \coprod_{i=1}^{d} \gamma_i \Gamma$. Let $f \in M_k(\Gamma)$, and define

$\mathcal{N}f = \prod_{i=1}^{d} f|_k \gamma_i$. Claim $\mathcal{N}f \in M_{kd}(\Gamma(1))$. ; and $\mathcal{N}f = 0 \Leftrightarrow f = 0$

Holomorphic on $\mathfrak{H}$ ✓. $\mathcal{N}f|_{kd}\gamma = \prod f|_k \gamma_i \gamma$ and $\gamma_i\gamma = \delta\gamma_j$ for some $j = \pi(i)$ some $\delta \in \Gamma$. ($\pi \in S_d$)

Holomorphic at ∞ (each $f|\gamma_i$ is). ✓ $= \prod f|_k \gamma_{\pi(i)} = \mathcal{N}f$. ✓

$k<0 \Rightarrow \mathcal{N}f = 0 \Rightarrow f = 0$.

$k \geq 0$. Suppose $\dim M_k > N$. Pick distinct $z_1, \dots, z_N \in \mathcal{D} - \{i, \rho\}$. Then $\exists f \in M_k - 0$ with $f(z_1) = \dots = f(z_N) = 0$. Then $0 \neq \mathcal{N}f \in M_{kd}(\Gamma(1))$, $\mathcal{N}f(z_1) = \dots = \mathcal{N}f(z_N) = 0$

If $N > \frac{kd}{12}$ this $\Rightarrow \mathcal{N}f = 0$ i.e. $f = 0$. So $\dim M_k \le 1 + \frac{kd}{12}$.

$k = 0 \Rightarrow \dim M_0 \le 1 \Rightarrow M_0 = \mathbb{C}$ $\square$

The inner product. Let $f, g \in S_k(\Gamma)$.

Then $y^k f(z)\overline{g(z)}$ is Γ-invariant, and bounded on $\mathfrak{H}$.

Also $\frac{dx\,dy}{y^2}$ is $GL_2(\mathbb{R})^+$-invariant, so we can define the Petersson scalar product

$$\langle f, g \rangle = \frac{1}{v(\Gamma)} \int_{\Gamma\backslash\mathfrak{H}} y^k f(z)\overline{g(z)} \cdot \frac{dx\,dy}{y^2} \in \mathbb{C} \qquad (*)$$

Here $\int_{\Gamma\backslash\mathfrak{H}}$ means integrate over any fundamental domain $\mathcal{D}_\Gamma$ for Γ, and $v(\Gamma) = \int_{\Gamma\backslash\mathfrak{H}} \frac{dx\,dy}{y^2} = (\overline{\Gamma(1)}:\overline{\Gamma}) \cdot \int_{\mathcal{D}} \frac{dx\,dy}{y^2} < \infty$

If we replace Γ with $\Gamma' \subset \Gamma$ finite index, then a FD for Γ' is a union of $(\overline{\Gamma}:\overline{\Gamma'})$ FDs. for Γ.

$\Rightarrow$ RHS of $(*)$ is independent of Γ.

Prop: (i) $\langle\ ,\ \rangle$ is a Hermitian inner product on $S_k(\Gamma)$.

(ii) $\gamma \in GL_2(\mathbb{Q})^+ \Rightarrow \langle f|_k\gamma, g|_k\gamma \rangle = \langle f, g \rangle$

(iii) $f, g \in S_k(\Gamma(1))$. Then $\langle T_p f, g \rangle = \langle f, T_p g \rangle$.

Proof (i). $\mathbb{C}$-linear in f, and $\overline{\langle f, g \rangle} = \langle g, f \rangle$; $\langle f, f \rangle = 0 \Leftrightarrow \int y^{k-2}|f|^2 dx\,dy = 0 \Leftrightarrow f = 0$.

(ii) Let $f' = f|_k\gamma$, $g' = g|_k\gamma \in S_k(\Gamma')$, $\Gamma' = \gamma^{-1}\Gamma\gamma \cap \Gamma$

Then $y^k f' \cdot \overline{g'} = y^k \cdot (\det\gamma)^k |cz+d|^{-2k} f(\gamma(z))\overline{g(\gamma(z))} = (\operatorname{Im}\gamma(z))^k \cdot f(\gamma(z))\overline{g(\gamma(z))}$.

As $\frac{dx\,dy}{y^2}$ is $GL_2(\mathbb{R})^+$-invariant,

$$\langle f', g' \rangle = \frac{1}{v(\Gamma')} \int_{\mathcal{D}_{\Gamma'}} \left(y^k f \bar{g} \frac{dx\,dy}{y^2} \right)\Big|_{\gamma(z)} = \frac{1}{v(\Gamma')} \int_{\gamma(\mathcal{D}_{\Gamma'})} y^k f \bar{g} \frac{dx\,dy}{y^2}$$

and $\gamma(\mathcal{D}_{\Gamma'})$ is a FD. for $\gamma\Gamma'\gamma^{-1} = \gamma\Gamma\gamma^{-1} \cap \Gamma$.

So $v(\Gamma') = \int_{\mathcal{D}_{\Gamma'}} \frac{dx\,dy}{y^2} = \int_{\gamma(\mathcal{D}_{\Gamma'})} \frac{dx\,dy}{y^2} = v(\gamma\Gamma'\gamma^{-1})$ (middle = by invariance of $\frac{dx\,dy}{y^2}$)

$\Rightarrow \langle f', g' \rangle = \langle f, g \rangle$.

(iii) As T_n is a polynomial in the T_p's with integer coefficients,

enough to check this for T_p.

Lemma $\langle T_p f, g\rangle = p^{k/2-1}(p+1)\langle f|_k\delta, g\rangle$ for any $\delta \in M_2(\mathbb{Z})$ with $\det(\delta) = p$.

Proof. Let $\Gamma(1)\begin{pmatrix}p & \\ & 1\end{pmatrix}\Gamma(1) = \coprod_{0\le j\le p} \Gamma(1)\delta\gamma_j$, $\gamma_j \in \Gamma(1)$.

Then $\langle T_p f, g\rangle = p^{k/2-1}\langle \sum_j f|_k\delta\gamma_j, g\rangle = p^{k/2-1}\sum_j \langle f|_k\delta\gamma_j, g|_k\gamma_j\rangle$ $\quad (g|_k\gamma_j = g)$

$= p^{k/2-1}(p+1)\langle f|_k\delta, g\rangle$ by (ii). $\square$

Let δ be as in lemma, & $\delta^a = p\delta^{-1} \in M_2(\mathbb{Z})$. Then as $g|_k\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix} = g$,

$$\langle T_p f, g\rangle = p^{k/2-1}(p+1)\langle f, g|_k\delta^{-1}\rangle = p^{k/2-1}(p+1)\langle f, g|_k\delta^a\rangle$$

$$= p^{k/2}(p+1).\overline{\langle g|_k\delta^a, f\rangle} = \overline{\langle T_p g, f\rangle} = \langle f, T_p g\rangle. \qquad \square$$

Example: Eisenstein series. $N\ge 1$, $k\ge 3$, $\underline{r} = (r_1, r_2) \in \mathbb{Q}^2$ (row vectors).

$$G_{\underline{r},k}(z) = \sum_{\underline{m}\in\mathbb{Z}^2}{}' \frac{1}{[(m_1+r_1)z + m_2+r_2]^k}$$ where $'$ means omit $\underline{m}+\underline{r}=0$ term.

(Converges absolutely as for $k>2$). Obviously depends only on $\underline{r}$ modulo $\mathbb{Z}^2$ (if $\underline{r}\in\mathbb{Z}^2$, $= G_k(z)$.)

Thm. (i) If $\gamma\in\Gamma(1)$, then $G_{\underline{r},k}|_k\gamma = G_{\underline{r}\gamma,k}$

(ii) $G_{\underline{r},k}(z) \in M_k(\Gamma(N))$ if $N\underline{r}\in\mathbb{Z}^2$.

Proof. (i) If $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $\underline{u}\in\mathbb{R}^2$

$$\frac{1}{(u_1z+u_2)^k}\Big|_k g = \frac{(\det g)^{k/2}}{((au_1+cu_2)z + (bu_1+du_2))^k} = \frac{(\det g)^{k/2}}{(v_1z+v_2)^k}, \quad \underline{v} = \underline{u}g$$

$$\text{So } G_{\underline{r},k}|_k\gamma = \sum_{\underline{m}}{}' \frac{1}{[((\underline{m}+\underline{r})g)_1 z + ((\underline{m}+\underline{r})g)_2]^k} = \sum_{\underline{m}'}{}' \frac{1}{((m_1'+r_1')z + m_2'+r_2')^k} = G_{\underline{r}',k}$$

$\underline{r}' = \underline{r}\gamma$, $\underline{m}' = \underline{m}\gamma$.

Recall: $\underline{r} \in \mathbb{Q}^2$, $k \geq 3$: $\quad G_{\underline{r},k}(z) = \sum'_{\underline{m} \in \mathbb{Z}^2} \frac{1}{[(m_1+r_1)z + m_2 + r_2]^k}$

Showed $\forall \gamma \in \Gamma(1)$, $G_{\underline{r},k}|_k \gamma = G_{\underline{r}\gamma,k}$. Now show:

(ii) $\underline{N\underline{r} \in \mathbb{Z}^2 \Rightarrow G_{\underline{r},k} \in M_k(\Gamma(N))}$.

If $\gamma \in \Gamma(N)$ and $N\underline{r} \in \mathbb{Z}^2$, $N\underline{r}\gamma \equiv N\underline{r}$ (mod N) $\Rightarrow \underline{r}\gamma \equiv \underline{r}$ mod $\mathbb{Z}^2$

So by (i), $G_{\underline{r},k}|\gamma = G_{\underline{r}\gamma,k} = G_{\underline{r},k}$. So STP $G_{\underline{r},k}$ holomorphic at cusps.

By (i), enough to show that $G_{\underline{r},k}$ is holo. at ∞ $\forall \underline{r}$.

$$G_{\underline{r},k} = \sum_{\substack{m_1+r_1>0 \\ m_2}} + \sum_{\substack{m_1+r_1=0 \\ m_2}} + \sum_{\substack{m_1+r_1<0 \\ m_2}}$$

First sum $= \sum_{m_1>-r_1} \sum_{m_2 \in \mathbb{Z}} \frac{1}{[(m_1+r_1)z + r_2 + m_2]^k} = \sum_{m_1>r_1} \sum_{d \geq 1} \frac{(-2\pi i)^k}{(k-1)!} d^{k-1} e^{2\pi i d r_2} q^{(m_1+r_1)d}$

2nd sum is constant.

3rd $= \sum_{m_1<-r_1} \sum_{m_2 \in \mathbb{Z}} \frac{(-1)^k}{((-m_1-r_1)z - r_2 - m_2)^k}$ = series in positive powers of $q^{-(m_1+r_1)}$

$\therefore$ holomorphic at ∞. $\square$

Next example: theta series

In lecture 3/4 we introduced the theta function $\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} q^{n^2/2}$

It's convenient to introduce 2 more ϑ-functions, called ϑ_2 and ϑ_4 traditionally:-

$$\vartheta_2(z) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 z} = 2q^{1/8} \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

$$\vartheta_4(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 z} = 1 + 2\sum_{n \geq 1} (-1)^n q^{n^2/2}.$$

Define $\vartheta_3 = \vartheta$ of old.

Thm (i) $\vartheta_2(z+1) = e^{\pi i/4} \vartheta_2(z)$, $\vartheta_3(z \pm 1) = \vartheta_4(z)$

(ii) $\vartheta_3(-1/z) = (z/i)^{1/2} \vartheta_3(z)$, $\quad \vartheta_4(-1/z) = (z/i)^{1/2} \vartheta_2(z)$

Proof (i) obvious; (ii) first part we already proved. 2nd part: modify proof for ϑ_3.

$h_t(x) = e^{-\pi t(x+1/2)^2} = g_t(x + 1/2)$. Recall $\hat{g}_t(y) = t^{-1/2} e^{-\pi y^2/t}$

$$\therefore \hat{h}_t(y) = \int e^{-2\pi i x y} g_t(x+1/2)\,dx = \int e^{-2\pi i x y + \pi i y} g_t(x)\,dx = e^{\pi i y} \hat{g}_t(y)$$

$$\therefore \vartheta_2(it) = \sum_{n \in \mathbb{Z}} h_t(n) = \sum_{n \in \mathbb{Z}} \hat{h}_t(n) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot t^{-1/2} e^{-\pi n^2/t} = t^{-1/2}\,\vartheta_4(i/t).$$

Coroll (i) Let $F = \begin{pmatrix} \vartheta_2^4 \\ \vartheta_3^4 \\ \vartheta_4^4 \end{pmatrix}$. Then $F(z+1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} F(z)$, $\quad z^{-2}F(-1/z) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} F$

(ii) $\vartheta_j^4 \in M_2(\Gamma)$ for a sgp. $\Gamma \subset \Gamma(1)$ of finite index. In particular,

$\vartheta_j^4|_2 \gamma$ is holomorphic at ∞ for every $\gamma \in GL_2(\mathbb{Q})^+$.

Proof (i). follows from thm. 20−2

(ii) Since $\overline{\Gamma(1)} = \langle S, T\rangle$, by (i) there is a homomorphism

$\rho : \Gamma(1) \longrightarrow GL_3(\mathbb{Z})$ with $F|_2\gamma = \rho(\gamma)F$, and $\rho(\gamma) = \pm$ a permutation matrix.

So if $\gamma \in \Gamma = \ker\rho$, $\vartheta_j^4|_2\gamma = \vartheta_j^4$; and as each ϑ_j is holomorphic at ∞,

so is $\vartheta_j^4|\gamma$ for any $\gamma \in \Gamma(1)$. So $\vartheta_j^4 \in M_2(\Gamma)$. $\square$

<u>Theorem</u>. $f(z) = \vartheta(2z)^4 \in M_2(\Gamma_0(4))$; and $f|_2 W = -f$ where $W = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$.
$(= \vartheta_3(2z)^4)$

<u>Lemma</u> $\Gamma_0(4)$ is generated by $-I$, $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 4 & \end{pmatrix}\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 4 & \end{pmatrix}^{-1}$

<u>Proof</u> STP $\overline{\Gamma_0(4)} = \langle T, U\rangle$, $T = \pm\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $U = \pm\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \in \overline{\Gamma_0(4)}$.

Let $\gamma = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma_0(4)}$. Consider $s(\gamma) = a^2 + b^2$. As c is even and $\det\gamma = 1$, a is odd.

So $s(\gamma) = 1 \iff \gamma = \pm\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = U^{c/4}$.

<u>Claim</u>: If $s(\gamma) \neq 1$, $\exists\, \delta \in \{T^{\pm 1}, U^{\pm}\}$ s.t. $s(\gamma\delta) < s(\gamma)$. Applying this repeatedly shows that $\gamma \in \langle T, U\rangle$.

To prove claim: $s(\gamma) \neq 1 \Rightarrow b \neq 0$, and a odd $\Rightarrow |a| \neq |2b|$. 2 cases:-

- $|a| < |2b|$. Then $\min\{|b \pm a|\} < |b| \Rightarrow s(\gamma T^{\pm 1}) = a^2 + (b \pm a)^2 < s(\gamma)$.
- $|a| > |2b|$. Then $\min\{|a \pm 4b|\} < |a| \Rightarrow s(\gamma U^{\pm 1}) = (a \pm 4b)^2 + b^2 < s(\gamma)$ $\square$

Recall: we are proving that $f(z) = \vartheta(2z)^4 \in M_2(\Gamma_0(4))$, and $f|_2 W_4 = -f$, $W_4 = \begin{pmatrix}0&-1\\4&0\end{pmatrix}$. (21-1)

Last time: $\overline{\Gamma_0(4)} = \langle T = \pm\begin{pmatrix}1&1\\0&1\end{pmatrix}, U = \pm\begin{pmatrix}1&0\\4&1\end{pmatrix} = W_4 T^{-1} W_4^{-1} \rangle$

We also saw that $\vartheta^4|_2 \gamma$ is holomorphic at ∞, $\forall \gamma \in GL_2(\mathbb{Q})^+$.

So to prove thm, STP: $f|_2 T = f = f|_2 U$

$\vartheta(z+2) = \vartheta(z) \Rightarrow f|_2 T = f$

$f|_2 W = 4.(4z)^{-2} f(-1/4z) = \frac{1}{4z^2}\cdot \vartheta(-1/2z)^4 = \frac{1}{4z^2}\cdot\left[\left(\frac{2z}{i}\right)^{1/2}\vartheta(2z)\right]^4 = -f(z)$

So $f|_2 U = f|_2 W|_2 T|_2 W^{-1} = (-1).(-1)f = f$. □

Look more closely at $\Gamma_0(4)$

First look at $\Gamma_0(2)$. Have: $(\Gamma(1):\Gamma_0(2)) = 3$ (since $\#SL_2(\mathbb{F}_2) = 6$, image$(\Gamma_0(2) \to SL_2(\mathbb{F}_2)) = \{1, \begin{pmatrix}1&1\\&1\end{pmatrix}\}$.)

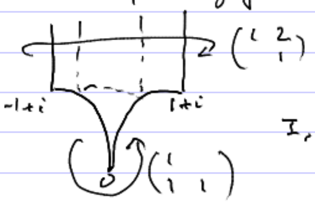
Coset reps. I, $\begin{pmatrix}1&\\1&1\end{pmatrix}$, $\begin{pmatrix}&1\\-1&\end{pmatrix}$.

$\Gamma_0(2) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is a homomorphism, kernel $\Gamma_0(4)$ since
$\begin{pmatrix}a&b\\2c&d\end{pmatrix} \mapsto c \bmod 2$ $\quad \begin{pmatrix}a&b\\2c&d\end{pmatrix}\begin{pmatrix}a'&b'\\2c'&d'\end{pmatrix} = \begin{pmatrix}*&*\\2(a'c+dc')&*\end{pmatrix}$ and $a' \equiv d \equiv 1 \pmod 2$.

$\therefore (\Gamma_0(2):\Gamma_0(4)) = 2$, coset reps $\begin{pmatrix}1&\\&1\end{pmatrix}$ $\begin{pmatrix}1&\\2&1\end{pmatrix}$
$\Rightarrow \overline{\Gamma_0(2)} = \langle T, U^2 = \begin{pmatrix}1&0\\2&1\end{pmatrix} = W_2\begin{pmatrix}1&-1\\&1\end{pmatrix}W_2^{-1}\rangle$ $\quad W_2 = \begin{pmatrix}&-1\\2&\end{pmatrix}$

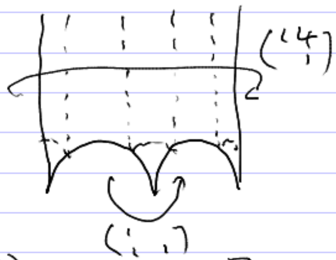
Fundamental domains for conjugate groups $\Gamma^0(2)$, $\Gamma^0(4)$:-

$\Gamma^0(2)$



coset reps I, $\begin{pmatrix}1&1\\&1\end{pmatrix}$, $\begin{pmatrix}&-1\\1&\end{pmatrix}$.

$\Gamma^0(4)$



coset reps $\begin{pmatrix}1&b\\&1\end{pmatrix}$ ($b \bmod 4$) $\begin{pmatrix}&-1\\1&\end{pmatrix}$, $\begin{pmatrix}2&-1\\1&\end{pmatrix}$.

Consider $g(z) = E_2|_2\begin{pmatrix}2&\\&1\end{pmatrix} - E_2 = 2E_2(2z) - E_2(z) = 1 + 24\sum \sigma_1(n)(q^n - 2q^{2n})$

Claim $g \in M_2(\Gamma_0(2))$, $g|_2 W_2 = -g$, $W_2 = \begin{pmatrix}&-1\\2&\end{pmatrix}$

Well, $g|_2 W_2 = \frac{2}{4z^2} g(-1/2z) = \frac{1}{z^2}E_2(-\frac{1}{z}) - \frac{2}{(2z)^2}E_2(\frac{-1}{2z})$

$\left(z^{-2}E_2(-\frac{1}{z}) = E_2(z) + \frac{12}{2\pi i z}\right)$ $\quad = E_2(z) + \frac{12}{2\pi i z} - \left(2E_2(2z) + 2\cdot\frac{12}{2\pi i\cdot 2z}\right) = -g$.

And $g|_2 T = g(z+1) = g$, so $g|_2 U^2 = g|W_2 T^{-1} W_2^{-1} = g$

g holomorphic at ∞, and so holomorphic at $0 = W_2(\infty)$

So $h = \frac{1}{2} g|\begin{pmatrix}2&\\&1\end{pmatrix} = 2E_2(4z) - E_2(2z) \in M_2(\Gamma_0(4)) \supset M_2(\Gamma_0(2))$

Now $\dim M_2(\Gamma_0(4)) \le 1 + \frac{2.(\Gamma(1):\Gamma_0(4))}{12} \le 2$, and h, g obviously lin. independent.

So: $\boxed{M_2(\Gamma_0(4)) = \mathbb{C}g \oplus \mathbb{C}h}$, and in particular, $f = ag + bh$ for some a, b.

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$f = \vartheta(2z)^4 = (1+2q+2q^4+\dots)^4 = 1+8q+24q^2+32q^3+\dots$

$g = 1+24\sum\sigma_1(n)(q^n-2q^{2n}) = 1+24q+24q^2+96q^3+\dots$

$h = g(2z) \qquad = 1 \qquad +24q^2 \qquad +$

$$\Rightarrow \quad f = \tfrac{1}{3}(g+2h) = \tfrac{1}{3}(4E_2(4z)-E_2(z)) = 1+8\sum_{n\geq 1}\left(\sigma_1(n)-4\sigma_1(n/4)\right).q^n$$

OTOH, $f = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^4 = \sum_{a,b,c,d\in\mathbb{Z}} q^{a^2+b^2+c^2+d^2}$ $\qquad (\sigma_1(n/4) \overset{\text{def}}{=} 0 \text{ if } 4\nmid n)$

$$= \sum_{n\geq 0} \#\{(a,\dots,d)\in\mathbb{Z}^4 \mid n = a^2+b^2+c^2+d^2\}.q^n.$$

<u>Coroll.</u> $\forall n\geq 1$, $\#\{(a,b,c,d)\in\mathbb{Z}^4 \mid n = a^2+b^2+c^2+d^2\}$

$$= 8(\sigma_1(n) - 4\sigma_1(n/4)).$$

$$= 8\sum_{\substack{d\mid n\\ 4\nmid d}} d.$$

In particular this is always >0 (Lagrange's 4-squares theorem).

<u>Proof of last $=$:</u> if $d\mid n$ then either $4\nmid d$ or $4\mid d \Rightarrow \frac{d}{4}\mid\frac{n}{4}$

$$\text{so } \sum_{d\mid n} d = \sum_{\substack{d\mid n\\ 4\nmid d}} d + \sum_{e\mid \frac{n}{4}} 4e. \qquad \square$$

<u>Remark.</u> Looking instead at $\vartheta(2z)^2 \in M_1(\Gamma_1(4))$, can get a formula for $\#\{(a,b)\in\mathbb{Z}^2 \mid n = a^2+b^2\}$ which implies the 2-squares theorem.

<u>Remark.</u> $W_N = \begin{pmatrix} & -1\\ N & \end{pmatrix}$. Then $\gamma = \begin{pmatrix} a & b\\ Nc & d\end{pmatrix} \in \Gamma_0(N) \Rightarrow W_N\gamma W_N^{-1} = \begin{pmatrix} d & -c\\ -Nb & a\end{pmatrix} \in \Gamma_0(N)$

$\Rightarrow$ if $f \in \begin{cases} M_k(\Gamma_0(N))\\ S_k\end{cases}$ then so is $f|_k W_N$, and $f|_k W_N^2 = f|_k\begin{pmatrix} -N & \\ & -N\end{pmatrix} = f$

(as $-I \in \Gamma_0(N)$)

So $M_k(\Gamma_0(N)) = M_k(\Gamma_0(N))^+ \oplus M_k(\Gamma_0(N))^-$

$\pm$-eigenspaces for W_N

& likewise for S_k (and decomposition is orthogonal for inner product, as it is $GL_2(\mathbb{Q})^+$-invariant).

W_N is the <u>Atkin-Lehner involution</u>.

§11. Hecke theory for $\Gamma_0(N)$

Thm $f \in S_k(\Gamma_0(N))^\varepsilon$ i.e. $f|_k W_N = \varepsilon f$, $\varepsilon \in \{\pm 1\}$.

Then $L(f,s) = \sum_{n \geq 1} a_n(f) n^{-s}$ is entire, with F.E.

$$\Lambda(f,s) = \Gamma_{\mathbb{C}}(s) L(f,s) = \varepsilon (-1)^{k/2} \Lambda(f, k-s).$$

Hecke operators for $\Gamma_0(N)$, $N>1$ are a bit more complicated.

Thm (Strong multiplicity 1 for $SL_2(\mathbb{Z})$)

Let $f, g \in S_k(\Gamma(1))$ be normalized Hecke eigenfns: $f|T_p = \lambda_p f$, $\lambda_p = a_p(f)$
$g|T_p = \mu_p f$, $\mu_p = a_p(g)$

Suppose $\lambda_p = \mu_p$ $\forall p \notin$ finite set S.

Then $f = g$.

[Idea of proof. $\Lambda(f,k-s) = (-1)^{k/2}\Lambda(f,s)$ & likewise for g. $\Rightarrow \frac{L(f,k-s)}{L(g,k-s)} = \frac{L(f,s)}{L(g,s)}$

$$\Rightarrow \prod_{p \in S} \frac{1-\lambda_p p^{-s} + p^{k-1-2s}}{1-\mu_p p^{-s} + p^{k-1-2s}} \cdot \frac{1-\mu_p p^{s-k} + p^{2s-k-1}}{1-\lambda_p p^{s-k} + p^{2s-k-1}} = 1.$$

Looking at poles/zeroes $\Rightarrow$ (factor at p) $= 1$ $(p \in S)$ $\Rightarrow \lambda_p = \mu_p$.

and $\lambda_p = \mu_p$ $\forall p \Rightarrow a_n(f) = a_n(g)$ $\forall n$ $\Rightarrow f = g$.]

Defn. If $p \nmid N$: $T_p f = p^{k/2-1}\left[f|_k \begin{pmatrix} p & \\ & 1 \end{pmatrix} + \sum_{b=0}^{p-1} f|_k \begin{pmatrix} 1 & b \\ & p \end{pmatrix} \right]$

If $p \mid N$: $U_p f = p^{k/2-1} \sum_{b=0}^{p-1} f \Big| \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$
(also called T_p)

Effect on q-expansions: $p \nmid N$: $a_n(T_p f) = a_{np}(f) + p^{k-1} a_{n/p}(f)$

$p \mid N$: $a_n(U_p f) = a_{np}(f)$

Propn $T_p, U_p : S_k(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(N))$, and they commute with one another.

Proof (sketch) check that $p \nmid N \Rightarrow \Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} p & \\ & 1 \end{pmatrix} \amalg \coprod_{b=0}^{p-1} \Gamma_0(N)\begin{pmatrix} 1 & b \\ & p \end{pmatrix}$

$p \mid N \Rightarrow \Gamma_0(N)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(N) = \coprod_{b=0}^{p-1} \Gamma_0(N)\begin{pmatrix} 1 & b \\ & p \end{pmatrix}$

For commutativity, can check on q-expansions which is easy

However these don't generate $\mathcal{H}(GL_2(\mathbb{Q})^+, \Gamma_0(N))$.

Ex. $S_{12}(\Gamma_0(2)) \ni f = \Delta(z)$, $g = \Delta|_{12}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2^6 \Delta(2z) = \Delta|_{12} W_2$

since $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & -1 \\ 2 & \end{pmatrix}$

So matrix of W_2 is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$f = \sum \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4380q^5 - 6048q^6 \ldots$ 22-2

$g = 2^6 \sum \tau(n) q^{2n} = 64[q^2 - 24q^4 + 252q^6 - \ldots]$

$\Rightarrow U_2 g = 2^6 \sum \tau(n) q^n = 64f.$

$U_2 f = \sum \tau(2n) q^n = -24q - 1472q^2 - 6048q^3 + \ldots = -24f - 32g$

So matrix of U_2 is $\begin{pmatrix} -24 & 64 \\ -32 & 0 \end{pmatrix}$ since $1472 = -24^2 + 32.64$

So U_2, W_2 don't commute, and Δ generates a 2-dim rep^n of $\mathcal{H}(GL_2(\mathbb{Q})^+, \Gamma_0(2))$

Similarly can consider $\Delta(dz) \in S_{12}(\Gamma_0(N))$ for any $d \mid N$.

Point: • Δ is a modular form of level 1.

• $\{\Delta|_k \gamma \quad \gamma \in G = GL_2(\mathbb{Q})^+\}$ span a large (∞-dimensional) representation V of G; $V^{\Gamma(1)} = \mathbb{C}\Delta$ is 1-dimensional.

In general :- $N \geq 1$. $S_k(\Gamma_0(N))$

$\forall M \mid N$, $\forall d \mid N/M$, have $S_k(\Gamma_0(M)) \xrightarrow{\alpha_{M,d}} S_k(\Gamma_0(N))$

$f \longmapsto f(dz)$

- the span of $\bigcup_{\substack{M \mid N \\ d \mid N/M}} \mathrm{im}(\alpha_{M,d})$ is a subspace $S_k(\Gamma_0(N))^{\text{old}}$ old forms

- Complement (for inner Π) is a space $S_k(\Gamma_0(N))^{\text{new}}$. new forms

This behaves just like $S_k(\Gamma(1))$:

- image of Hecke alg. $\mathcal{H}(G, \Gamma_0(N))$ is a commutative, $*$-closed subalgebra of $\mathrm{End}\, S_k^{\text{new}}$,

so diagonalisable

- "strong multiplicity one" holds for S_k^{new}

even better: for any collection of Hecke eigenvalues $\{\lambda_p \mid p \notin S\}$ $\exists$ at most one N, and at most one normalised eigenform $f \in S_k(\Gamma_0(N))^{\text{new}}$ with $T_p f = \lambda_p f \quad \forall p \nmid N, p \notin S$.