Modular forms part III — lecture notes

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These are the notes from 2008 with corrections and edited to reflect better the content and presentation of the course in 2016.

1 Elliptic functions

Generalities

Function theory on an elliptic curve $E/\mathbb{C}$; since $E(\mathbb{C}) = \mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$ this amounts to studying functions on $\mathbb{C}$ which are invariant under Λ.

Definition. V a finite-dimensional real v.s.; then a *lattice* in V is a discrete subgroup $\Lambda \subset V$ of rank $\dim V$.

Interested mainly in case $V = \mathbb{C}$, then Λ is a discrete sgp of $\mathbb{C}$ of rank 2, and so $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, where ω_i are lin.ind. over $\mathbb{R}$. (See example sheet for proof of these and similar facts.) The basis $\{\omega_i\}$ then determines a *fundamental domain* which is the parallelogram $\mathcal{P} = \{x_1\omega_1 + x_2\omega_2 \mid x_i \in [0, 1)\}$, which is a set of coset representatives for $\Lambda \subset \mathbb{C}$. WLOG we can assume that

$$\omega_2/\omega_1 \in \mathcal{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}.$$

This is equivalent to the natural orientation of the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ being given by taking the vertices in order $0, \omega_1, \omega_1 + \omega_2, \omega_2$. (Picture here.)

It's also convenient to write $\omega_3 = -\omega_1 - \omega_2$, so that the 3 elements of $\mathbb{C}/\Lambda$ of order 2 are $(\omega_i/2) + \Lambda$.

The quotient $\mathbb{C}/\Lambda$ is compact (for example, it is the continuous image of the closure $\mathcal{P}^c$).

Definition. An *elliptic function* w.r.t. Λ is a meromorphic function $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ which is Λ-invariant; i.e. $f(z + \omega) = f(z)$ for all $\omega \in \Lambda$.

First prove some general facts (which are special cases of general function theory on compact Riemann surfaces). Obvious remark: if f is an elliptic function then for $a \in \mathbb{C}$, the quantities $f(a)$, $\mathrm{ord}_{z=a} f(z)$, $\mathrm{res}_{z=a} f(z)$ depend only on the class of a mod Λ.

Theorem 1.1. *Let $0 \neq f : \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ be meromorphic.*

(i) If f has no poles then f is constant. If $f \neq 0$ then f has only a finite number of poles and zeroes mod Λ.

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(ii) $\sum_{a\in\mathbb{C}/\Lambda} \operatorname{res}_{z=a} f(z) = 0$.

(iii) $\sum_{a\in\mathbb{C}/\Lambda} \operatorname{ord}_{z=a}(f(z)) = 0$.

(iv) $\sum_{a\in\mathbb{C}/\Lambda} a \operatorname{ord}_{z=a}(f(z)) \equiv 0 \; mod\; \Lambda$.

Proof. (i) is Liouville's theorem; since $\mathbb{C}/\Lambda$ is compact, f is bounded hence constant. (Or use maximum modulus principle, which shows that a holomorphic function on any compact RS is constant.)
(ii) Assume first there are no poles of f on the boundary $\partial\mathcal{P}$. Then

$$2\pi i \sum_{a\in\mathbb{C}/\Lambda} \operatorname{res}_{z=a} f(z) = \int_{\partial\mathcal{P}} f(z)\,dz$$

and the integrals on the opposite sides of the parallelogram cancel in pairs, so this is 0. If f has poles on the boundary we can find some $b \in \mathbb{C}$ such that no poles lie on the translate $b + \partial\mathcal{P}$, and integrate around this curve instead.
(iii/iv) Same argument as (ii) applied to $f'(z)/f(z)$ and $zf'(z)/f(z)$. We just do the case (iv); we know that $\operatorname{res}_{z=a} f'(z)/f(z) = \operatorname{ord}_{z=a} f(z)$, and so

$$\sum_{a\in\mathbb{C}/\Lambda} a \operatorname{ord}_{z=a}(f(z)) = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} z\frac{f'(z)}{f(z)}\,dz.$$

Splitting the integral into its 4 parts, we have

$$\begin{aligned}\int_0^{\omega_1} + \int_{\omega_1+\omega_2}^{\omega_2} z\frac{f'(z)}{f(z)}\,dz &= \int_0^{\omega_1} z\frac{f'(z)}{f(z)} - (z+\omega_2)\frac{f'(z+\omega_2)}{f(z+\omega_2)}\,dz \\ &= -\omega_2\int_0^{\omega_1}\frac{f'(z)}{f(z)}\,dz = -2\pi i\omega_2 N_1\end{aligned}$$

where $N_1 = (2\pi i)^{-1}\int_0^{\omega_1} f'(z)/f(z)\,dz \in \mathbb{Z}$. Likewise

$$\int_{\omega_1} \omega_1+\omega_2 + \int \omega_2 0 z\frac{f'(z)}{f(z)}\,dz = 2\pi i\omega_1 N_2, \quad N_2 \in \mathbb{Z}$$

giving

$$\int_{\partial\mathcal{P}} z\frac{f'(z)}{f(z)} \in 2\pi i\Lambda.$$

□

1.1 Weierstraß theory

Notation: write

$$\sum_{\omega\in\Lambda}{}' := \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}}$$

Proposition 1.2. *Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $s \in \mathbb{R}$. Then:*

$$\sum_{\omega\in\Lambda}{}' \frac{1}{\|\omega\|^s} \text{ converges iff } s > d.$$

Proof. Let $\{\omega_i\}$ be a basis for Λ. Then there exist constants $0 < c < C$ such that for any $0 \neq x = (x_i) \in \mathbb{R}^d$

$$0 < c\|x\|_\infty < \left\|\sum x_i\omega_i\right\| < C\|x\|_\infty$$

where $\|x\|_\infty = max\,|x_i|$ (since any two norms on a finite-dimensional real vector space are equivalent). Therefore $\sum'_{\omega\in\Lambda}\|\omega\|^{-s}$ converges iff $\sum'_{x\in\mathbb{Z}^d}\|x\|_\infty^{-s}$ does. But if $1 \leq n \in \mathbb{Z}$ then

$$\#\{x \in \mathbb{Z}^d \mid \|x\|_\infty = n\} = (2n+1)^d - (2n-1)^d \sim 2d(2n)^{d-1}$$

and therefore $\sum'_{x\in\mathbb{Z}^d}\|x\|_\infty^{-s}$ converges according as $\sum_{n\geq 1} n^{d-1-s}$ does. □

Corollary. *If $\Lambda \subset \mathbb{C}$ is a lattice and $2 < k \in \mathbb{Z}$, then the series*

$$G_k(\Lambda) := \sum_{\omega\in\Lambda}{}' \frac{1}{\omega^k}$$

converges. If $k \geq 3$ is odd, then $G_k(\Lambda) = 0$.

Proof. The first part is the case $d = 2$ of the Proposition. For the last part, since $\Lambda \subset \mathbb{C}$ is a subgroup

$$G_k(\Lambda) = \sum_{\omega\in\Lambda}{}' \frac{1}{\omega^k} = \sum_{-\omega\in\Lambda}{}' \frac{1}{\omega^k} = (-1)^k G_k(\Lambda).$$

□

To construct an elliptic function, the simplest thing would be to try to make a function which just one singularity, which needs to be at least a double pole by Thm.1.1(ii). The obvious try would be to consider the series $\sum_{\omega\in\Lambda} 1/(z-\omega)^2$ — but this is not convergent. But we can subtract off the divergences:[2]

[2] Analogous to the series $\pi\cot\pi z = 1/z + \sum_{0\neq n\in\mathbb{Z}} 1/(z+n) - 1/n$

Theorem 1.3 (Weierstraß $\wp$-function)**.** *i) The series*

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda}{}' \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

represents an elliptic function.

ii) $\wp$ *is even, and its only poles are at* $\omega \in \Lambda$*, of order* 2*. Moreover, for any* $a \in \mathbb{C}$ *the function* $\wp(z) - a$ *has either two simple zeroes* $z, -z \not\equiv z \pmod{\Lambda}$ *or one double zero* $z \equiv \omega_i/2$ $(i \in \{1,2,3\})$*.*

iii) In a neighbourhood of zero.

$$\wp(z) = \frac{1}{z^2} + \sum_{r=1}^{\infty}(2r+1)G_{2r+2}(\Lambda)z^{2r}$$

iv) The function $\wp(z)$ *satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60G_4(\Lambda)$ *and* $g_3 = 140G_6(\Lambda)$*.*

Corollary 1.4. *(i) The mapping* $\wp\colon \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ *identifies* $\mathbb{P}^1(\mathbb{C})$ *with the quotient of* $\mathbb{C}/\Lambda$ *by* $z \mapsto -z$*.*

(ii) The mapping $\mathbb{C}/\Lambda \to \mathbb{P}^2(\mathbb{C})$ *given by* $z \mapsto (\wp(z), \wp'(z), 1)$ *is a biholomorphic equivalence between* $\mathbb{C}/\Lambda$ *and the curve in* $\mathbb{P}^2(\mathbb{C})$ *with equation* $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$*.*

(i) is just a restatement of (ii) above, and (ii) is an exercise.

Proof. (i,ii) If $|\omega| > 2|z|$ then

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{z(2\omega - z)}{\omega^2(\omega-z)^2}\right| < \frac{(5/2)\,|\omega z|}{(1/4)\,|\omega|^4} = \frac{10\,|z|}{|\omega^3|}$$

which shows that the series converges uniformly on compact subsets of $\mathbb{C}\setminus\Lambda$, and so is holomorphic there. Clearly it has a double pole at every point of Λ. It is also obviously even. To show it is elliptic, consider the derivative:

$$\wp'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}$$

which is obviously elliptic, so $\wp'(z+\omega_i) = \wp'(z)$ for $i = 1, 2$. Therefore $\wp(z+\omega_i) - \wp(z) = c_i$ is constant. As $\wp$ is even, putting $z = \omega_i/2$ gives $c_i = 0$, hence $\wp(z+\omega_i) = \wp(z)$ — i.e. $\wp$ is elliptic.

Now Thm.1.1(iii) applied to $\wp(z) - a$ shows that it has exactly two zeroes z, $z' in \mathbb{C} \setminus \Lambda$, counted with multiplicity, and moreover that $z + z' \equiv 0 \bmod \Lambda$. Since $z \equiv -z$ iff $z \equiv 0$ or $z \equiv \omega_i/2$, (ii) follows.

(iii) For $|z|$ sufficiently small,

$$\frac{1}{(z-\omega)^2} = \sum_{n=0}^{\infty}(n+1)\frac{z^n}{\omega^{n+2}}$$

and therefore

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{n=1}^{\infty}\sum_{\omega}{}'(n+1)\frac{z^n}{\omega^{n+2}} = \frac{1}{z^2} + \sum_{r=1}^{\infty}(2r+1)G_{2r+2}z^{2r} \\ &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + O(z^6)\end{aligned}$$

with $G_k = G_k(\Lambda)$.

(iv) Consider the functions

$$\wp'(z)^2 = \left(\frac{-2}{z^3} + 6G_4z + 20G_6z^3 + O(z^5)\right)^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + O(z^2)$$

and

$$\wp(z)^3 = \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + O(z^2).$$

Then

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6 = O(z^2)$$

and so by Thm.1.1(i), this vanishes. □

Write $e_i = \wp(\omega_i/2)$. Then the function $f_i(z) := \wp(z) - e_i$ has the double zero $z = \omega_i/2$. So $\wp'$ vanishes at each $\omega_i/2$, and these must be all the zeros of $\wp'$ in $\mathbb{C}/\Lambda$, by Thm.1.1(ii).

Therefore the function $(\wp')^2/f_1f_2f_3$ has no zeroes and poles, so by 1.1(i) is a constant. Comparing with the differential equation gives the constant value 4, and

$$4\wp(z)^3 - g_2\wp(z) - g_3 = 4\prod_{i=1}^{3}(\wp(z) - e_i), \quad e_i := \wp(\omega_i/2).$$

Also if $i \neq j$ then $e_i \neq e_j$ (since f_i cannot vanish at $\omega_j/2$) so we see:

Corollary 1.5. *The discriminant*

$$\Delta(\Lambda) := g_2^3 - 27g_3^2 = 16\prod_{i<j}(e_i - e_j)^2$$

is non-zero.

Another easy consequence of the above is that the field of all elliptic functions for Λ is $\mathbb{C}(\wp, \wp')$.

2 Modular forms of level 1

Motivation. Let $\mathcal{L}$ be the set of all lattices in $\mathbb{C}$. Then $G_k\colon \mathcal{L} \to \mathbb{C}$ satisfies

$$G_k(\alpha\Lambda) = \sum_{\omega\in\Lambda}{}^{'}(\alpha\omega)^{-k} = \alpha^{-k}G_k(\Lambda).$$

More generally, let $F\colon \mathcal{L} \to \mathbb{C}$ be a function on lattices which satisfies $F(\alpha\Lambda) = \alpha^{-k}F(\Lambda)$ for some $k \in \mathbb{Z}$. Then

$$F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_1^{-k}F(\mathbb{Z} + \mathbb{Z}\tau), \quad \tau = \omega_1/\omega_2.$$

Also, if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \quad \text{and} \quad \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \gamma \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$$

so that

$$\tau' = \frac{\omega_2'}{\omega_1'} = \frac{a\tau + b}{c\tau + d}$$

then $\mathbb{Z}\omega_1' + \mathbb{Z}\omega_2' = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ so

$$F(\mathbb{Z}\omega_1' + \mathbb{Z}\omega_2') = F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2), \quad F(\mathbb{Z} + \mathbb{Z}\tau') = (c\tau + d)^k F(\mathbb{Z} + \mathbb{Z}\tau).$$

Notation: take the usual left action of $GL_2(\mathbb{R})$ on $\mathbb{C} \setminus \mathbb{R}$ by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tau \in \mathbb{C} \setminus \mathbb{R}$$

Then

$$\operatorname{Im}\gamma(\tau) = \det\gamma \frac{\operatorname{Im}\tau}{|c\tau + d|^2}, \quad \gamma'(\tau) = \frac{\det\gamma}{(c\tau + d)^2}$$

so that $\operatorname{Im}\tau$, $\operatorname{Im}\gamma(\tau)$ have the same sign iff $\det\gamma > 0$. Write also

$$j(\gamma, \tau) = c\tau + d.$$

Then for γ, $\delta \in GL_2(\mathbb{R})$

$$j(\gamma\delta, \tau) = j(\gamma, \delta(\tau))j(\delta, \tau), \quad j(\gamma^{-1}, \tau) = j(\gamma, \gamma^{-1}(\tau))^{-1} \tag{1}$$

(the 2nd identity from the first taking $\delta = \gamma^{-1}$.)
Now for $\gamma \in GL_2(\mathbb{R})^+$, $f\colon \mathcal{H} \to \mathbb{C}$ and $k \in \mathbb{Z}$ write

$$(f|_k\gamma)(\tau) := \det(\gamma)^{k/2} j(\gamma, \tau)^{-k} f(\gamma(\tau)).$$

Lemma. *$\gamma\colon f \to f|_k f$ is a (right) group action of $GL_2(\mathbb{R})^+$ on functions on $\mathcal{H}$.*

Proof: follows from (1).

Definition. Let $f\colon \mathcal{H} \to \mathbb{P}^1(\mathbb{C})$ be meromorphic and $k \in \mathbb{Z}$. We say that f is *modular of weight* k if

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Equivalently, $f|_k\gamma = f$ for all $\gamma \in SL_2(\mathbb{Z})$.
Suppose now that f is holomorphic for $\operatorname{Im}\tau > R$, some $R > 0$. Then

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad f(\tau + 1) = f(\tau)$$

in other words,

$$f(\tau) = g(e^{2\pi i\tau}) = \sum_{n\in\mathbb{Z}} a_n(f)q^n, \quad q = e^{2\pi i\tau} \tag{2}$$

for some holomorphic function g on $\{q \in \mathbb{C} \mid 0 < |q| < r\}$, where $\log r = -2\pi R$. The series (2) is called the q-expansion (or Fourier expanson) of f at ∞.

Definition. f is *meromorphic* (resp. *holomorphic*) at infinity if it is holomorphic for $\operatorname{Im}\tau \gg 0$ and $a_n(f) = 0$ for $n \ll 0$ (resp. for all $n < 0$).

Definition. (i) A *modular form* of weight $k \in \mathbb{Z}$ is a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ which is modular of weight k and holomorphic at infinity. It is a *cusp form* if moreover $a_0(f) = 0$.
(ii) A *modular function* is a meromorphic function $f\colon \mathcal{H} \to \mathbb{P}^1(\mathbb{C})$ which is meromorphic at infinity.

Examples

First note that if f is a modular form of weight k and k is odd, then by taking $\gamma = -1$ we see that $f(\tau) = -f(\tau)$. So any non-zero modular form has even weight.
Eisenstein series: For $k \geq 4$ even, set

$$G_k(\tau) = G_k(\mathbb{Z} + \mathbb{Z}\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau + n)^k}$$

— here the prime means omit $(m, n) = (0, 0)$. By the convergence of the series this is holomorphic for $\tau \in \mathcal{H}$.

Proposition 2.1. *$G_k(\tau)$ is a modular form of weight k, and has q-expansion*

$$G_k(\tau) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n\right)$$

Here $\zeta(s) = \sum_{n\geq 1} n^{-s}$ is the Riemann zeta-function, and $\sigma_r(n) = \sum_{0<d|n} d^r$. Finally, the B_k are the *Bernoulli numbers*, defined by the identity

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$$

Proof. That $G_k(\tau)$ is modular of weight k has already been noted. To show it is holomorphic at infinity we compute its q-expansion. Begin with

$$\pi \cot \pi z = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z-m} + \frac{1}{z+m} \right) \tag{3}$$

$$= \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \tag{4}$$

$$= \pi i - \frac{2\pi i}{1 - e^{2\pi i z}} \tag{5}$$

$$= -\pi i - 2\pi i \sum_{d\geq 1} e^{2\pi i d z} \tag{6}$$

Differentiating $(k-1)$ times, for $k > 1$:

$$(-1)^{k-1}(k-1)! \sum_{m=-\infty}^{\infty} \frac{1}{(z-m)^k} = -(2\pi i)^k \sum_{d\geq 1} d^{k-1} e^{2\pi i d z}$$

or

$$\sum_{m\in\mathbb{Z}} \frac{1}{(z-m)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d\geq 1} d^{k-1} e^{2\pi i d z}$$

So if k is even

$$G_k(\tau) = 2\sum_{n=1}^{\infty} \frac{1}{n^k} + 2\sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau + n)^k}$$

$$= 2\zeta(k) + \sum_{m=1}^{\infty} \frac{2(2\pi i)^k}{(k-1)!} \sum_{d\geq 1} d^{k-1} e^{2\pi i m d \tau}$$

$$= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n$$

Now (3) gives, setting $t = 2\pi i z$,

$$\frac{1}{2} + \frac{1}{e^t - 1} = \frac{1}{t} + \sum_{m=1}^{\infty} \frac{1}{t - 2\pi i m} + \frac{1}{t + 2\pi i m}$$

so

$$\begin{aligned}
\frac{t}{e^t - 1} &= 1 - \frac{t}{2} + \sum_{m=1}^{\infty} \frac{t}{t + 2\pi i m} + \frac{t}{t - 2\pi i m} \\
&= 1 - \frac{t}{2} - 2 \sum_{m,n \geq 1} \left(\frac{t}{2\pi i m} \right)^{2n} \\
&= 1 - \frac{t}{2} - \sum_{n=1}^{\infty} 2(2\pi i)^{-2n} \zeta(2n) t^{2n}
\end{aligned}$$

using the identity

$$\frac{t}{t \pm 2\pi i m} = - \sum_{n \geq 1} \left(\frac{\pm t}{2\pi i m} \right)^n$$

□

Definition. For $k \geq 4$ even, the *normalised Eisenstein series* is

$$E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

The first few nonzero Bernoulli numbers are

$$B_0 = 1,\ B_1 = -\frac{1}{2}, B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},$$
$$B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730},\ B_{14} = \frac{7}{6}$$

giving

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

$$E_8(\tau) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$$

$$E_{10}(\tau) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n$$

$$E_{12}(\tau) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$$

$$E_{14}(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n$$

The modular group

We've already considered the action of $GL_2(\mathbb{R})^+$ on $\mathcal{H}$, which factors through $PGL_2(\mathbb{R})^+$. Note that since $GL_2(\mathbb{R})^+ = \mathbb{R}^*.SL_2(\mathbb{R})$ ($\mathbb{R}^*$ embedded as diagonal matrices), we have $PGL_2(\mathbb{R})^+ = PSL_2(\mathbb{R})$. We'll see later why GL is needed.

Note the fact that $PSL_2(\mathbb{R})$ is the group of holomorphic automorphisms of $\mathcal{H}$ (or equivalently of the unit disc). Moreover it acts transitively, and the stabiliser of i is $SO(2)/\pm 1$, which is a maximal compact subgroup. The can be assembled in the *Iwasawa decomposition* which I'll just write for SL_2:

$$SL_2(\mathbb{R}) = KAN = NAK, \quad K = SO(2), \quad A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

since

$$\tau = x + iy = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} (i)$$

If $\Gamma \subset SL_2(\mathbb{R})$ then we often write $\overline{\Gamma} \subset PSL_2(\mathbb{R})$ for its image, so that $\overline{\Gamma} = \Gamma/(\Gamma \cap \{\pm 1\})$. We are mainly interested in $\Gamma = SL_2(\mathbb{Z})$ and its subgroups. The quotient $\overline{\Gamma} = PSL_2(\mathbb{Z})$ is the *modular group*. First task is to describe its action on $\mathcal{H}$ as explicitly as possible. Let S, $T \in \overline{\Gamma}$ denote the elements

$$S = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.2. *Let $\Gamma = SL_2(\mathbb{Z})$, $\overline{\Gamma} = PSL_2(\mathbb{Z})$. Then the subset*

$$\mathcal{D} = \left\{ \tau \in \mathcal{H} \mid -\frac{1}{2} < \operatorname{Re}(\tau) \le \frac{1}{2},\ |\tau| \ge 1,\ |\tau| = 1 \implies \operatorname{Re}(\tau) \ge 0 \right\}$$

is a fundamental domain for Γ, i.e. $\mathcal{D} \xrightarrow{\sim} \Gamma\backslash\mathcal{H}$ is bijective. Moreover, if $\tau \in \mathcal{D}$ and $\overline{\Gamma}_\tau \ne 1$, then:

$$\begin{aligned} &\text{either } \tau = i, \quad \overline{\Gamma}_\tau = \langle S \rangle \simeq \mathbb{Z}/2 \\ &\text{or } \tau = \rho = e^{\pi i/3}, \quad \overline{\Gamma}_\tau = \langle TS \rangle \simeq \mathbb{Z}/3. \end{aligned}$$

Finally, $\overline{\Gamma} = \langle S, T \rangle$.

Proof. (i) Let $\overline{\Gamma}^* = \langle S, T \rangle$. First show that if $\tau \in \mathcal{H}$ then $\exists \gamma \in \overline{\Gamma}^*$ with $\gamma(\tau) \in \mathcal{D}$. Since $\mathbb{Z} + \mathbb{Z}\tau$ is a lattice, $\{|c\tau + d| \mid c,\ d \in \mathbb{Z}\}$ is a discrete subset of $\mathbb{R}$, so

$$\left\{ \operatorname{Im}\gamma(\tau) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2} \mid \gamma \in \overline{\Gamma}^* \right\}$$

is a discrete subset or $\mathbb{R}_{>0}$, bounded above. So can choose $\gamma \in \overline{\Gamma}^*$ with $\operatorname{Im}\gamma(\tau)$ maximal, and (replacing γ with $T^m\gamma$ is necessary) may assume $|\operatorname{Re}\gamma(\tau)| \le 1/2$.

If $|\gamma(\tau)| < 1$ then

$$\operatorname{Im}(S\gamma(\tau)) = \operatorname{Im}(-1/\gamma(\tau)) = \frac{\operatorname{Im}(\gamma(\tau)}{|\gamma(\tau)|^2} > \operatorname{Im}\gamma(\tau)$$

contradicting maximality. So $|\gamma(\tau)| \geq 1$ i.e. $\gamma(\tau) \in \mathcal{D}^c$. If $\operatorname{Re}\gamma(\tau) = -1/2$ then $T\gamma(\tau) \in \mathcal{D}$. If $-1/2 < \operatorname{Re}\gamma(\tau) < 0$ and $|\gamma(\tau)| = 1$ then $S\gamma(\tau) \in \mathcal{D}$.

Remark: This leads to the following simple algorithm to compute γ (explained in the lectures, where the proof of the theorem was not given): first find a power T^n such that $|\operatorname{Re} T^n\tau| \leq 1/2$. If $|T^n\tau| < 1$ then put $\tau_1 = ST^n\tau$, so that $\operatorname{Im}(\tau_1) = 1/\operatorname{Im}(\tau) > \operatorname{Im}(\tau)$. Replace τ by τ_1 and repeat; eventually the sequence of imaginary parts terminates, so we end up with a point τ' with $|\operatorname{Re}\tau'| \leq 1/2$ and $|\tau'| \geq 1$. Then one of τ', $S\tau'$, $T\tau'$ is in $\mathcal{D}$.

(ii) Next show that if t, $\tau' \in \mathcal{D}$ and $\tau' = \gamma(\tau)$, $\gamma \in \overline{\Gamma}$ then $\tau = \tau'$ and γ is as in statement of theorem.

We can assume that

$$\operatorname{Im}\tau' = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2} \geq \operatorname{Im}\tau$$

i.e. that $|c\tau + d| \leq 1$. Then

$$1 \geq |\operatorname{Im}(c\tau + d)| = |c| \operatorname{Im}(\tau) \geq |c| \frac{\sqrt{3}}{2}$$

forcing $c \in \{0, \pm 1\}$.

$$c = 0 \implies \gamma = \pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \ \tau' = \tau + m, \ m \in \mathbb{Z} \implies m = 0, \ \gamma = \pm 1, \ \tau = \tau'.$$

If $c = 1$ then $|\tau| \geq 1$ and $|\tau + d| \leq 1$. Then 2 possibilities:

$$d = 0, \ |\tau| = 1, \ \gamma = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \ \tau' = a - 1/\tau.$$

Then $a = 0$ and $\tau = \tau' = i$ or $a = 1$, $\tau = \tau' = \rho$.

$$d = 1, \ |\tau| \geq 1, |\tau - 1| \leq 1, \ \operatorname{Re}(\tau) \leq 1/2.$$

Then $\tau = \rho$, $|c\tau + d| = |\tau - 1| = 1$ so $\operatorname{Im}\tau' = \operatorname{Im}\tau = \sqrt{3}/2$, so $\tau = \tau' = \rho$. Then

$$\frac{a\tau + b}{\tau - d} = \tau = \rho \implies \rho^2 - (a+1)\rho - b = 0 \implies b = -1, \ a = 0.$$

$\square$

Important consequence is that the quotient $\Gamma\backslash\mathcal{H}$ has finite invariant measure:

Proposition 2.3. *(i) The measure $d\mu = y^{-2}dx\,dy$ on $\mathcal{H}$ is invariant under $PSL_2(\mathbb{R})$.*
(ii) If $\Gamma \subset SL_2(\mathbb{Z})$ is a subgroup of finite index, then $\mu(\overline{\Gamma}\backslash\mathcal{H}) < \infty$.

Proof. (i) Consider the associated 2-form

$$\eta = \frac{dx \wedge dy}{y^2} = \frac{i\,d\tau \wedge d\bar{\tau}}{2\,\mathrm{Im}(\tau)^2}.$$

Then if $\gamma(\tau) = (a\tau + b)/(c\tau + d)$, $\gamma \in SL_2(\mathbb{R})$, the formulae $\gamma'(\tau) = 1/(c\tau + d)^2$, $\mathrm{Im}\,\gamma(\tau) = \mathrm{Im}(\tau)/\left|c\tau + d\right|^2$ show that η is invariant under γ.
(ii) If $(PSL_2(\mathbb{Z}) : \overline{\Gamma}) = M$ then

$$\begin{aligned}\mu(\overline{\Gamma}\backslash\mathcal{H}) &= M\mu(PSL_2(\mathbb{Z})\backslash\mathcal{H}) \\ &= M\mu(\mathcal{D}) \\ &< M \int_{\substack{y\geq 1\\ |x|\leq\sqrt{3}/2}} \frac{dx\,dy}{y^2} < \infty.\end{aligned}$$

(Exercise: show that $\mu(PSL_2(\mathbb{Z})\backslash\mathcal{H}) = \pi/3$.) □

Now return to modular forms. We let M_k and S_k denote the spaces of modular and cusp forms, respectively, of weight k. From the definition we have an exact sequence

$$0 \to S_k \to M_k \xrightarrow{f \mapsto a_0(f)} \mathbb{C} \tag{7}$$

so S_k either equals M_k or is a subspace of codimension 1. Let's first prove:

Proposition 2.4. *$S_0 = 0$ and $M_0 = \mathbb{C}$.*

Proof. Obviously $\mathbb{C} \subset M_0$, so enought to show $S_0 = 0$. If $f \in S_0$ the f is holomorphic on $\mathcal{H}$ and satisfies $f(x + iy) \to 0$ as $y \to \infty$, uniformly in x. So on $\mathcal{D}^c$, f attains a maximum on the boundary (unit circle plus vertical lines). But as weight is modular of weight 0 it's invariant under $SL_2(\mathbb{Z})$, so since $\mathcal{D}$ is a fundamental domain, this maximum must be the maximum of $|f|$ on $\mathcal{H}$, so f is constant by maximum modulus, hence in fact zero. □

Now recall that we have, for every even $k \geq 4$, the normalised Eisenstein series; in particular the series of weights 4 and 6:

$$E_4 = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n$$
$$E_6 = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^n.$$

Define

$$\Delta = \frac{E_4^3 - E_6^2}{1728} == q - 24q^2 + \cdots = \sum_{n=1}^{\infty} \tau(n) q^n$$

which is therefore a cusp form of weight 12.
Write in terms of the unnormalised series $G_4 = G_4(\mathbb{Z}+\mathbb{Z}\tau) = g_2/60$, $G_6 = g_3/140$. We have

$$2\zeta(4) = \frac{\pi^4}{45}, \quad 2\zeta(6) = \frac{2\pi^6}{945} \tag{8}$$

$$E_4 = \frac{3}{8\pi^4} g_2, \quad E_6 = \frac{27}{16\pi^6} g_3 \tag{9}$$

$$E_4^3 - E_6^2 = \frac{3^3}{2^9\pi^{12}} (g_2^3 - 27g_3^2) \tag{10}$$

and so comparing with Corollary 1.5 we see that Δ is nonvanishing on $\mathcal{H}$. Use this to determine the spaces M_k completely. Let's first note the simple fact:

Multiplication by Δ is an isomorphism $M_{k-12} \xrightarrow{\sim} S_k$ for every $k \in \mathbb{Z}$.

Indeed, we have $0 \neq \Delta \in S_{12}$, so $\Delta M_{k-12} \subset S_k$, and since $\Delta(\tau) \neq 0$ for $\tau \in \mathcal{H}$, if $f \in S_k$ the f/Δ is both holomorphic on $\mathcal{H}$ and holomorphic at infinity.

Theorem 2.5. *(i) If $k < 0$ then $M_k = 0$. If $k \geq 4$ then $M_k = S_k \oplus \mathbb{C}.E_k$.*
(ii) If $k \geq 0$ is even

$$\dim M_k = \begin{cases} 1 + \left[\frac{k}{12}\right] & \text{if } k \not\equiv 2 \text{ mod } 12 \\ \frac{k-2}{12} & \text{if } k \equiv 2 \text{ mod } 12 \end{cases}$$

(iii) A basis for M_k is $\{E_4^a E_6^b \Delta^c\}$ where $4a + 6b + 12c = k$, $a, b, c \geq 0$ and $b = 0$ if $k \equiv 0 \pmod 4$, $b = 1$ if $k \equiv 2 \pmod 4$. In particular,

$$\bigoplus_{k \in \mathbb{Z}} M_k = \mathbb{C}[E_4, E_6].$$

Proof. (i) If $f \in M_k$, $k < 0$ then $f^{12}\Delta^{(-k)} \in S_0 = \{0\}$, so $f = 0$. Since S_k is the kernel of the map $M_k \to \mathbb{C}$ given by $f \mapsto a_0(f)$ and $a_0(E_k) = 1$, the second part follows.
(ii) Assume k even. We have $S_k = \Delta M_{k-12}$, hence $S_k = 0$ for $k < 12$, and therefore $M_k = \mathbb{C}.E_k$ for $4 \leq k \leq 10$. Let's show that $M_2 = 0$. If not, and $0 \neq f \in M_2$ then $f^2 \in M_4$ so $f^2 = aE_4$. Likewise $f^3 E_6$. But as $a_0(E_k) = 1$, this would imply that $E_4^3 = E_6^2$, whereas $E_4^3 - E_6^2 = 1728\Delta \neq 0$.
The dimension formula is therefore true for $0 \leq k \leq 10$. But if $k \geq 12$, $M_k = \mathbb{C}.E_k + \Delta M_{k-12}$ so $\dim M_k = 1 + \dim M_{k-12}$, hence (by induction) the formula holds for all k.

(iii) Let $k \geq 4$ be even. Take b as in the first statement, and $a = (k - 6b)/4$. Then $M_k = \mathbb{C}.E_4^a E_6^b \oplus \Delta M_{k-12}$. So by induction on k the statement holds. For the second, as $\Delta = 12^{-3}(E_4^3 - E_6^2)$, every element of M_k is a linear combination of $\{E_4^a E_6^b \mid a, b \leq 0,\ 4a + 6b = k\}$. But it is easy to see that the number of such pairs (a, b) equals the right hand side of the dimension formula, so the mononomials are linearly independent. □

Remark. The previous result shows there is no non-trivial homogeneous relation between E_4 and E_6. It is not hard to see there is no inhomogeneous relation — in other words, E_4 and E_6 are algebraically dependent as functions on $\mathcal{H}$. It is enough to check that if $f_1, \ldots, f_r$ are modular forms of weights $k_1 < \cdots < k_r$ with $\sum f_j = 0$ then each $f_j = 0$. But then $f_j(iy) = (iy)^{-k_j} f_j(i/y)$ and as $y \to 0$, $f(i/y) \to a_0(f_j)$, so as the weights are different we have $a_0(f_j) = 0$ for all j. But then dividing the relation by Δ gives one with smaller weights, so by induction all f_j are zero.

Rationality and integrality

Let $R \subset \mathbb{C}$ be a subring. Define $M_k(R) = M_k \cap R[[q]] = \{f \in M_k | a_n(f) \in R \forall n\}$, $S_k(R) = S_k \cap M_k(R)$.
We have $E_4 \in M_4(\mathbb{Z})$ and $E_6 \in M_6(\mathbb{Z})$. Also easy to see that $\Delta \in S_{12}(\mathbb{Z})$, since expanding $E_4^3 - E_6^2$ one easily reduces to checking that $\sigma_5(n) - \sigma_3(n)$ is always divisible by 12, which follows from the congruence $d^5 - d^3 \equiv 0 \pmod{12}$.

Proposition 2.6. *Let $d = \dim M_k$. There exists a basis $\{g_j \mid 0 \leq j < d\}$ for M_k such that $g_j \in M_k(\mathbb{Z})$ and*

$$a_n(g_j) = \delta_{nj} \quad \textit{for all } j, n \in \{0, \ldots, d-1\}.$$

Proof. Let $b = 0$ if $4|k$ and 1 otherwise, and consider the basis $h_j = E_4^{(k-6b-12j)/4} E_6^b \Delta^j \in M_k(\mathbb{Z})$ given in 2.5(iii). Then $a_n(h_j) = 0$ if $n < j$ and $a_j(h_j) = 1$, so by elementary operations we can replace $\{h_j\}$ by a basis with the desired properties. □

Corollary. *(i) For every R, the map $f \mapsto (a_j(f))_{0 \leq j < d}$ is an isomorphism of R-modules $M_k(R) \xrightarrow{\sim} R^d$. Likewise, $f \mapsto (a_j(f))_{1 \leq j < d}$ is an isomorphism $S_k(R) \xrightarrow{\sim} R^{d-1}$.*
(ii) If $1/6 \in R$ then $M_*(R) = \bigoplus M_k(R) = R[E_4, E_6]$, and in general

$$M_*(R) = R[E_4, E_6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta).$$

Proof. (i) The inverse map is $(r_j) \mapsto \sum r_j g_j$.
(ii) Since $g_j \in \mathbb{Z}[E_4, E_6, \Delta]$ this is clear. □

Nice example: $\Delta = \sum_{n\geq 1} \tau(n) q^n$, $\tau(n) \in \mathbb{Z}$. Then we have

$$E_{12} = 1 + \frac{65520}{691} \sum \sigma_{11}(n) q^n = E_4^3 + c\Delta, \quad c \in \mathbb{Q}.$$

So for every $n \geq 1$,

$$\frac{65520}{691} \sum \sigma_{11} \equiv c\tau(n) \pmod{\mathbb{Z}}.$$

Putting $n = 1$ shows that $c \equiv 65220/691 \pmod{\mathbb{Z}}$, giving Ramnujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

There is a very elagant explanation of these and similar congruences in terms of mod ℓ Galois representations, due to Serre and Swinnerton-Dyer (see papers by them in *Modular Forms of One Variable III*).

Finally, a useful estimate for the Fourier coefficients of a cusp form. First prove the following nice characterisation of cusp forms:

Proposition 2.7. *Let $f\colon \mathcal{H} \to \mathbb{C}$ be holomorphic and modular of weight $k > 0$. Then $f \in S_k$ if and only if $y^{k/2} |f(\tau)|$ is bounded on $\mathcal{H}$, $y = \mathrm{Im}(\tau)$.*

Proof. The transformation law shows that $y^{k/2} |f(\tau)|$ is invariant under Γ, so it is bounded on $\mathcal{H}$ iff it is bounded on Δ. But this holds iff it is bounded as $y \to \infty$, which implies that $|f| \to 0$ as $y \to \infty$. So the Fourier series $f = \sum a_n(f) q^n \to 0$ as $q \to 0$, meaning that $a_n = 0$ for all $n \leq 0$, i.e. f is a cusp form. Conversely if $f \in S_k$ then $|f/q|$ is bounded as $q \to 0$, so certainly $y^N |f|$ is bnounded at $y \to \infty$ for any N. □

Corollary 2.8. *If $f \in S_k$ then $|a_n(f)| \ll n^{k/2}$.*

Proof. Since $f(x + iy) = \sum a_n(f) e^{2\pi i n x - 2\pi n y}$ we have, for any $y > 0$,

$$a_n(f) = \int_0^1 e^{2\pi n y - 2\pi i n x} f(x + iy)\, dx$$

hence

$$\begin{aligned} |a_n(f)| &\leq \sup_{x\in[0,1]} |f(x + iy)|\, e^{2\pi n y} \\ &\ll y^{-k/2} e^{2\pi n y} \end{aligned}$$

by the previous result. Take $y = 1/n$. □

Notice that for $k > 2$ this estimate is false for any $f \in M_k$ which isn't a cusp form. In fact, f is then the sum of a cusp form and a non-zero multiple of E_k, so $|a_n(f)| \gg \sigma_{k-1}(n) \geq n^{k-1}$.

Later will prove the better estimate $O(n^{k/2-1/4})$ for the Fourier coefficients of a cusp form. This is about the best one can do by analytic means, but using arithmetic geometry Deligne proved the best possible bound of $O(n^{(k-1)/2+\epsilon})$ (Ramanujan-Petersson conjecture).