Modular forms and L-functions (Lent 2017) — example sheet #3

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(In these questions, Γ denotes a finite index subgroup of $\Gamma(1)$.)

1. Fix an even weight $k \geq 4$. Let $\mathbb{T} \subset \operatorname{End} S_k$ be the subalgebra of endomorphisms generated over $\mathbb{Z}$ by the Hecke operators T_n, $n \geq 1$. Let $S_k(\mathbb{Z}) = S_k(\Gamma(1), \mathbb{Z})$ denote the submodule of cusp forms of level 1 and weight k with integral Fourier coefficients. Show that $S_k(\mathbb{Z})$ is stable under $\mathbb{T}$, and that the map
$$\begin{aligned} S_k(\mathbb{Z}) \times \mathbb{T} &\to \mathbb{Z} \\ (f, T_n) &\mapsto a_1(T_n f) \end{aligned}$$
gives an isomorphism between $S_k(\mathbb{Z})$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z})$, which is an isomorphism of $\mathbb{T}$-modules.
(Hint consider the basis $\{g_j \mid 1 \leq j \leq m = \dim S_k\}$ for which $a_n(g_j) = \delta_{jn}$ for $1 \leq n \leq m$.)

2. Let (G, Γ) satisfy property (H). Suppose there is a map $\sigma\colon G \to G$ such that
   (i) $\sigma^2(g) = g$ and $\sigma(gh) = \sigma(h)\sigma(g)$ for all $g, h \in G$; and
   (ii) σ fixes every double coset $\Gamma g \Gamma$.

   Show that the Hecke algebra of (G, Γ) is then commutative. Use this to give another proof of the commutativity of the Hecke algebra for $(GL_2(\mathbb{Q})^+, SL_2(\mathbb{Z}))$.

3. Let $\Lambda \subset \mathbb{R}^k$ be a lattice (i.e. a discrete subgroup of maximal rank). Say that Λ is *self-dual* if the set
$$\Lambda^* = \{x \in \mathbb{R}^k \mid x \cdot y \text{ is an integer for every } y \in \Lambda\}$$
is precisely Λ, and that Λ is *even* if $\|x\|^2 \in 2\mathbb{Z}$ for every $x \in \Lambda$. (Here $\|-\|$ denotes Euclidean norm.)
Use the Poisson summation formula to show that if $\Lambda \subset \mathbb{R}^k$ is an even self-dual lattice, then the theta series
$$\theta_\Lambda(z) = \sum_{x \in \Lambda} \exp(\pi i \|x\|^2 z)$$
is a modular form of weight $k/2$ and level 1. (In particular, k is divisible by 4.)

4. Let k be a positive integer divisible by 4. Let Λ be the set of all $x = (x_1 \ldots, x_k) \in \mathbb{R}^k$ satisfying
$$2x_i \in \mathbb{Z}, \qquad x_i - x_j \in \mathbb{Z}, \qquad \sum_{i=1}^{k} x_i \in 2\mathbb{Z}.$$
Show that Λ is a self-dual lattice. [Λ is usually denoted E_k.]
(ii) Suppose further that k is divisible by 8. Show that Λ is even.
(iii) Finally let $k = 8$. Show that $\theta_\Lambda(z) = E_4(z)$. Hence (or directly) show that there are exactly 240 elements $x \in \Lambda$ with $\|x\|^2 = 2$.

5. Let $p > 2$ be prime. Draw a fundamental domain for $\Gamma^0(p)$ as given in the lectures, and show that the identifications of points along the boundary are given as follows:
   - the vertical lines $\operatorname{Im}(z) = \pm p/2$ are identified by the translation $z \mapsto z + p$.
   - the circular arcs $C_a = \{|z - a| = 1\}$, for integers a with $0 < |a| < p/2$, are identified as follows: C_a is identified with C_b iff $ab \equiv -1 \pmod{p}$.

6. (i) Show that Γ has a fundamental domain which is a *connected* union of translates of the standard fundamental domain for $\Gamma(1)$.

   (ii) * Show that if Γ has no elements of finite order other than ± 1 then Γ is a free group. (Use the fact that any group acting without fixed points on a tree is free.)

7. (i) Show that $\Gamma(N)$ is torsionfree if $N \geq 3$ and that the only elements of finite order of $\Gamma(2)$ are $\{\pm 1\}$.

   (ii) Show that if $N \geq 4$ then $\Gamma_1(N)$ is torsionfree.

8. Let

$$\Gamma^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

   (i) Show that if $f \in M_k(\Gamma)$, then the function $f^*(z) = \overline{f(-\bar{z})}$ belongs to $M_k(\Gamma^*)$.

   (ii) Show that if $\Gamma = \Gamma^*$ (for example, any one of $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$) then $M_k(\Gamma)$ has a basis all of whose elements have real Fourier coefficients.

9. (i) Let ν be the number of cusps of Γ. Show that there is a linear map $M_k(\Gamma) \to \mathbb{C}^\nu$ whose kernel is $S_k(\Gamma)$.

   (ii) By considering the Eisenstein series $E_{\underline{r},k}$ for suitable $\underline{r} \in N^{-1}\mathbb{Z}^2$, show that if $k > 2$ then there exists $f \in M_k(\Gamma(N))$ whose q-expansion at ∞ has contant term 1, but which vanishes at all other cusps of $\Gamma(N)$. Deduce that $\dim M_k(\Gamma(N)) = \nu + \dim S_k(\Gamma(N))$ if $k \geq 3$.

10. (i) Show that if every cusp of Γ has width one then Γ must be $\Gamma(1)$.

    (ii) * Show that if Γ is a congruence subgroup containing -1, then $\Gamma \supset \Gamma(N)$ where N is the least common multiple of the widths of the cusps of Γ. (This gives a way to tell whether or not a given group is a congruence subgroup.)