Modular forms and L-functions (Lent 2017) — example sheet #2

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1. (i) The *Bernoulli polynomials* $B_n(X)$ are defined by the formula

$$\sum_{k=0}^{\infty} B_k(X)\frac{t^k}{k!} = \frac{te^{tX}}{e^t-1}.$$

Let $\psi\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be any periodic function. Use the analytic continuation to write the values at negative integers of the L-series $L(\psi, s)$ in terms of the values $B_k(j/N)$.

(ii) Consider the (inverse) Fourier transform

$$\widehat{B}_{n,N}(\zeta) = \sum_{j=0}^{N-1} \zeta^j B_n(j/N) \qquad (\zeta^N = 1)$$

Show that if $\zeta \neq 1$, then $\widetilde{B}_n(\zeta) \stackrel{\text{def}}{=} N^{1-n} B_{n,N}(\zeta) = P_n(\zeta)/(\zeta-1)^n$ for some polynomial P_n which does not depend on N. (The substitution $u = e^{-t/N}$ may be useful.)

(iii) Obtain for $D > 1$ the *distribution relation*: if $\zeta^N = 1$ then

$$\sum_{\eta^D=\zeta} \widetilde{B}_n(\eta) = D^n \widetilde{B}_n(\zeta)\,.$$

2. Fix a positive real number D. Let $\mathcal{X}_D$ be the set of real symmetric 2×2 matrices of determinant D. Let $SL_2(\mathbb{R})$ act on $\mathcal{X}_D$ by $g\colon X \to gXg^t$ (g^t = transpose of g). Describe the orbits of this action, and identify one of them with the upper halfplane.

3. Write $E_6(z)\Delta(z) = \sum_{n=1}^{\infty} c_n q^n$. Show that $c_n \equiv \sigma_{17}(n) \pmod{43867}$. Obtain similar congruences for the coefficients of $E_4\Delta$, $E_8\Delta$, $E_{10}\Delta$ and $E_{14}\Delta$.

[NB: $B_{16} = -3617/510$, $B_{18} = 43867/798$, $B_{20} = -174611/330$, $B_{22} = 854513/138$, $B_{26} = 8553103/6$.]

4. (i) Let f be a modular function (= a weakly modular form of weight 0). Show that $\mathrm{ord}_{\tau=i} f \equiv 0 \pmod 2$ and $\mathrm{ord}_{\tau=\rho} f \equiv 0 \pmod 3$.

(ii) Let $f \in M_k$. Show that if $k \not\equiv 0 \pmod 4$ then $f(i) = 0$, and that if $k \not\equiv 0 \pmod 3$ then $f(\rho) = 0$, where $\rho = e^{\pi i/3}$.

5. Let $f \in M_k$ and $g \in M_l$ be modular forms. Show that $lf'g - kfg' \in M_{k+l+2}$.

6. Let $f\colon \mathcal{H} \to \mathbb{C}$ satisfy $f|_k\gamma = f$ for all $\gamma \in \Gamma(1)$. Show that $y^{k/2}\,|f(x+iy)|$ is invariant under $z = x+iy \mapsto \gamma(z)$. Show that if moreover f is holomorphic on $\mathcal{H}$ and $k > 0$, then f is a cusp form if and only if $y^{k/2}\,|f|$ is bounded on $\mathcal{H}$ (or equivalently, is bounded on $\mathcal{D}$).

7. Define

$$G_2(z) = \sideset{}{'}\sum_{m=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2} \right)$$

where the inner sum is over all integers n, except where $m = 0$, in which case the term $n = 0$ is omitted. By rewriting the inner sum, show that the series converges to

$$\frac{\pi^2}{3}\left(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n\right).$$

Explain why $G_2(z)$ is not a modular form of weight 2. (Later we will prove that G_2 satisfies a somewhat more complicated transformation law for $z \mapsto -1/z$.)