Modular forms and L-functions (Lent 2017) — example sheet #1

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1. Prove that if $f(x) = e^{-\pi x^2}$ then $\widehat{f}(x) = f(x)$.

2. Let G be a finite abelian group, and χ, χ' characters of G. Show that

$$\sum_{g \in G} \overline{\chi(g)} \chi'(g) = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ \#G & \text{if } \chi = \chi' \end{cases}$$

3. Show that every continuous homomorphism $\chi \colon \mathbb{R}_{>0}^{\times} \to \mathbb{C}^{\times}$ is of the form $\chi(x) = x^s$ for some $s \in \mathbb{C}$. [Hint: first describe all continuous homomorphisms $\mathbb{R} \to \mathbb{C}$.] Show also that $\widehat{\mathbb{R}} = \{\chi_y \colon x \mapsto e^{2\pi i x y} \mid y \in \mathbb{R}\}$.

4. Show that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \Lambda(n) n^{-s}$$

   where Λ is the *Von Mangoldt function*:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ k \geq 1,\ p \text{ prime} \\ 0 & \text{if } n = 1 \text{ or } n \text{ is not a prime power} \end{cases}$$

   Show also that $\sum_{d|n} \Lambda(d) = \log n$.

5. Evaluate $\zeta(2k)$ is terms of Bernoulli numbers. Deduce that $(-1)^{k-1} B_{2k} > 0$ for every $k \geq 1$. (It is not easy to prove this directly from the generating function definition!)

6. A subgroup $\Lambda \subset \mathbb{R}^n$ is *discrete* if is is discrete as a topological space (i.e. for any ball $B \subset \mathbb{R}^n$, $\Lambda \cap B$ is finite). Show that Λ is discrete iff $\Lambda = \sum \mathbb{Z} x_i$ where $\{x_i\} \subset \mathbb{R}^n$ is an $\mathbb{R}$-linearly independent set.