3 Hecke operators

Let $\mathcal{L}$ be the free abelian group on symbols $[\Lambda]$, where Λ is a lattice.
Define operators $T(n)$, $R(n) \in \operatorname{End}_{\mathbb{Z}} \mathcal{L}$ for $n \geq 1$ by

$$R(n)\colon [\Lambda] \mapsto [n\Lambda]$$
$$T(n)\colon [\Lambda] \mapsto \sum_{\Lambda' \underset{n}{\subset} \Lambda} [\Lambda']$$

Theorem 3.1. *(i) $R(m)R(n) = R(mn)$, $R(m)T(n) = T(n)R(m)$ for all m, n, and $R(1) = 1 = T(1)$.*
(ii) If $(m,n) = 1$ then $T(mn) = T(m)T(n)$.
(iii) For p primte and $r \geq 1$,

$$T(p^{r+1}) = T(p^r)T(p) - pR(p)T(p^{r-1}).$$

Corollary. *The subalgebra of $\operatorname{End}_{\mathbb{Z}} \mathcal{L}$ generated by $\{R(n), T(n)\}_{n\geq 1}$ is commutative, and is generated by $T(p)$, $R(p)$ for p primes.*

Proof. (i) is clear. For (ii), we have

$$T(m)T(n)\colon [\mathcal{L}] \mapsto \sum_{\Lambda' \underset{n}{\subset} \Lambda} \sum_{\Lambda'' \underset{m}{\subset} \Lambda'} [\Lambda''] = \sum_{L'' \underset{mn}{\subset} \Lambda} \#\{\Lambda' \mid \Lambda'' \underset{m}{\subset} \Lambda' \underset{n}{\subset} \Lambda\} \cdot [\Lambda'']$$

But as $(m,n) = 1$ the abelian group Λ/Λ'' has exactly one subgroup of index n, so $\#\{\Lambda' \mid \Lambda'' \underset{m}{\subset} \Lambda' \underset{n}{\subset} \Lambda = 1$.
(iii) Have

$$T(p^r)T(p(([\Lambda]) = \sum_{L'' \underset{p^{r+1}}{\subset} \Lambda} \#\{\Lambda' \mid \Lambda'' \underset{p^r}{\subset} \Lambda' \underset{p}{\subset} \Lambda\} \cdot [\Lambda'']$$

and

$$\Lambda'' \subset p\Lambda \implies p+1 \text{ possible } \Lambda' \ (\Leftrightarrow \text{subgps of } \Lambda/p\Lambda \simeq (\mathbb{Z}/p)^2)$$
$$\Lambda'' \not\subset p\Lambda \implies \Lambda/\Lambda'' \text{ cyclic and } \Lambda' = \Lambda'' + p\Lambda \text{ unique}$$

hence

$$\begin{aligned}
T(p^r)T(p)([\Lambda]) &= \sum_{\substack{\Lambda'' \underset{p^{r+1}}{\subset} \Lambda \\ \Lambda'' \subset p\Lambda}} (p+1)[\Lambda''] + \sum_{\substack{\Lambda'' \underset{p^{r+1}}{\subset} \Lambda \\ \Lambda'' \not\subset p\Lambda}} [\Lambda''] \\
&= \sum_{\substack{\Lambda'' \underset{p^{r+1}}{\subset} \Lambda \\ \Lambda'' \subset p\Lambda}} p[\Lambda''] + \sum_{\Lambda'' \underset{p^{r+1}}{\subset} \Lambda} [\Lambda''] \\
&= \sum_{\Lambda'' \underset{p^{r-1}}{\subset} p\Lambda} p[\Lambda''] + \sum_{\Lambda'' \underset{p^{r+1}}{\subset} \Lambda} [\Lambda''] \\
&= pR(p)T(p^{r-1})([\Lambda]) + T(p^{r+1}([\Lambda]).
\end{aligned}$$

□

If R is any ring with 1, by a formal Dirichlet series we mean a formal expression

$$\sum_{n\geq 1} A_n\, n^{-s}, \quad A_n \in R$$

Two such series are added and multiplied in the obvious way, using the formal relation $m^{-s}n^{-s} = (mn)^{-s}$. If $A^{(i)} = 1 + \sum_{n\geq n_i} A_n^{(i)}\, n^{-s}$ are formal Dirichlet series with $n_i \to \infty$ then the infinite product $\prod A^{(i)}$ makes sense, and if R is commutative we can also define the inverted Dirichlet series if $A_1 = 1$ by

$$\frac{1}{A} = \sum_{k\geq 0} (1-A)^k$$

We apply this with R the subring of $\operatorname{End}_{\mathbb{Z}} \mathcal{L}$ generated by Hecke operators.

Theorem 3.2 (Euler product)**.**

$$\begin{aligned}
\sum_{n\geq 1} T(n)\, n^{-s} &= \prod_p \left(1 + T(p)p^{-s} + T(p^2)p^{-2s} + \cdots\right) \\
&= \prod_p \frac{1}{1 - T(p)p^{-s} + R(p)p^{1-2s}}.
\end{aligned}$$

Proof. (ii) implies that $T(n) = \prod T(p_i^{r_i})$ if $n = \prod p_i^{r_i}$, from which the first equality follows. For the second, multiply both sides by $(1 - T(p)p^{-s} + R(p)p^{1-2s})$ and use (iii). □

Next define the operators on modular forms.

$$f \in M_k \quad \Leftrightarrow \quad F\colon \text{(lattices)} \to \mathbb{C},\ F(\alpha\Lambda) = \alpha^{-k}F(\Lambda)$$

and extend F by linearity to a function on $\mathcal{L}$.

Definition. $T_n^k F = n^{k-1} F \circ T_n$, so that

$$(T_n^k F)(\Lambda) = n^{k-1} \sum_{\Lambda' \underset{n}{\subset} \Lambda} F(\Lambda')$$

Notice that $L' \underset{n}{\subset} \Lambda \iff \Lambda \underset{n}{\subset} n^{-1}\Lambda'$, so that T_n and Fourier expansions.

$$(T_n^k F)(\Lambda) = n^{-1} \sum_{\Lambda'' \underset{n}{\supset} \Lambda} F(\Lambda'')$$

Restating the previous results:

Theorem 3.3. *$\{T_n^k\}_{n\geq 1}$ generates a commutative algebra of operators on the space of $(-k)$-homogeneous functions on lattices, satisfying:*

$$\begin{aligned} T_{mn}^k &= T_m^k T_n^k \text{ if } (m,n) = 1 \\ T_{p^{r+1}}^k &= T_p^k T_{p^r}^k - p^{k-1} T_{p^{r-1}}^k, \; p \text{ prime}, \; r \geq 1 \\ \sum_{n\geq 1} T_n^k \, n^{-s} &= \prod_p \frac{1}{1 - T_p^k p^{-s} + p^{k-1-2s}} \end{aligned}$$

Example: Eisenstein series $G_k(\Lambda) = \sum'_{\omega} \omega^{-k}$, $k \geq 4$.

Theorem 3.4. *$T_n^k G_k = \sigma_{k-1}(n) G_k$ for every $n \geq 1$.*

Proof. First consider $n = p$. Then Λ has $(p+1)$ sublattices Λ' of index p. If $\omega \in \Lambda$ then:

$$\begin{aligned} \omega \in p\Lambda &\implies \omega \in \text{every } \Lambda' \\ \omega \notin p\Lambda &\implies \omega \in \Lambda' = \mathbb{Z}\omega + p\Lambda \text{ only.} \end{aligned}$$

So:

$$\begin{aligned} (T_p^k G_k)(\Lambda) &= \sum_{\Lambda' \underset{p}{\subset} \Lambda} \sideset{}{'}\sum_{\omega \in \Lambda'} \frac{p^{k-1}}{\omega^k} \\ &= \sideset{}{'}\sum_{\omega \in p\Lambda} \frac{(p+1)p^{k-1}}{\omega^k} + \sum_{\omega \in \Lambda \setminus p\Lambda} \frac{p^{k-1}}{\omega^k} \\ &= \sideset{}{'}\sum_{\omega \in p\Lambda} \frac{1}{(\omega/p)^k} + p^{k-1} \sideset{}{'}\sum_{\omega \in \Lambda} \frac{1}{\omega^k} \\ &= (1 + p^{k-1}) G_k(\Lambda) = \sigma_{k-1}(p) G_k(\Lambda). \end{aligned}$$

In general, we have the formal identity:

$$\sum_{n\geq 1} \sigma_r(n) n^{-s} = \prod_p \frac{1}{(1-p^{-s})(1-p^{r-s})} = \prod_p \frac{1}{1-(1+p^r)p^{-2}+p^{r-2s}} \tag{11}$$

Proof: we have

$$\begin{aligned} \text{RHS} = \zeta(s)\zeta(s-r) &= \sum_{m,d\geq 1} m^{-s} d^{r-s} = \sum_{m,d\geq 1} d^r (md)^{-s} \\ &= \sum_{n\geq 1} \sum_{d|n} d^r n^{-s} = \sum_{n\geq 1} \sigma_r(n) n^{-s} \qquad (n = md) \end{aligned}$$

This gives

$$\begin{aligned} \left(\sum T_n^k n^{-s}\right) G_k &= \left(\prod_p \frac{1}{1 - T_p^k p^{-s} + p^{k-1-2s}}\right) G_k \\ &= \left(\prod_p \frac{1}{1-(1+p^{k-1})p^{-s} + p^{k-1-2s}}\right) G_k = \left(\sum_{n\geq 1} \sigma_{k-1}(n) n^{-s}\right) G_k \end{aligned}$$

□

Now translate all this into the language of functions on $\mathcal{H}$.
Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice with $\omega_1/\omega_2 \in \mathcal{H}$. Suppose $\Lambda' \underset{n}{\subset} \Lambda$. Then $\Lambda' \cap \mathbb{Z}\omega_2 = \mathbb{Z}\omega_2'$ where $\omega_2' = d\omega_2$ for some $d \geq 1$.
Now $\Lambda'/\mathbb{Z}\omega_2' \subset \Lambda/\mathbb{Z}\omega_2 \simeq \mathbb{Z}$ so $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ for some $\omega_1' = a\omega_1 + b\omega_2$, with $|ad| = (\Lambda;\Lambda') = n$.
Fixing $\omega_1'/\omega_2' \in \mathcal{H} \implies a \geq 1$; then ω_1' is unique mod $\mathbb{Z}\omega_2'$, and is unique if we require $0 \leq b < d$. Putting this together,

$$T(n)\colon [\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2] \mapsto \sum_{\substack{a,d\geq 1 \\ ad=n \\ 0\leq b<d}} [\mathbb{Z}(a\omega_1 + b\omega_2) + \mathbb{Z}(d\omega_2)]$$

Equivalently:

$$\Lambda' \underset{n}{\subset} \Lambda \iff \Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2', \ \begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \gamma \begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}, \ \det\gamma = n$$

and

$$SL_2(\mathbb{Z})\backslash\{\gamma \mid \det\gamma = n\} = \left\{\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \mid a,d \geq 1, \ ad = n, \ b \bmod d\right\}$$

Either way,

$$\begin{aligned}(T_n^k f)(\tau) :=&(T_n^k F)(\mathbb{Z}\tau + \mathbb{Z}) \\ =& n^{k-1} \sum_{a,b,d} F((a\tau+b)\mathbb{Z} + d\mathbb{Z})\end{aligned}$$

giving

Proposition 3.5.

$$(T_n^k f)(\tau) = n^{k-1} \sum_{\substack{a,d\geq 1 \\ ad=n \\ 0\leq b<d}} d^{-k} f\left(\frac{a\tau+b}{d}\right) = n^{k/2-1} \sum_{a,b,d} f\Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Effect of T_n on q-expansion:

Theorem 3.6. *Let* $f = \sum_{n\geq 0} a_n q^n \in M_k$. *Then*

$$T_m^k f = \sum_{n\geq 0} b_n q^n \in M_k$$

$$b_0 = \sigma_{k-1}(n) a_0, \quad b_n = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2}$$

In particular, $T_m^k(S_k) \subset S_k$.

Remark. For $m = p$ prime it's convenient to write the formula as

$$b_n = a_{np} + p^{k-1} a_{n/p}$$

with the convention that $a_{n/p} = 0$ if $n/p \notin \mathbb{Z}$.

Proof.

$$(T_m^k f)(\tau) = m^{k-1} \sum_{\substack{n\geq 0 \\ d|m \\ b \bmod d}} d^{-k} a_n e^{2\pi i n((m/d)\tau + b)/d}$$

and since

$$\sum_{b \bmod d} e^{2\pi i b n/d} = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod d \\ d & \text{if } n \equiv 0 \end{cases}$$

the sum becomes

$$\begin{aligned} m^{k-1} \sum_{n\geq 0,\ d|(m,n)} a_n d^{-k+1} e^{2\pi i m n \tau/d^2} & \\ &= \sum_{n\geq 0,\ d|(m,n)} (m/d)^{k-1} a_n q^{mn/d^2} \\ &= \sum_{n'\geq 0,\ e|(m,n')} e^{k-1} a_{mn'/e^2} q^n \end{aligned}$$

writing $n' = mn/d^2$, $e = m/d$. Since $T_m^k f$ is modular of weight k, this shows it is in M_k, and then clearly $T_m^k(S_k) \subset S_k$ as well. □

Corollary 3.7. *Suppose* $0 \neq f = \sum_{n\geq 0} a_n q^n \in M_k$ *is an eigenfunction for all* $\{T_n^k\}_{n\geq 1}$ *with* $T_n^k f = \lambda_n f$. *Then:*

(i) $f \in S_k \implies a_1 \neq 0$ *and* $a_n/a_1 = \lambda_n$.

(ii) $f \notin S_k \implies f = a_0 E_k$.

Such an f is called a *Hecke eigenform*, or simply *eigenform*.

Proof. (i) Coefficient of q in $T_n^k f = \lambda_n f$ is

$$\lambda_n a_1 = \sum_{d|(n,1)} d^{k-1} a_{n/d} = a_m.$$

So if $a_1 = 0$ then $a_n = 0\ \forall n \geq 1 \implies f = 0$.
(ii) Constant coefficient of $T_n^k f$ is

$$\lambda_n a_0 = \sigma_{k-1}(n) a_0$$

sp $a_0 \neq 0$ (i.e. $f \notin S_k$) $\implies \lambda_n = \sigma_{k-1}(n)$. Put $g = f - a_0 E^k = \sum_{n\geq 1} b_n q^n \in S_k$. Then

$$T_n^k E_k = \sigma_{k-1} E_k \implies T_g^k = \sigma_{k-1}(n) g \implies b_n = \sigma_{k-1} b_1$$

But $|b_n| \ll n^{k/2}$ and $\sigma_{k-1}(n) > n^{k-1} \implies g = 0$ (since $k \geq 4$). □

Next show T_n^k can be simultaneously diagonalised. Need:

Definition. let $f, g \in M_k$, at least one in S_k. The *Petersson inner product* of f and g is

$$\langle f, g \rangle = \int_{\mathcal{D}} f(\tau)\overline{g(\tau)} y^{k-2}\, dx\, dy, \quad \tau = x + iy$$

Remarks. (i) Since fg is a cusp form, $|fg| \ll e^{-2\pi y}$ as $y \to \infty$ so the integral converges.
(ii) Since $\langle f, f \rangle = \int_{\mathcal{D}} |f|^2 y^{k-2}\, dx\, dy$ it's clear that $\langle -, - \rangle$ is an inner product on S_k.
(iii) Note that the differential form $f(\tau)\overline{g(\tau)} y^{k-2}\, dx\, dy$ is invariant under Γ, so this is a reasonable definition.

Theorem 3.8. $\langle T_n^k f, g \rangle = \langle f, T_n^k g \rangle$, *i.e. the Hecke operators are self-adjoint wrt* $\langle -, - \rangle$.

Proof maybe to be given (time permitting) later.

Theorem 3.9. *(i) There exists a basis of S_k comsisting of Hecke eigenforms.*
(ii) The eigenvalues of T_n^k on S_k are totally real algebraic, and the characteristic polynomial of T_n^k has rational integer coefficients.

Proof. (i) since $\{T_n^k\}$ commute and are self-adjoint. This also shows that the eigenvalues are real.
(ii) Recall that $M_k(\mathbb{Z}) = M_k \cap \mathbb{Z}[[q]]$ is a $\mathbb{Z}$-lattice in M_k, which by the corollary to Thm.3.6 is stable under the T_n^k. So the characteristic polynomial of T_n^k has integer coefficients, so the eigenvalues of T_n^k are algebraic integers, closed under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. So as they are real they are totally real. □

We also get:

Corollary 3.10 (Multiplicity one)**.** *The representations of the algebra generated by the $\{T_n^k\}$ acting on S_k occur with multiplicity one.*

Proof. Since S_k is completely reducible as a module for the Hecke algebra, it's enough to show that if $f, g \in S_k \setminus \{0\}$ and

$$T_n^k f = \lambda_n f,\ T_n^k g = \lambda_n g\ \forall n \geq 1$$

then $f = cg$.But since $a_n(f) = \lambda_n a_1(f)$ and $a_n(g) = \lambda_n a_1(g)$, up to a scalar f, g have the same q-expansions. □

Remark. A stronger result holds: if $f \in S_k$, $g \in S_l$ and for all but finitely many p, $T_p^k f = \lambda_p f$ and $T_p^l g = \lambda_p$ then $k = l$ and f, g are linearly dependent. This is the "strong multiplicity one" theorem.

Worked example: $S_{24} = \mathbb{C}\Delta^2 + \mathbb{C}E_4^3\Delta$. Have q-expansions:

$$\begin{aligned}
\Delta &= q - 24q^2 + 252q^3 - \cdots \\
E_4^3 &= 1 + 720q + 179280q^2 + 16954560q^3 - \dots \\
\Delta^2 &= q^2 - 48q^3 + 1080q^4 - \dots \\
E_4^3\Delta &= q + 696q^2 + 162252q^3 + 12831808q^4 + \dots
\end{aligned}$$

and $T_2^k f = \sum b_n q^n$ with $b_n = a_{2n} + 2^{k-1} a_{n/2}$, hence

$$\begin{aligned}
T_2(E_4^3\Delta) &= 696q + 21220416q^2 + \dots & &= 696E_4^3\Delta + 20736000\Delta^2 \\
T_2(\Delta^2) &= q + 1080q^2 + \dots & &= E_4^3\Delta + 384\Delta^2
\end{aligned}$$

so matrix of T_2 is

$$T_2^{23} = \begin{pmatrix} 696 & 1 \\ 20736000 & 384 \end{pmatrix}$$

which has characteristic polynomial

$$x^2 - 1080x - 20468736 = (x - 540 + 12\sqrt{144169})(x - 540 - 12\sqrt{144169})$$

(144169 is prime). In particular we see that the Hecke eigenforms of weight 24 do *not* have rational coefficients. In fact, there are no known counterexamples to the following:

Conjecture. If $f \in S_k$ is a Hecke eigenform with rational coefficients then $k < 24$ or $k = 26$.

Corollary 3.11. *Let $f = \sum_{n \geq 1} a_n q^n$ be a Hecke eigenform. Then*

$$\sum_{n \geq 1} a_n \, n^{-s} = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}$$

As a formal identity this follows from the Euler product for the T_n^k. Since $|a_n| \ll n^{k/2}$ both sides are actually convergent for $\mathrm{Re}(s) > k/2 + 1$.