Q1. Suppose $g \in M_2(\mathbb{Z})$, $\det g = N > 0$. The rows of g generate a subgroup $\Lambda_g \subset \mathbb{Z}^2$ of index N. Two matrices g, g' have $\Lambda_g = \Lambda_{g'}$ iff $\exists\, \gamma \in GL_2(\mathbb{Z})$ with $\gamma g = g'$ (in fact $\gamma \in SL_2(\mathbb{Z})$ if $\det g' > 0$).

So $\{\text{cosets } \Gamma g \mid g \in M_2(\mathbb{Z}),\ \det g = N\} \simeq \{\text{subgroups } \Lambda \subset \mathbb{Z}^2 \text{ of index } N\} = (*)$

$$g \longleftrightarrow \Lambda_g$$

and $\{\text{cosets } \Gamma h \mid \Gamma h \subset \Gamma g \Gamma\} \simeq \{\text{subgroups } \Lambda \subset \mathbb{Z}^2 \text{ of index } N \text{ s.t. } \Lambda = \Lambda_{g\gamma} \text{ for some } \gamma \in \Gamma\}$.

and $(*)$ is a finite set.

In general, $g = \frac{1}{n} g'$, $n \geq 1$, $g' \in M_2(\mathbb{Z})$ and if $\Gamma g' \Gamma = \coprod \Gamma h_j'$ then $\Gamma g \Gamma = \coprod \Gamma h_j$ with $h_j = \frac{1}{n} h_j'$.

By classification of f. gen. abelian groups, $\exists \{\underline{e}, \underline{f}\}$, basis for $\mathbb{Z}^2$ such that $\Lambda_g = \langle m\underline{e}, n\underline{f} \rangle$ with $m, n \geq 1$, $mn = N$ and $n \mid m$ (and then $\mathbb{Z}^2/\Lambda_g \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$). Let $\underline{e}, \underline{f}$ be the rows of $\gamma \in GL_2(\mathbb{Z})$; replacing $\underline{e}$ by $\pm\underline{e}$ may assume $\gamma \in SL_2(\mathbb{Z})$. Then Λ_g is generated by the rows of $\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}\gamma$, so $g = \gamma' \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}\gamma$, some $\gamma' \in GL_2(\mathbb{Z})$.

Comparing dets of each side $\Rightarrow \gamma' \in SL_2(\mathbb{Z})$. Hence

$$\Gamma g \Gamma = \Gamma \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \Gamma.$$

The subgroups of index p in $\mathbb{Z}^2$ are generated by rows of h where

$$h = \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} \ (0 \leq a < p) \quad \text{or} \quad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

and $SL_2(\mathbb{Z})$ permutes them transitively, so

$$\Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \amalg \coprod_{0 \leq a < p} \Gamma \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}.$$

Let $F = f|_k [\Gamma g \Gamma] = \sum f|_k h_j$ where $\Gamma g \Gamma = \coprod \Gamma h_j$.

Suppose $\Gamma g \Gamma = \coprod \Gamma h_j'$. After renumbering, $\Gamma h_j = \Gamma h_j'$ so that $h_j' = \gamma_j h_j$, $\gamma_j \in \Gamma$, and $f|_k h_j' = f|_k \gamma_j h_j = f|_k h_j$. So independent of choice of coset representatives.

So if $\gamma \in \Gamma$ then $F|_k \gamma = \sum_j f|h_j \gamma = F$ since $\Gamma g \Gamma = \coprod \Gamma h_j \gamma$.

ie, F is modular of weight k.

Now every coset Γh_j has a representative which is upper triangular. — if $h_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then choose $\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $ac' + cd' = 0$, then $\gamma h_j = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

If $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $f = \sum_{n \geq 0} A_n q^n$ then $f|_k h = (\frac{a}{d})^{\frac{k}{2}} \sum_{n \geq 0} A_n e^{2\pi i bn/d} \cdot q^{a/d}$,

so the q-expansion of F has no negative powers of q, ie $F \in M_k$.

In the last case we have

$$F = f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{p-1} f|_k \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} = p^{1-\frac{k}{2}} T_p f. \qquad \square$$

(when $\det \gamma = N$ is not prime one can express $f|[\Gamma\gamma\Gamma]$ in terms of the $T_m f$ for $m | N$.)

Q2. If Λ is even then $\Theta_\Lambda(\tau) = \sum_{x \in \Lambda} q^{\frac{1}{2}\|x\|^2}$ is a power series in q so $\Theta_\Lambda(\tau) = \Theta_\Lambda(\tau + 1)$. If Λ is self-dual then $\Theta_\Lambda(-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}}^{\,k} \Theta_\Lambda(\tau)$

It's enough to show that $k \equiv 0 \pmod 8$; then $\Theta_\Lambda(-1/\tau) = \tau^{k/2} \Theta_\Lambda(\tau)$ $\Rightarrow \Theta \in M_{k/2}$. Suppose $k/8 = l/2^r$ with $r \geq 1$, l odd. Let $f = \Theta_\Lambda^{2^{r-1}}$.

Then $f(-\frac{1}{\tau}) = -\tau^{2l} f(\tau)$, ie. $f|_{2l} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = -f$, $f|_{2l} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = f$.

Let $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$; then $f|_{2l} \gamma = -f$

But $\gamma^3 = 1 \Rightarrow f = f|_{2l} \gamma^3 = -f \Rightarrow f = 0$.

[It also follows from classification for self-dual lattices that $k \equiv 0 \ (8)$]

Q3. (i, ii) Λ obviously a subgroup: and $\frac{1}{2}\mathbb{Z}^k \supset \Lambda \supset 2\mathbb{Z}^k \Rightarrow \Lambda$ a lattice.

Let $\underline{x} = \frac{1}{2}\underline{y}$, $\underline{y} \in \mathbb{Z}^k$, $y_i \equiv y_j \bmod 2$ and $\sum_{i=1}^k y_i \equiv 0 \pmod 4$.

Then $y_i^2 \equiv y_j^2 \bmod 4 \Rightarrow \sum y_i^2 \equiv k y_1^2 \equiv 0 \pmod 4 \Rightarrow \|x\|^2 \in \mathbb{Z}$.

If $8 | k$ then $\|x\|^2 \in 2\mathbb{Z}$.

(iii). $(x,y) = \dfrac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} \in \mathbb{Z} \quad \forall x, y \in \Lambda$; So $\Lambda \subset \Lambda'$.

Now $\Lambda \subset \{x \in \frac{1}{2}\mathbb{Z}^k \mid x_i \equiv x_j \ (2)\} = \Lambda_1 \subset \frac{1}{2}\mathbb{Z}^k$

$\Lambda = \ker(\Lambda_1 \to \mathbb{Z}/2\mathbb{Z})$, $x \mapsto \sum x_i$

$\therefore (\Lambda_1 : \Lambda) = 2$, $(\frac{1}{2}\mathbb{Z}^k : \Lambda_1) = 2^{k-1} \Rightarrow (\frac{1}{2}\mathbb{Z}^k : \Lambda) = 2^k$

so $m(\Lambda) = \frac{1}{2^k} m(\frac{1}{2}\mathbb{Z}^k) = 1$, so as $m(\Lambda') = m(\Lambda)^{-1} = 1$, $\Lambda = \Lambda'$.

$k = 8 \Rightarrow \Theta_\Lambda \in M_4$, const. term $= 1 \Rightarrow \Theta_\Lambda = E_4$. So $\#\{x \in \Lambda \mid \|x\|^2 = 2\} = $ coeff. of $q = 240$

Q 4

(i) $$\zeta(s) - \int_1^\infty \frac{1}{x^s}\,dx = \sum_{n=1}^\infty \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\,dx \right)$$

$$= \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s}\,dx$$ and if $x \in [n, n+1]$, $\sigma = Re(s)$, then

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \le |s| \frac{1}{n^{\sigma+1}}$$ (*) see end of page

Also $\int_1^\infty \frac{1}{x^s}\,dx = \frac{1}{s-1}$ so $\zeta(s) - \frac{1}{s-1}$ has A.C. to a nbd of $s=1$

and $$\lim_{s\to 1} \zeta(s) - \frac{1}{s-1} = \lim_{N\to\infty} \left[\sum_{n=1}^{N-1} \left(\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx \right) \right]$$

$$= \lim_{N\to\infty} 1 + \tfrac{1}{2} + \dots + \frac{1}{N-1} - \log N = \gamma.$$

(ii) $$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s) = \pi^{-1/2+s/2}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-s)$$

$$\therefore \text{ STP } \frac{\pi^{-1/2+s}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)$$

which follows from :-
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \qquad (I)$$
$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = 2^{1-2s}\pi^{1/2}\Gamma(2s) \qquad (II)$$
((reflection and duplication formula)

Indeed, $$RHS = \frac{2^s\pi^s\,\Gamma(1-s)}{\Gamma(s/2)\Gamma(1-s/2)}$$ by I

$$= \frac{2^s\pi^s}{\Gamma(\frac{s}{2}).\Gamma(1-\frac{s}{2})} \times \pi^{-1/2}.2^{-1+(1-s)}\Gamma\left(\tfrac{1-s}{2}\right)\Gamma\left(\tfrac{1-s}{2}+\tfrac{1}{2}\right)$$ by II

$= RHS.$

(iii) $$\Gamma(s) = s^{-1}e^{-\gamma s}\prod_{n=1}^\infty \left(1+\frac{s}{n}\right)^{-1}e^{s/n}$$

and $\left(1+\frac{s}{n}\right)^{-1}e^{s/n} = 1 + O(s^2)$ at $s=0$

(ie) $\Gamma(s) = s^{-1} - \gamma + O(s)$ at $s=0$

$\Rightarrow \Gamma(1-s) = -s\Gamma(-s) = 1 + s.\gamma + O(s^2)$ at $s=0$

So from (ii) we have at $s=0$:

$$\zeta(s) = (1+s\log 2)\,\pi^{-1}(1+s\log\pi)\,\frac{\pi s}{2}.(1+s\gamma).\left(\frac{1}{-s}+\gamma\right) + O(s^2)$$

$$= -\tfrac{1}{2}(1+s\log 2\pi) + O(s^2) \Rightarrow \zeta'(0) = -\tfrac{1}{2}\log 2\pi.$$

(*) $f(z) = \frac{1}{z^s}$ is analytic for $Re(z)>0$, and $f'(z) = \frac{-s}{z^{s+1}}$, hence

$$|f(z) - f(n)| = \left| \int_n^z \frac{-s}{w^{s+1}}\,dw \right| \le \frac{|s|}{n^{\sigma+1}}$$ if $z \in [n, n+1]$.

Q5. We proved in lectures :-

$$G(\tau,0) = E_{\Lambda_\tau}(0) = -1 \quad (\text{Thm. } 4.5)$$

$$(\text{KLF}) \qquad G'(\tau,0) = 4\zeta'(0) - \log\left(y|\Delta|^{1/6}\right)$$
$$= -\log\left(4\pi^2 y|\Delta|^{1/6}\right) \quad \text{using Q.4}$$

and we have the functional equation
$$\pi^{-s}\Gamma(s)\,G(\tau,s) = \pi^{s-1}\Gamma(1-s)\,G(\tau,1-s)$$

i.e. $$G(\tau,1-s) = \pi^{1-2s}\frac{\Gamma(s)}{\Gamma(1-s)}\,G(\tau,s).$$

We know $G(\tau,s) = \frac{\pi}{s-1} + (\text{holomorphic in } s)$ so $G(\tau,1-s) = -\frac{\pi}{s} + C + O(s)$ for C to be determined.

$$\pi^{1-2s} = \pi - 2\pi\log\pi . s + O(s^2).$$

$$\left.\begin{aligned}\Gamma(s) &= \tfrac{1}{s} - \gamma + O(s)\\ \Gamma(1-s) &= 1 + \gamma s + O(s^2)\end{aligned}\right\} \text{ see sol}^{\text{n}} \text{ to Q4}$$

So $$G(\tau,1-s) = \pi\left(1 - 2\log\pi . s + O(s^2)\right).\left(\tfrac{1}{s} - \gamma + O(s)\right).\left(1+\gamma s + O(s^2)\right)^{-1}$$
$$\times\left(-1 - \log\left(4\pi^2 y|\Delta|^{1/6}\right)s + O(s^2)\right)$$

$$= -\frac{\pi}{s}\left(1 + \left(-2\log\pi - \gamma - \gamma + \log\left(4\pi^2 y|\Delta|^{1/6}\right)\right)s + O(s^2)\right)$$

$$= -\frac{\pi}{s} + 2\pi\gamma - \log\left[4y|\Delta|^{1/6}\right] + O(s).$$

Q6. We know $\left(\begin{smallmatrix}\text{by Mellin}\\ \text{transform}\end{smallmatrix}\right)$ that $\tilde{G}(\tau,s) = G(\tau,s) - \frac{\pi}{s-1}$ along with its partial derivatives has AC. to the left of $s=1$. So

$$\frac{\partial}{\partial\tau}\tilde{G}(\tau,1) = \lim_{s\to 1}\frac{\partial}{\partial\tau}\tilde{G}(\tau,s) = \lim_{s\to 1}\frac{\partial}{\partial\tau}G(\tau,s) \qquad \text{as } \frac{\partial}{\partial\tau}\left(\frac{1}{s}\right) = 0$$

$$= \lim_{s\to 1}\sum{}'\frac{\partial}{\partial\tau}\left(\frac{y^s}{|m\tau+n|^{2s}}\right)$$

If $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$,
$$\frac{\partial}{\partial\tau}\left(\frac{y^s}{|c\tau+d|^{2s}}\right) = \frac{\partial}{\partial\tau}\left(\text{Im}\,\gamma(\tau)\right)^s = \frac{\partial}{\partial\tau}\left(\frac{\gamma(\tau) - \overline{\gamma(\tau)}}{2i}\right)^s$$

$$= \frac{1}{2i}s\,\gamma'(\tau).\,\text{Im}\,\gamma(\tau)^{s-1} = \frac{s}{2i}\,\frac{y^{s-1}}{|c\tau+d|^{2s-2}\,(c\tau+d)^2}$$

so true for $(m,n) = l.(c,d)$ also i.e.

$$\frac{\partial}{\partial\tau}\tilde{G}(\tau,1) = \lim_{s\to 1}\underbrace{sy^{s-1}}_{\downarrow\, 1}\;\frac{1}{2i}\sum{}'\frac{1}{(m\tau+n)^2|m\tau+n|^{2s-2}}.$$

Now $\tilde{G}(\tau,1)$ is invariant under $SL_2(\mathbb{Z})$

$$\Rightarrow \frac{\partial}{\partial\tau}\tilde{G}(\tau,1) = \frac{\partial}{\partial\tau}\tilde{G}(\gamma(\tau),1) = \left(\frac{\partial\tilde{G}}{\partial\tau}\right)(\gamma(\tau),1).\,\gamma'(\tau)$$

i.e. $$G_2^*(\gamma(\tau)) = \gamma'(\tau)^{-1}\,G_2^*(\tau) \quad \text{so } G_2^* \text{ modular of wt } 2.$$

Q7. $G_2^*(\tau) = \lim_{s\to 1} 2i \frac{\partial}{\partial \tau} G(\tau, s)$ has Fourier series $\sum_{k\in\mathbb{Z}}' B_k(y)\, e^{2\pi i k x}$

Constant term:

$$B_0(y) = \lim_{s\to 1} 2i \frac{\partial}{\partial \tau} A_0(\tau,s) = \lim_{s\to 1} 2i \frac{\partial}{\partial \tau}\left(\frac{\pi^s}{\Gamma(s)}\left(2\xi(2s)y^s + 2\xi(2s-1)y^{1-s}\right)\right)$$

$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. $\xi(2) = \pi^{-1}\cdot\frac{\pi^2}{6} = \pi/6$ $\xi(2s-1) \sim \pi^{-1/2}\Gamma(\tfrac12)\cdot\frac{1}{2(s-1)}$ at $s=1$

$\sim \frac{1}{2(s-1)}$

$2i\frac{\partial}{\partial\tau} y^s = s y^{s-1}$ $\left(\frac{\partial}{\partial \tau} = \frac12\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}\right)$

So $$B_0(y) = \lim_{s\to 1} \pi\left(\frac{\pi}{3} + \frac{1}{s-1}\cdot(1-s)y^{-s}\right) = \frac{\pi^2}{3} - \frac{\pi}{y}.$$

$k\neq 0$: $$B_k(y) = \int_0^1 G_2^*(\tau)\cdot e^{-2\pi i k x}\,dx = \int_0^1 \lim_{s\to 1}\sum{}' \frac{e^{-2\pi i k x}}{|m\tau+n|^{2s-2}(m\tau+n)^2}\,dx$$

and just as for $A_k(y,s)$, only terms $0<m|k$ contribute; sum then cvgs absolutely for $s=1$

i.e. $$B_k(y) = \int_0^1 2\sum_{\substack{1\le m|k\\ n\in\mathbb{Z}}} \frac{1}{(m\tau+n)^2}\, e^{-2\pi i k x}\,dx$$

$$2\sum_{1\le m|k}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2} = 2(2\pi i)^2\sum_{1\le m|k}\sum_{d=1}^{\infty} d q^{md}$$

$$\int_0^1 q^n e^{-2\pi i k x}\,dx = \begin{cases} e^{-2\pi k y} & n=k \\ 0 & n\neq k\end{cases}$$

$$\Rightarrow B_k(y) = \begin{cases} -8\pi^2\sum_{d|k} d\cdot e^{-2\pi k y} & k>0 \\ 0 & k<0\end{cases}$$

So $$G_2^*(\tau) = \frac{\pi^2}{3}\left(1 - 24\sum\sigma_1(n)q^n\right) - \frac{\pi}{y} = G_2(\tau) - \pi/y$$

Now $G_2^*(-1/\tau) = \tau^2 G_2^*(\tau)$, so as $G_2(\tau) = G_2^*(\tau) + \frac{\pi}{y}$,

$$G_2(-1/\tau) - \tau^2 G_2(\tau) = \pi\Big/\underbrace{\left(\frac{y}{x^2+y^2}\right)}_{\mathrm{Im}(-1/\tau)} - \tau^2\cdot\frac{\pi}{y} = \frac{\pi}{y}\left(x^2+y^2-\tau^2\right) = \frac{\pi}{y}\left(-2ixy+y^2\right)$$
$$= -2\pi i\tau.$$

Now $$G(\tau,s) = \frac{\pi}{s-1} + 2\pi\gamma - \pi\log 4y|\Delta|^{1/6} + O(s-1)$$

$$\Rightarrow G_2^*(\tau) = -2\pi i\frac{\partial}{\partial\tau}\log 4y(\Delta\bar\Delta)^{1/12} = \frac{\pi}{y} - \frac16\pi i\frac{\partial}{\partial\tau}\log\Delta \quad \left(\text{as } \frac{\partial}{\partial\tau}\bar\Delta = 0\right)$$

$$\therefore G_2(\tau) = -\frac16\pi i\frac{\partial}{\partial\tau}\log\Delta = \frac13\pi^2 q\frac{d}{dq}\log\Delta \qquad \left(\frac{\partial}{\partial\tau} = 2\pi i q\frac{\partial}{\partial q}\right)$$

i.e. $$q\frac{d}{dq}\log\Delta = \frac{3}{\pi^2}G_2 = 1 - 24\sum_{n\ge1}\sigma_1(n)q^n$$

$$\therefore \log\Delta = \log q - 24\sum_{n\ge1}\sigma_1(n)q^n/n + c = \log q - 24\sum_{d,m\ge1}\frac{q^{md}}{d} + c$$

$$= \log q + 24\sum_{m\ge1}\log(1-q^m) + c \Rightarrow \Delta = e^c\cdot q\prod_{m\ge1}(1-q^m)^{24}$$, and know leading term $= q$, so $c=0$.

Q8 Decompose the strip $\{\tau \in \mathfrak{H} \mid -\tfrac12 \le \mathrm{Re}\,\tau \le \tfrac12\}$ as

$$\{-\tfrac12 \le \mathrm{Re}\,\tau \le \tfrac12\} = \bigcup_i \gamma_i \mathcal{D}^{cl}$$

, where $\{\gamma_i\}$ is a set of coset reps. for $\Gamma_\infty \subset \Gamma$. ($\mathcal{D}^{cl}$ = closure of standard fundamental domain)

So
$$\int_{|\mathrm{Im}\,\tau| \le \frac12} F(\tau)\,\frac{d\tau\, d\bar\tau}{y^2} = \int_{\mathcal{D}} \sum_i F(\gamma_i \tau)\,\frac{d\tau\, d\bar\tau}{y^2}$$
any function F

↑ recall this is Γ-invariant

so
$$\int_{\mathcal{D}} E(\tau, s)\, f(\tau)\,\overline{g(\tau)}\, y^k\, \frac{dx\,dy}{y^2} \quad (*) \qquad E(\tau,s) = \sum_i \mathrm{Im}\,\gamma_i(\tau)^s$$

$$= \int_{|\mathrm{Im}\,\tau| \le \frac12} f(\tau)\,\overline{g(\tau)}\, y^{k+s}\, \frac{dx\,dy}{y^2}$$

$$= \int_{\substack{|x| \le \frac12 \\ 0 < y < \infty}} \sum_{m,n \ge 1} a_m \bar{b}_n\, q^m \bar{q}^n\, y^{k+s-2}\, dx\, dy$$

If $m \ne n$, the x-integral $\displaystyle\int_{-1/2}^{1/2} e^{2\pi i (m-n)x - 2\pi(m+n)y}\, dx = 0$

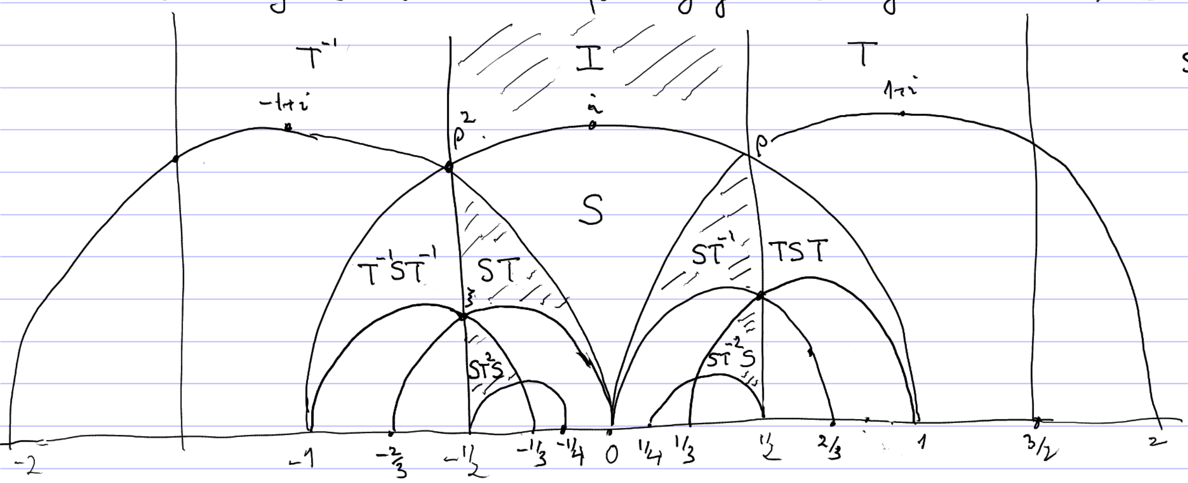
$\therefore$ just left with $\displaystyle\sum_{n \ge 1} \int_0^\infty a_n \bar{b}_n\, e^{-4\pi n y}\, y^{k+s-1}\, dy/y$

$$= \sum_{n \ge 1} \int_0^\infty a_n \bar{b}_n\, e^{-y} \left(y/4\pi n\right)^{k+s-1} \frac{dy}{y}$$

$$= (4\pi)^{1-k-s}\, \Gamma(s+k-1)\, D(f, g, k+s-1).$$

Now $(*)$ has merom. ctn. and FE (same as that for $E(\tau, s)$) so same is true for $D(f, g, s)$.

__Remark:__ why is such a decomposition possible? The following diagram shows that no Γ-translate of $\mathcal{D}$ has an interior point lying on either of the lines $\mathrm{Re}(\tau) = \pm\tfrac12$.



$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ $\quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\xi = ST(\rho) = -\frac12 + \frac{i}{2\sqrt3}$

– since union of translates of $\mathcal{D}^{cl}$ by I, ST, ST^{-1}, ST^2S, $ST^{-2}S$ contains $\{\,|\mathrm{Re}\,\tau| = \tfrac12\}$.
(There is a prettier picture – but without labelling the group elements – on the Wikipedia page "Modular group")