4 L-series

Γ-function and Mellin transform

Recall:

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \, \frac{dt}{t}$$

converges on right half-plane $\mathrm{Re}(s) > 0$, where it defines an analytic function. Integration by parts gives $s\Gamma(s) = \Gamma(s+1)$, which allows one to meromorphically continue the function to $\mathbb{C}$: if $\mathrm{Re}(s) > -N$ then

$$\Gamma(s) = \frac{1}{s(s+1)\dots(s+N-1)}\Gamma(s+N).$$

$\Gamma(n) = (n-1)!$ for $n \geq 1$; simple poles at each nonpositive integer, and residue at $s = 0$ is 1. Moreover $\Gamma(s)$ has no zeroes in $\mathbb{C}$ (this follows from either of the infinite product expansions

$$\Gamma(s)^{-1} = s\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)\left(1+\frac{1}{n}\right)^{-s} \qquad \text{(Euler)}$$

$$= e^{\gamma s} s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right) e^{-s/n} \qquad \text{(Weierstrass)}$$

see e.g. chapter 5 sect.2.4–5 of Ahlfors). Here $\gamma = \lim_{n\to\infty}(1+1/2+\cdots+1/n-\log n)$ is the *Euler-Mascheroni constant.*

Let f be a function on $\mathcal{H}$. The **Mellin transform** of f is defined to be

$$M(f,s) = \int_0^\infty f(iy) y^s \, \frac{dy}{y}$$

(assuming that this converges). So $\Gamma(s)$ is the Mellin transform of $e^{i\tau}$. From the definition, if $a > 0$ then

$$M(f(a\tau), s) = a^{-s} M(f,s).$$

Suppose that $f(\tau)$ can be written as an absolutely convergent series $\sum_\alpha c_\alpha e^{2\pi i\alpha\tau}$, where α runs over some discrete subset of $(0,\infty)$. Then

$$M(f,s) = \sum_\alpha c_\alpha M(e^{2\pi i\alpha\tau}, s) = \sum_\alpha \frac{c_\alpha}{(2\pi\alpha)^s} M(e^{i\tau}, s)$$

$$= (2\pi)^{-s}\Gamma(s)\sum_\alpha \frac{c_\alpha}{\alpha^s}$$

(at least formally).

Example: let $f(\tau) = \sum_{n=1}^{\infty} e^{\pi i n^2 \tau}$. Then

$$M(f,s) = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{(n^2/2)^s} = \pi^{-s}\Gamma(s)\zeta(2s).$$

L-series of cusp forms

Now suppose $f = \sum_1^{\infty} a_n q^n \in S_k$ is a cusp form. Then as $f(iy)$ tends exponentially to 0 as $y \to \infty$, and $y^{k/2}\,|f(iy)|$ is bounded as $y \to 0$. the integral defining $M(f,s)$ is absolutely convergent and represents an analytic function for $\mathrm{Re}(s) > k/2$.

The above formula shows that $M(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$ where $L(f,s) = \sum a_n n^{-s}$ (strictly speaking we should only call it an L-function when f is a normalised eigenform).

Theorem 4.1. *If $f \in S_k$ then $M(f,s)$ has an analytic continuation to $\mathbb{C}$, and satisfies the* ***functional equation***

$$M(f,s) = (-1)^{k/2} M(f,k-s).$$

Proof. Split the defining integral into the intervals $[0,1]$ and $[1,\infty)$, and use the relation $f(-1/\tau) = \tau^k f(\tau)$:

$$\begin{aligned} M(f,s) &= \int_1^{\infty} + \int_0^1 f(iy) y^s \,\frac{dy}{y} \\ &= \int_1^{\infty} f(iy) y^s \,\frac{dy}{y} + \int_0^1 (iy)^{-k} f(i/y) y^s \frac{dy}{y} \\ &= \int_1^{\infty} f(iy) y^s \,\frac{dy}{y} + (-1)^{k/2} \int_1^{\infty} f(iy) y^{k-s} \frac{dy}{y} \end{aligned}$$

using the substitution $y \mapsto 1/y$ in the second integral. As $f(iy)$ decays exponentially as $y \to \infty$, the integrals over $[1,\infty)$ are analytic for all s, and their sum obviously transforms in the desired way under $s \mapsto k-s$. □

Poisson summation and applications

Let $\langle -,- \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the usual inner product, $\|-\|$ the Euclidean length. Let $f \colon \mathbb{R}^n \to \mathbb{C}$ be a nice[3] function. The **Fourier transform** of f is

$$\hat{f} \colon \mathbb{R}^n \to \mathbb{C}, \qquad \hat{f}(u) = \int_{\mathbb{R}^n} e^{-2\pi i \langle u,v \rangle} f(v)\, dv$$

The **Fourier inversion formula** for $\mathbb{R}^n$ says: $\hat{\hat{f}}(v) = f(-v)$.

[3] "Nice" here means that the derivatives $f^{(m)}$ ($m \in \mathbb{N}^n$) satisfy: for every polynomial function P on $\mathbb{R}^n$, $P(v) f^{(m)}(v)$ is bounded.

Theorem 4.2 (Poisson summation formula)**.** *Let $\Lambda \subset \mathbb{R}^n$ be a lattice, $\Lambda' = \{y \in \mathbb{R}^n | \langle x, y\rangle \in \mathbb{Z}\ \forall x \in \Lambda\}$ the dual lattice. Then*

$$\sum_{x\in\Lambda} f(x) = m(\Lambda)^{-1} \sum_{y\in\Lambda'} \hat{f}(y)$$

where $m(\Lambda) = \mathrm{vol}(\mathbb{R}^n/\Lambda) = m(\Lambda')^{-1}$.

Proof. Let $g(v) = \sum_{x\in\Lambda} f(v+x) \colon \mathbb{R}^n/\Lambda \to \mathbb{C}$. Then g can be written as a Fourier series:

$$g(v) = \sum_{y\in\Lambda'} c_y\, e^{2\pi i\langle y,v\rangle}$$

with coefficients

$$\begin{aligned} c_y &= m(\Lambda)^{-1} \int_{V/\Lambda} g(v) e^{-2\pi i\langle y,v\rangle}\, dv \\ &= m(\Lambda)^{-1} \int_{V} f(v) e^{-2\pi i\langle y,v\rangle}\, dv = m(\Lambda)^{-1}\hat{f}(y) \end{aligned}$$

Then $\sum_{x\in\Lambda} f(x) = g(0) = \sum_{y\in\Lambda'} c_y = m(\Lambda)^{-1}\sum_{y\in\Lambda'}\hat{f}(y)$. The identity $m(\Lambda')m(\Lambda) = 1$ is easy linear algebra (or Fourier inversion). □

Now we define the **theta function** of Λ to be

$$\Theta_\Lambda(\tau) = \sum_{x\in\Lambda} e^{\pi i\|x\|^2\tau} \qquad (\tau \in \mathcal{H}).$$

The series converges absolutely (cf. Prop.1.2 but easier) to an analytic function on $\mathcal{H}$.

Theorem 4.3.

$$\Theta_\Lambda(\tau) = (\tau/i)^{-n/2} m(\Lambda)^{-1}\Theta_{\Lambda'}(-1/\tau).$$

Proof. By analytic continuation it's enough to prove this for $\tau = iy$. We let $f(v) = e^{-\pi\langle v,v\rangle} = \prod_i e^{-\pi v_i^2}$. Then $\hat{f} = f$ (standard result[4] for $n = 1$, and follows by separation of variables for general n).
Moreover as $(c\Lambda)' = c^{-1}\Lambda'$, by Poisson summation

$$\Theta_\Lambda(iy) = \sum_{x\in y^{1/2}\Lambda} f(x) = m(y^{1/2}\Lambda)^{-1} \sum_{x\in y^{-1/2}\Lambda'} f(x) = y^{-n/2} m(\Lambda)^{-1}\Theta_{\Lambda'}(i/y).$$

□

[4]Proof: if $f(x) = \exp(-\pi x^2)$ then

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x y}\, dx = e^{-\pi y^2}\int_{-\infty}^{\infty} e^{-\pi(x+iy)^2}\, dx = f(y)\int_{-\infty+iy}^{\infty+iy} e^{-\pi z^2}\, dz\,.$$

Now shift the path of integration to $[-\infty, \infty]$ and use $\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$.

As $\Theta_{\Lambda'}(\tau) \to 1$ as $\operatorname{Im}\tau \to \infty$, deduce:

Corollary 4.4. *$\Theta_\Lambda(iy) \sim m(\Lambda)^{-1} y^{-n/2}$ as $y \to 0$.*

The **(Epstein) zeta function of the lattice** Λ is

$$E_\Lambda(s) = \sum_{0 \neq x \in \Lambda} \frac{1}{\|x\|^{2s}}.$$

It converges absolutely for $\operatorname{Re}(s) > n/2$ by Propn.1.2.

Theorem 4.5. *$\mathcal{E}_\Lambda(s) = \pi^{-s}\Gamma(s)E_\Lambda(s)$ has a meromorphic continuation to $\mathbb{C}$, analytic apart from simple poles at $s = 0$, $n/2$ with residues -1, $m(\Lambda)^{-1}$ respectively. It satisfies the **functional equation***

$$\mathcal{E}_\Lambda(s) = m(\Lambda)^{-1}\mathcal{E}_{\Lambda'}(\frac{n}{2} - s).$$

In particular, $E_\Lambda(0) = -1$.

Proof. Consider the function

$$\sum_{0 \neq x \in \Lambda} e^{\pi i \tau \|x\|^2} = \Theta_\Lambda(\tau) - 1.$$

It decays exponentially as $\operatorname{Im}\tau \to \infty$, and together with Cor.4.4 this shows that the integral defining its Mellin transform converges for $\operatorname{Re}(s) > n/2$, and equals $\pi^{-s}\Gamma(s)E_\Lambda(s)$.
As in the proof of Thm.4.1 we can break up the integral in the Mellin transform as $\int_0^1 + \int_1^\infty$. Then if $\operatorname{Re}(s) > n/2$

$$\begin{aligned}
\int_0^1 &= \int_0^1 \Theta_\Lambda(iy)y^s \frac{dy}{y} - \frac{1}{s} = -\frac{1}{s} + \int_0^1 m(\Lambda)^{-1}\Theta_{\Lambda'}(i/y)y^{s-n/2}\frac{dy}{y} \\
&= -\frac{1}{s} + m(\Lambda)^{-1}\int_1^\infty \Theta_{\Lambda'}(iy)y^{n/2-s}\frac{dy}{y} \\
&= -\left(\frac{1}{s} + \frac{m(\Lambda)^{-1}}{n/2 - s}\right) + m(\Lambda)^{-1}\int_1^\infty \left(\Theta_{\Lambda'}(iy) - 1\right) y^{n/2-s}\frac{dy}{y}
\end{aligned}$$

So $\mathcal{E}(\Lambda, s)$ equals

$$-\left(\frac{1}{s} + \frac{m(\Lambda)^{-1}}{n/2 - s}\right) + \int_1^\infty \left(\Theta_\Lambda(iy) - 1\right) y^s + m(\Lambda)^{-1}\left(\Theta_{\Lambda'}(iy) - 1\right) y^{n/2-s} \cdot \frac{dy}{y}$$

and the integral is analytic for all $s \in \mathbb{C}$. This gives the residues, and using the FE for Θ and $m(\Lambda)^{-1} = m(\Lambda')$ we get the FE. □

Corollary 4.6. *(Riemann ζ-function.) Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $\xi(s)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 1$ and 0 with residues 1, -1, and $\xi(s) = \xi(1-s)$. In particular, $\zeta(s)$ has a meromorphic continuation with a simple pole at $s = 1$ as its only singularity, and $\zeta(-2k) = 0$ for every positive integer k.*

Proof. Take $\Lambda = \mathbb{Z} \subset \mathbb{R}$, so that $E_\Lambda(s) = 2\zeta(2s)$ and $Z(s) = (1/2)\mathcal{E}_\Lambda(s/2)$. The last part (zeroes of $\zeta(s)$) follows from the description of the poles of $\Gamma(s)$. □