Solution to Q.10, ex. sheet 1.

In the lectures we proved (Prop. 2.6) that there exists a basis $g_1, \dots, g_m$ of S_k with $g_j \in S_k(\mathbb{Z})$ for all j, and

$$a_i(g_j) = \delta_{i,j} \quad \text{for} \quad 1 \le i \le m.$$

So $S_k(\mathbb{Z}) = \bigoplus_{j=1}^{m} \mathbb{Z} g_j$ (if $f \in S_k(\mathbb{Z})$ then $f = \sum_{j=1}^{m} a_j(f).g_j$)

Let $\mathbb{T} \subset \operatorname{End} S_k$ be the $\mathbb{Z}$-subalgebra generated by the Hecke operators $(T_n)_{n \ge 1}$. We have the formula $a_m(T_n f) = \sum_{d \mid (m,n)} d^{k-1} a_{mn/d^2}(f)$

so $f \in S_k(\mathbb{Z}) \Rightarrow \forall T \in \mathbb{T},\ Tf \in S_k(\mathbb{Z})$.

Consider the pairing

$$\langle\, ,\, \rangle : S_k(\mathbb{Z}) \times \mathbb{T} \longrightarrow \mathbb{Z}$$
$$(f, T) \longmapsto a_1(Tf).$$

So $\langle f, T_n \rangle = a_1(T_n f) = a_n(f)$.

If $1 \le j \le m$, $1 \le n \le m$ then $\langle g_j, T_n \rangle = a_n(g_j) = \delta_{n,j}$.

So, if $\mathbb{T}' \subset \mathbb{T}$ is the $\mathbb{Z}$-submodule generated by $T_1, \dots, T_m$,

$\langle\, ,\, \rangle : S_k(\mathbb{Z}) \times \mathbb{T}' \to \mathbb{Z}$ is a perfect pairing, i.e. induces isomorphisms

$$S_k(\mathbb{Z}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{T}', \mathbb{Z}), \qquad \mathbb{T}' \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(S_k(\mathbb{Z}), \mathbb{Z}).$$
$$f \longmapsto (T \mapsto \langle f, T \rangle) \qquad T \longmapsto (f \mapsto \langle f, T \rangle)$$

(as (g_j), $(T_n)_{n \le m}$ are dual bases for $S_k(\mathbb{Z})$ and $\mathbb{T}'$).

Now as the $\{T_n\}$ commute and are simultaneously diagonalisable, $\mathbb{T}$ has rank at most m (* below). As $\mathbb{T}'$ has rank m and $\mathbb{T}$ is torsion-free, this means that $\mathbb{T}' \subset \mathbb{T} \subset \mathbb{T} \otimes \mathbb{Q} = \bigoplus_{n=1}^{m} \mathbb{Q}.T_n \subset \operatorname{End} S_k$

Suppose $T = \sum_{n \ge 1}^{m} c_n T_n \in \mathbb{T}$, so $c_n \in \mathbb{Q}$. Then

$\mathbb{Z} \ni \langle g_j, T \rangle = \sum c_n \langle g_j, T_n \rangle = c_n$. So $T \in \mathbb{T}'$ i.e. $\mathbb{T} = \mathbb{T}'$.

Finally we show that $\langle\, ,\, \rangle$ induces an isomorphism of $\mathbb{T}$-modules

$$S_k(\mathbb{Z}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z})$$

The $\mathbb{T}$-module structure on $\operatorname{Hom}(\mathbb{T}, \mathbb{Z})$ is given by

$T \in \mathbb{T}$, $\varphi \in \operatorname{Hom}(\mathbb{T}, \mathbb{Z})$: the $(T\varphi)(Y) = \varphi(TY)$ $\forall Y \in \mathbb{T}$.

(As $\mathbb{T}$ is commutative there is no difference between left and right modules).

So taking $\varphi = \langle f, - \rangle$ we need to check that $\forall T, Y \in \mathbb{T}$,

$$\langle f, TY \rangle = \langle Tf, Y \rangle$$

But $\langle f, T\Upsilon\rangle = a_1((T\Upsilon)f) = a_1(\Upsilon(Tf)) = \langle Tf, \Upsilon\rangle$.

(*) Choosing a basis of $S_k(\mathbb{Q})$ [e.g. the basis (g_i)] identifies $\mathbb{T}$ with a subalgebra of $\mathrm{Mat}_{m\times m}(\mathbb{Q})$. To show $\mathbb{T}$ has rank $\le m$ it's enough to show that $\mathbb{T}\otimes_{\mathbb{Z}}\mathbb{C} \subset \mathrm{Mat}_{m\times m}(\mathbb{C})$ has dimension $\le m$. But simultaneously diagonalising the T_n's shows that $\mathbb{T}\otimes_{\mathbb{Z}}\mathbb{C}$ is conjugate to an algebra of diagonal matrices, hence has dimension $\le m$.

Sketch solutions of other questions

1. If $\Lambda \subset \mathbb{R}^n$ is discrete, let $x_1 \in \Lambda\setminus\{0\}$ be an element with $\|x_1\|$ minimal, and inductively choose $x_1, x_2, \dots x_m$ such that

$$(k>1)\quad x_k \in \Lambda \setminus \sum_{1\le i<k} \mathbb{R}.x_i \quad \text{with } \|x_k\| \text{ minimal.}$$

Stop when $\Lambda \subset \sum_{1\le i\le m} \mathbb{R}x_i$. By construction $x_1, \dots x_m$ are linearly independent over $\mathbb{R}$.

Set $\Lambda_k = \bigoplus_{i=1}^{k} \mathbb{Z}x_i$ and check by induction on k that $\Lambda \cap \bigoplus_{i=1}^{k}\mathbb{R}x_i = \Lambda_k$.

2. This is easy.

3. $$\wp''(z) = 6/z^4 + \sum_{k=2}^{\infty}(2k-1)(2k-2)(2k-3)G_{2k}z^{2k-4}$$

$$= 6\wp(z)^2 - \frac{g_2}{2} = -\frac{g_2}{2} + 6\left(1/z^2 + \sum_{m=2}^{\infty}(2m-1)G_{2m}z^{2m-2}\right)^2$$

$$= -\frac{g_2}{2} + 6/z^4 + 12\sum_{m=2}^{\infty}(2m-1)G_{2m}z^{2m-4} + 6\left(\sum_{m\ge 2}(2m-1)G_{2m}z^{2m-2}\right)^2$$

$$\Rightarrow \underbrace{\left[(2k-1)(2k-2)(2k-3) - 12(2k-1)\right]}_{>0 \text{ if } k\ge 4} G_{2k} = \text{poly. in } G_4, \dots G_{2k-2} \text{ with positive integral coefficients}$$

4. Identify $X = \begin{pmatrix} a & b\\ b & c\end{pmatrix}$ with quadratic form $aU^2 + 2bUV + cV^2 = (U\ V)X\begin{pmatrix}U\\V\end{pmatrix}$

Congruence $g: X \mapsto gXg^t$ of matrices $\Longleftrightarrow$ equivalence of quadratic forms under change of variables $(U\ V) \mapsto (U\ V)g$

$\therefore$ Under $SL_2(\mathbb{R})$, each X with $\det X = D$ is equiv. to one of $\pm\begin{pmatrix}\sqrt{D} & \\ & \sqrt{D}\end{pmatrix}$.

Stabilizer of $\begin{pmatrix}\sqrt{D} & \\ & \sqrt{D}\end{pmatrix}$ is $SO(2) = \{g \in SL_2(\mathbb{R}) \mid gg^t = I\}$ so

orbit $\simeq SL_2(\mathbb{R})/SO(2) \simeq \mathfrak{h}$.

$$g.\begin{pmatrix}\sqrt{D} & \\ & \sqrt{D}\end{pmatrix}g^t = \frac{\sqrt{D}}{y}\begin{pmatrix}x^2+y^2 & x\\ x & 1\end{pmatrix} \longleftrightarrow \tau = g(i),\quad g = \begin{pmatrix}y^{1/2} & xy^{-1/2}\\ 0 & y^{-1/2}\end{pmatrix}$$

Gauss's reduction for positive definite binary quadratic forms states that each $SL_2(\mathbb{Z})$-orbit of quadratic forms $aU^2 + 2bUV + cV^2$ of discriminant $-4D = 4(b^2-ac) < 0$ which are positive definite contains a unique representative with $-a < 2b \le a < c$ or $0 \le 2b \le a = c$.

If $a = \sqrt{D}/y$, $b = x\sqrt{D}/y$, $c = (x^2+y^2)\frac{\sqrt{D}}{y}$, this is the condition that $\tau = x+iy$ belongs to the fundamental domain $\mathcal{D}$ for $SL_2(\mathbb{Z})$ acting on $\mathfrak{H}$.

(5) Show that the difference is an elliptic function with no poles.

(6) Copy the proof of $\tau(n) \equiv \sigma_{11}(n) \bmod 691$.

(7) (i) $\tau = \begin{cases} i \\ \rho = e^{\pi i/3} \end{cases}$ is a fixed point of $\gamma = \begin{cases} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\ \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix} \end{cases}$

Let $f(\tau) = (\tau-a)^r g(\tau)$, $g(a) \neq 0$ with $a = \begin{cases} i \\ \rho \end{cases}$

Then $f(\tau) = f(\gamma(\tau)) \Rightarrow (\tau-a)^r g(\tau) = (\gamma(\tau)-a)^r g(\gamma(\tau)) = \left(\frac{\tau-a}{a\tau}\right)^r g(\gamma(\tau))$ in either case.

so $g(\tau) = (1/a\tau)^r g(\gamma(\tau))$

$\tau = a \Rightarrow a^{2r} = 1 \Rightarrow r \equiv 0 \bmod \begin{cases} 2 & a = i \\ 3 & \rho. \end{cases}$

(ii) same but easier; compare $f(\gamma(a))$ and $f(a)$.

(8) Easy.

(9) Use $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} = -4\pi^2 \sum_{d=1}^{\infty} d e^{2\pi i d z}$, and observe that

$\sum_{m\ge 1}\sum_{d\ge 1} d q^{md}$ is absolutely convergent.

$G_2 \notin M_2$ since $M_2 = 0$; the proof $G_k \in M_k$ fails for $k=2$ as the double series is only conditionally convergent, so

$$\tau^{-2} G_2(-1/\tau) = \tau^{-2} \sum_m{}' \left(\sum_n \frac{1}{(-m/\tau + n)^2} \right) = \sum_m{}' \sum_n \frac{1}{(n\tau+m)^2}$$

but this isn't the same as the series for $G_2(\tau)$. (In fact we can compute explicitly the difference, see later in course)