5 Eisenstein series

The series $G_k(\tau)$ are the simplest examples. The basic Eisenstein series is a *non-holomorphic* modular form:

Definition. The real-analytic Eisenstein series is the function

$$G(\tau, s) = \sum_{m,n}{}' \frac{y^s}{|m\tau + n|^{2s}}, \qquad \mathrm{Re}(s) > 1,\ \mathrm{Im}(\tau) > 0$$

(The summation is over $(m, n) \in \mathbb{Z}^2 - \{(0, 0\}.)$ This is a function of two variables τ, s.

If we divide each pair by its HCF, we can write

$$G(\tau, s) = \sum_{d=1}^{\infty} \frac{1}{d^{2s}} \sum_{(m,n)=1} \frac{y^s}{|m\tau + n|^{2s}} = 2\zeta(2s)E(\tau, s)$$

where

$$E(\tau, s) = \frac{1}{2} \sum_{(m,n)=1} \frac{y^s}{|m\tau + n|^{2s}}$$

Proposition 5.1.

$$E(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(\tau))^s$$

where

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{Z} \right\} \subset \Gamma$$

is the stabiliser of ∞.

Proof. As $\mathrm{Im}\,\gamma(\tau) = y/|c\tau + d|^2$ it's enough to check that $\gamma \mapsto (c, d)$ defines a bijection

$$\Gamma_\infty \backslash \Gamma \xrightarrow{\sim} \{(m, n) \in \mathbb{Z}^2 \mid \gcd(m, n) = 1\}/(m, n) \sim (-m, -n)$$

which is elementary. □

Here are some basic properties of the Eisenstein series.

Proposition 5.2. *(i)* $G(\tau, s)$ *is holomorphic as a function of* s *for* $\mathrm{Re}(s) > 1$.
(ii) For every $\gamma \in \Gamma = SL_2(\mathbb{Z})$, $G(\gamma(\tau), s) = G(\tau, s)$.
(iii) $\Delta G(\tau, s) = s(1 - s)G(\tau, s)$.

Here

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the hyperbolic Laplacian, or *Laplace-Beltrami operator.*

Lemma. *(i)* $\Delta = -4y^2 \frac{\partial^2}{\partial \bar{\tau} \partial \tau}$.
(ii) Δ *is* $SL_2(\mathbb{Z})$*-invariant: for all* $\gamma \in SL_2(\mathbb{Z})$, $\Delta(f(\gamma(\tau)) = (\Delta f)(\gamma(\tau))$.

Proof. (i) Follows from $\partial/\partial\tau = 1/2(\partial/\partial x - i\partial/\partial y)$.
(ii) Enough to check this for $\gamma \colon \tau \mapsto \tau + 1 = (x+1) + iy$, which is obvious, and for $\gamma \colon \tau \mapsto -1/\tau$:

$$\begin{aligned} \Delta \left(f \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) \right) &= -4y^2 \frac{1}{\tau^2} \frac{1}{\bar{\tau}^2} \frac{\partial^2 f}{\partial \bar{\tau} \partial \tau} \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) \\ &= -4 \operatorname{Im} \left(-\frac{1}{\tau} \right)^2 \frac{\partial^2 f}{\partial \bar{\tau} \partial \tau} \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}} \right) \end{aligned}$$

□

In fact Δ is invariant under $SL_2(\mathbb{R})$, since it is generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad (b \in \mathbb{R})$$

Proof of 5.2. (i) follows from the uniform convergence of the series on sets $\operatorname{Re}(s) \geq 1 + \epsilon$ (Propn.1.2).
(ii) This is obvious from the formula (5.1).
(iii) For fixed s with $\operatorname{Re}(s) > 1$, the partial derivatives (of any order) of the series defining $G(\tau, s)$ are easily seen to be uniformly convergent on compact sets. So differentiating termwise and using the Γ-invariance of Δ and (5.1), it is enough to check that $\Delta \operatorname{Im}(\tau)^s = s(1-s) \operatorname{Im}(\tau)^s$ which is trivial. □

Let $\Lambda_\tau = y^{-1/2}(\mathbb{Z} \oplus \mathbb{Z}\tau) \subset \mathbb{C}$. Then evidently $G(\tau, s) = E_{\Lambda_\tau}(s)$. So we can apply our results about Epstein zeta functions to $G(\tau, s)$. First we note:

Lemma 5.3. *The dual lattice of* Λ_τ *is* $i\Lambda_\tau$*, and* $m(\Lambda_\tau) = 1$.

Proof. Let $\omega_1 = 1/y^{1/2}$, $\omega_2 = \tau/y^{1/2}$ be the basis for Λ_τ, and $\omega_1' = i\tau/y^{1/2}$, $\omega_2' = i/y^{1/2}$. Writing out in terms of the orthonormal basis $(1, i)$ for $\mathbb{C}$, it is simple to check these are dual bases. The second statement follows from the first. □

Notice that if $\Lambda \subset \mathbb{R}^n$ and $g \in SO(n)$, then $E_\Lambda(s) = E_{g\Lambda}(s)$ by the definition. So $E_{i\Lambda_\tau}(s) = E_{\Lambda_\tau}(s) = G(\tau, s)$.

Theorem 5.4. *(i) Let $\mathcal{E}(\tau, s) = \pi^{-s}\Gamma(s)G(\tau, s)$. Then $\mathcal{E}(\tau, s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ with two simple poles at $s = 1, 0$ with residue 1, -1, and satisfies $\mathcal{E}(\tau, 1-s) = \mathcal{E}(\tau, s)$.*
(ii) $G(\tau, s) - \pi/(s-1)$ is a C^∞ function of τ for every $s \in \mathbb{C}$, and we have $\Delta G = s(1-s)G$.

Proof. (i) This follows from 4.5 applied to Λ_τ.
(ii) The Mellin transform gives

$$\begin{aligned}\mathcal{E}(\tau, s) &= \frac{1}{s(s-1)} + \int_1^\infty (\Theta_{\Lambda_\tau}(it) - 1)(t^s + t^{1-s})\,\frac{dt}{t} \\ &= \frac{1}{s(s-1)} + \int_1^\infty \sum_{m,n}{}' e^{-\pi it|m\tau+n|^2/y}(t^s + t^{1-s})\,\frac{dt}{t}\end{aligned}$$

and we can differentiate under the integral sign (with respect to x and y) arbitrarily often and keep convergence. So $\Delta\mathcal{E} - s(1-s)\mathcal{E}$ is analytic for $s \in \mathbb{C} - \{0, 1\}$ and since it vanishes for $\mathrm{Re}(s) > 1$, it vanishes for all s.
As $G(\tau, s) = \pi^s\Gamma(s)^{-1}\mathcal{E}(\tau, s)$, the analogous results for G follow. □

Remark: In fact G is real analytic as a function of τ.

The constant term

As $G(\tau, s)$ is invariant under $\tau \mapsto \tau + 1$ we can write it as a Fourier series:

$$G(\tau, s) = \sum_{n\in\mathbb{Z}} A_n(y, s)e^{2\pi inx}$$

(absolutely convergent since G is C^∞) where

$$A_n(y, s) = \int_0^1 G(x + iy, s)e^{-2\pi inx}\,dx.$$

The constant term $A_0(y, s)$ is particularly important. Unlike the case of the holomorphic G_k, the constant term receives contributions both from the terms $m = 0$ and the terms $m \neq 0$ in the sum.

Theorem 5.5. *The constant term of $\mathcal{E}(\tau, s)$ is*

$$\pi^{-s}\Gamma(s)A_0(y, s) = 2\xi(2s)y^s + 2\xi(2s-1)y^{1-s}$$

A_0 must be invariant under $s \mapsto 1 - s$, which is compatible with the functional equation for $\zeta(s)$.

Proof. By analytic continuation we may assume $\mathrm{Re}(s) > 1$, and then we have

$$\begin{aligned}\pi^{-s}\Gamma(s)A_0(y,s) &= \int_0^1 \mathcal{E}(x+iy,s)\,dx = \int_0^1 M(\Theta_{\Lambda_\tau}(z)-1,s)\,dx \\ &= \int_0^1\int_0^\infty {\sum_{m,n}}' e^{\pi i|m\tau+n|^2 t/y}\, t^s\,\frac{dt}{t}\,dx\end{aligned}$$

by the Mellin transform formula. First compute the sum of the terms with $m=0$. This is

$$M\Big(2\sum_{n=1}^\infty e^{\pi i n^2 z/y}, s\Big) = 2\pi^{-s}\Gamma(s)\zeta(2s)y^s$$

(we use z for the variable in the Mellin transform). Now consider terms with $m \neq 0$. The sum is invariant under $m \mapsto -m$ so this is the Mellin transform of

$$\begin{aligned}2\sum_{m=1}^\infty \int_0^1 \sum_{n\in\mathbb{Z}} e^{\pi i|m\tau+n|^2 z/y}\,dx &= 2\sum_{m=1}^\infty\sum_{n\in\mathbb{Z}}\int_{n/m}^{n/m+1} e^{\pi i|m\tau|^2 z/y}\,dx \\ = 2\sum_{m=1}^\infty m\int_{-\infty}^\infty e^{\pi i|m\tau|^2 z/y}\,dx &= 2\sum_{m=1}^\infty m\int_{-\infty}^\infty e^{\pi i m^2(x^2 z/y+zy)}\,dx \\ = 2\sum_{m=1}^\infty m e^{\pi i m^2 yz}\int_{-\infty}^\infty e^{\pi i m^2 x^2 z/y}\,dx.\end{aligned}$$

When $z = it$ the integral on the right hand side is $m^{-1}(y/t)^{1/2}$, so the Mellin transform is

$$\begin{aligned}2y^{1/2}\int_0^\infty \sum_{m=1}^\infty e^{-\pi m^2 yt}t^{s-1/2}\frac{dt}{t} &= 2y^{1/2}\sum_{m=1}^\infty \Gamma(s-\frac{1}{2})(\pi m^2 y)^{1/2-s} \\ = 2y^{1-s}\pi^{-(2s-1)/2}\Gamma(\frac{2s-1}{2})\zeta(2s-1) &= 2\xi(2s-1)y^{1-s}.\end{aligned}$$

□

Corollary. *$G(\tau,s) - A_0(y,s)$ is an entire function of s.*

Proof. From the analytic continuation of $\xi(s)$ it is easy to see that $A_0(y,s)$ has simple poles at $s = 1, 0$ with residues $1, -1$, and is analytic elsewhere. So the result follows from 5.4. □

The nonconstant terms

One can compute the entire Fourier expansion in closed terms (see the notes from 2008) using Bessel functions. Here we'll just compute some asymptotics which will enable us to obtain nice formulae for G at $s = 0$ and 1.

From the computation of the constant term, we have:

$$G(\tau, s) = 2\xi(2s)y^s = 2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}} \frac{y^s}{|m\tau+n|^{2s}}$$

For fixed $m \geq 1$, the inner sum is invariant under $\tau \mapsto \tau + 1/m$, so its k-th Fourier coefficient vanishes unless $m|k$. So if $k \neq 0$ and $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
A_k(y,s) &= \int_0^1 2\sum_{1\leq m|k}\sum_{n\in\mathbb{Z}} \frac{y^s}{|m\tau+n|^{2s}}\, e^{-2\pi ikx}\, dx \\
&= 2y^s \sum_{1\leq m|k}\sum_{n\in\mathbb{Z}} \int_{n/m}^{1+n/m} \frac{e^{-2\pi ikx}}{|m\tau|^{2s}}\, dx \qquad \text{(substituting } x \mapsto x - \frac{n}{m}) \\
&= 2y^s \sum_{1\leq m|k} m^{1-2s} \int_{-\infty}^{\infty} \frac{e^{-2\pi ikx}}{(x^2+y^2)^s}\, dx \\
&= 2y^{1-s}\sigma_{1-2s}(|k|) I(-2\pi ky) \qquad \text{(writing } x = zy)
\end{aligned}$$

where

$$I(\alpha, s) = \int_{-\infty}^{\infty} \frac{e^{i\alpha z}}{(z^2+1)^s}\, dz$$

for $\mathrm{Re}(s) > 1$.

Lemma. *Let $\alpha \in \mathbb{R} - \{0\}$. Then $I(\alpha, s)$ extends to an entire function of s, and for $\mathrm{Im}(s)$ in a compact subset $K \subset \mathbb{R}$,*

$$|I(\alpha, s)| < c_K e^{-\alpha/4}$$

for a constant c_K depending only on K.

Proof. Replacing z by $-z$ we may assume $\alpha > 0$. Since $\mathrm{Re}(s) > 1$, by Jordan's Lemma, $I(\alpha, s) = \int_{C_1+C_2+C_3} e^{i\alpha z}/(z^2+1)^s\, dz$ where $C_1 = [-1 + i\infty, -1 + i/2]$, $C_2 = [-1+i/2, 1+i/2]$, $C_3 = [1+i/2, 1+i\infty]$, and we take the branch of $(z^2+1)^s$ which is analytic outside $\{z = iy \mid |y| \geq 1\}$. The following calculation shows that, for any compact $K \subset \mathbb{R}$, the integral converges absolutely uniformly for $\mathrm{Re}(s) \in K$ (so defines an entire function of s) and satisfies the estimate of the lemma.

Let $\sigma = \mathrm{Re}(s)$. Then

$$\left|\int_{C_2}\right| \leq 2\sup_{C_2}\left|\frac{e^{i\alpha z}}{(z^2+1)^s}\right| \leq \frac{2e^{-\alpha/2}}{(3/4)^\sigma} = ce^{-\alpha/2}$$

and

$$\begin{aligned}
\left|\int_{C_1}\right| &\leq \int_{1/2}^{\infty} \frac{e^{-\alpha y}}{|(1+iy)^2+1|^\sigma}\, dy = \int_{1/2}^{\infty} \frac{e^{-\alpha y}}{(y^4+4)^{\sigma/2}}\, dy \\
&\leq \int_{1/2}^{\infty} c' e^{-\alpha y/2}\, dy = c'\,\frac{e^{-\alpha/4}}{\alpha/2} = c'' e^{-\alpha/4}
\end{aligned}$$

for constants c, c', c'' depending continuously on σ. $\square$

It follows that

$$|A_k(y,s)| \le 2\,|y^s\sigma_{1-2s}(|k|)|\,c_K e^{-\pi|k|y/2}$$

and the same argument applies to the derivatives of $G(\tau,s)$ (with respect to any of x, y, s).

Corollary. *The function $G(\tau,s) - A_0(y,s)$, along with its derivatives, is bounded by e^{-cy} as $y\to\infty$, for some $c = c(s) > 0$.*

Application: the Kronecker Limit Formula (KLF)

We have $\mathcal{E}(\tau,s) = \pi^{-s}\Gamma(s)G(\tau,s) \sim -\frac{1}{s}$ at $s=0$, so $G(\tau,s)$ is analytic at $s=0$ and $G(\tau,0)=-1$. The KLF computes the derivative $G'(\tau,0)$ (here and below $'$ denotes s-derivative).

Theorem 5.6. *(KLF at $s=0$).*

$$G'(\tau,0) = 4\zeta'(0) - \log\bigl(y\,|\Delta(\tau)|^{1/6}\bigr).$$

In fact one can show from the functional equation for $\zeta(s)$ that $\zeta'(0) = -(1/2)\log 2\pi$.

Proof. (cf. Cambridge Ph.D. thesis of C.Chen). We will prove some characterising properties of each side, which will imply the equality. First, we have

$$A_0(y,s) = 2y^s\zeta(2s) + 2y^{1-s}\frac{\xi(2-2s)}{\pi^{-s}\Gamma(s)}.$$

At $s=0$, the first term has Taylor series

$$2\zeta(0) + s(2\log y\,\zeta(0) + 4\zeta'(0)) + O(s^2) = -1 + s(4\zeta'(0) - \log y) + O(s^2)$$

since $\zeta(0) = -1/2$. The second term vanishes at $s=0$, and the leading term in its Taylor series is $2y\,\xi(2)/(1/s) = \pi ys/3$ since $\zeta(2) = \pi^2/6$. Therefore by the Corollary above,

$$G'(\tau,0) = \frac{\pi y}{3} - \log y + 4\zeta'(0) + O(e^{-cy}).$$

Next,[5] $\underline{\Delta}G(\tau,s) = s(1-s)G(\tau,s)$, hence

$$\underline{\Delta}G'(\tau,0) = \bigl(s(1-s)G'(\tau,s) + (1-2s)G(\tau,s)\bigr)\big|_{s=0} = G(\tau,0) = -1.$$

Now look at the RHS. As $\Delta(\tau)$ is holomorphic, $\log|\Delta(\tau)| = \operatorname{Re}\log\Delta(\tau)$ is harmonic, so $\underline{\Delta}(\log|\Delta(\tau)|) = 0$. Therefore

$$\underline{\Delta}(\text{RHS}) = \underline{\Delta}(-\log y) = y^2\frac{\partial^2}{\partial y^2}(\log y) = -1.$$

[5]Here we use $\underline{\Delta}$ for the Laplace-Beltrami oerator, to avoid confusion with the modular form $\Delta(\tau)$...

As $y \to \infty$, $\Delta(\tau) = q + O(q^2) = q(1 + O(q))$, and so

$$\log|\Delta(\tau)| = \log|q| + \log(1 + O(q)) = -2\pi y + O(q).$$

Therefore

$$\text{RHS} = 4\zeta'(0) - \log y - \frac{1}{6}\log|\Delta(\tau)| = 4\zeta'(0) - \log y + \frac{\pi y}{3} + O(q).$$

Both sides of the identity are invariant under $SL_2(\mathbb{Z})$ (for if f is modular of weight k then $y^{k/2}|f|$ is invariant under $SL_2(\mathbb{Z})$). So the difference $F(\tau) = G(\tau, 0) - (\text{RHS})$ satisfies:

(i) $F(\tau)$ is a harmonic function on $\mathcal{H}$;

(ii) $F(\tau) = F(\gamma\tau)$ for every $\gamma \in SL_2(\mathbb{Z})$; and

(iii) $|F(\tau)| \to 0$ as $y \to \infty$, uniformly in x.

Let $\mathcal{D}^c$ be the closure of the standard modular fundamental domain (Thm. 2.2). Then by (ii) and (iii)

$$\sup_{\mathcal{H}} \text{Re}(F) = \sup_{\mathcal{D}^c} \text{Re}(F) = \sup_{\mathcal{D}^c \cap \{y \leq Y\}} \text{Re}(F)$$

for Y sufficiently large. By continuity the supremum is attained at some $\tau_0 \in \mathcal{D}^c$. So by the maximum principle for harmonic functions, $\text{Re}(F)$ is constant, hence by (iii) equals 0. The same argument holds for $\text{Im}(F)$, so $F = 0$. □