Modular forms (Lent 2016) — example sheet #2

1. (Double cosets). Let $G = GL_2(\mathbb{Q})^+$, $\Gamma = SL_2(\mathbb{Z})$. By a *double coset* of Γ in G we mean a subset of G of the form $\Gamma g \Gamma$ (equivalently, an orbit of G under the action of $\Gamma \times \Gamma$ given by $(\gamma', \gamma)g = \gamma' g \gamma^{-1}$.)

   Show that every double coset $\Gamma g \Gamma$ is a finite disjoint union of single cosets Γh_j. Show also that if $g \in G$ has integer entries, then
$$\Gamma g \Gamma = \Gamma \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \Gamma$$
   where $\mathbb{Z}^2/g\mathbb{Z}^2 \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Write down the decomposition into single cosets when $\det(g) = p$ is prime.

   Let $\Gamma g \Gamma = \bigcup \Gamma h_j$ be a double coset as above, and $f \in M_k$. Define $f|_k[\Gamma g \Gamma] = \sum_j f|_k h_j$. Show that $f|_k[\Gamma g \Gamma]$ belongs to M_k, and that it depends only on the double coset; and that if g has integer entries and determinant p, then $f|_k[\Gamma g \Gamma]$ is a constant multiple of $T_p f$.

2. Say that a lattice $\Lambda \subset \mathbb{R}^k$ is *even* if $\|x\|^2 \in 2\mathbb{Z}$ for every $x \in \Lambda$, and that Λ is *self-dual* if $\Lambda = \Lambda'$. Show that if $\Lambda \subset \mathbb{R}^k$ is an even self-dual lattice, then the theta series $\Theta_\Lambda(\tau)$ is a modular form of weight $k/2$.

3. Let k be a positive integer divisible by 4. Let Λ be the set of all $x = (x_1 \dots, x_k) \in \mathbb{R}^k$ satisfying
$$2x_i \in \mathbb{Z}, \quad (x_i - x_j) \in \mathbb{Z}, \quad \frac{1}{2}\sum_{i=1}^{k} x_i \in \mathbb{Z}.$$
   Show that Λ is a lattice, and that $\|x\|^2$ is always an integer for $x \in \Lambda$. [Λ is usually denoted E_k.]

   (ii) Suppose further that k is divisible by 8. Show that Λ is even.

   (iii) Finally let $k = 8$. Show that Λ is self-dual, and that $\Theta_\Lambda(\tau) = E_4(\tau)$. Hence (or directly) show that there are exactly 240 elements $x \in \Lambda$ with $\|x\|^2 = 2$.

4. (i) Show that $\zeta(s) = 1/(s-1) + \gamma + O(s-1)$ at $s = 1$. (Hint: write down a real integral which approximates the sum defining $\zeta(s)$.) Here γ is the *Euler–Mascheroni constant*
$$\gamma = \lim_{n\to\infty}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Big).$$
   (ii) Show that the functional equation for $\zeta(s)$ can be rewritten as
$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$
   (You will need to look up some identities for the Gamma function.)

   (iii) Use this, and the Weierstrass product for $\Gamma(s)$, to show that $\zeta'(0) = -\frac{1}{2}\log 2\pi$.

5. Show that at $s = 1$ the Kronecker Limit Formula takes the form
$$G(\tau, s) = \frac{\pi}{s-1} + (2\pi\gamma - \pi \log 4y\, |\Delta(\tau)|^{1/6}) + O(s-1).$$

6. Show that
$$G_2^*(\tau) := \lim_{s\to 1}(2i\partial/\partial\tau)G(\tau, s) = \lim_{s\to 1+} \sum_{(m,n)\neq(0,0)}{}' \frac{1}{(m\tau+n)^2\,|m\tau+n|^{2s-2}}$$
   is well-defined, and is modular of weight 2.

...continued overleaf

7. Show that $G_2^*(\tau)$ has Fourier expansion

$$\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n - \frac{\pi}{y}$$

and deduce that (cf. Sheet 1, Q.9)

$$G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi i \tau.$$

Use this and the result of the previous questions to prove that

$$\frac{1}{2\pi i} \frac{d}{d\tau} \log \Delta(\tau) = q \frac{d}{dq} \log \Delta(\tau) = \frac{3}{\pi^2} G_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1 q^n.$$

Hence show that

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

8. (Rankin-Selberg integral) Let f, $g \in S_k(SL_2(\mathbb{Z}))$, with q-expansions $\sum a_n q^n$ and $\sum b_n q^n$ respectively. By writing the strip $\{x + iy \in \mathcal{H} \mid -1/2 < x \leq 1/2\}$ as a union of translates of the fundamental domain $\mathcal{D}$, show that

$$\begin{aligned} &\int_{\mathcal{D}} E(\tau, s) f(\tau) \overline{g(\tau)} y^k \, \frac{dx\, dy}{y^2} \\ &= \int_{\substack{-1/2 \leq x \leq 1/2 \\ 0 < y < \infty}} f(\tau) \overline{g(\tau)} y^{k+s-2} \, dx\, dy \\ &= (4\pi)^{1-k-s} \Gamma(s+k-1) \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^{s+k-1}} \end{aligned}$$

Deduce that the Dirichlet series

$$D(f, g, s) = \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^s}$$

has a meromorphic continuation to the s-plane and satisfies a functional equation.