Chapter VII

Modular Forms

§1. *The modular group*

1.1. *Definitions*

Let H denote the upper half plane of $\mathbf{C}$, i.e. the set of complex numbers z whose imaginary part $Im(z)$ is >0.

Let $\mathbf{SL}_2(\mathbf{R})$ be the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with real coefficients, such that $ad-bc = 1$. We make $\mathbf{SL}_2(\mathbf{R})$ act on $\tilde{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ in the following way:

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\mathbf{SL}_2(\mathbf{R})$, and if $z \in \tilde{\mathbf{C}}$, we put

$$gz = \frac{az+b}{cz+d}.$$

One checks easily the formula

$$Im(gz) = \frac{Im(z)}{|cz+d|^2}. \tag{1}$$

This shows that H is *stable* under the action of $\mathbf{SL}_2(\mathbf{R})$. Note that the element $-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\mathbf{SL}_2(\mathbf{R})$ acts trivially on H. We can then consider that it is the group $\mathbf{PSL}_2(\mathbf{R}) = \mathbf{SL}_2(\mathbf{R})/\{\pm 1\}$ which operates (and this group acts *faithfully*—one can even show that it is the group of all analytic automorphisms of H).

Let $\mathbf{SL}_2(\mathbf{Z})$ be the subgroup of $\mathbf{SL}_2(\mathbf{R})$ consisting of the matrices with coefficients in $\mathbf{Z}$. It is a discrete subgroup of $\mathbf{SL}_2(\mathbf{R})$.

Definition 1.—*The group* $G = \mathbf{SL}_2(\mathbf{Z})/\{\pm 1\}$ *is called the modular group; it is the image of* $\mathbf{SL}_2(\mathbf{Z})$ *in* $\mathbf{PSL}_2(\mathbf{R})$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\mathbf{SL}_2(\mathbf{Z})$, we often use the same symbol to denote its image in the modular group G.

1.2. *Fundamental domain of the modular group*

Let S and T be the elements of G defined respectively by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. One has:

$$Sz = -1/z, \qquad Tz = z+1$$
$$S^2 = 1, \qquad (ST)^3 = 1$$

On the other hand, let D be the subset of H formed of all points z such that $|z| \geqq 1$ and $|Re(z)| \leqq 1/2$. The figure below represents the transforms of D by the elements:

$\{1, T, TS, ST^{-1}S, S, ST, STS, T^{-1}S, T^{-1}\}$ of the group G.

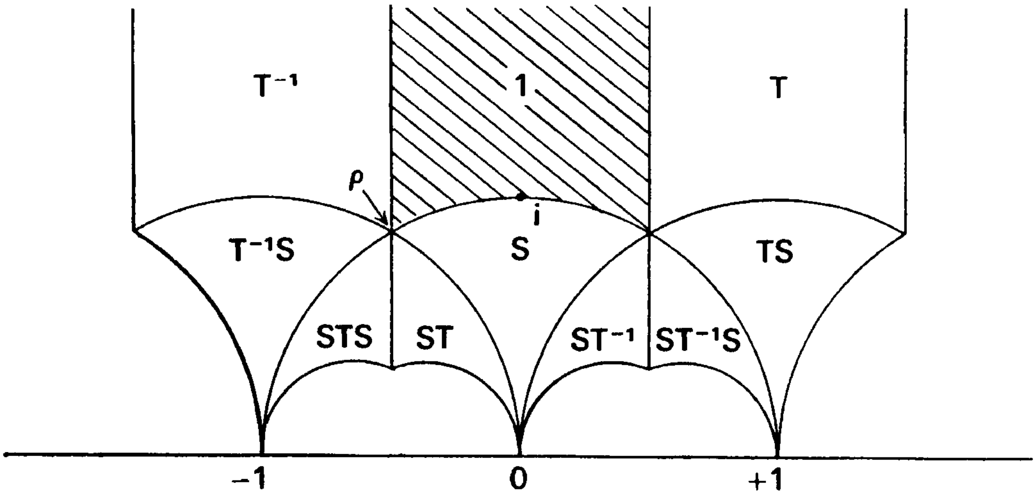

Fig. 1

We will show that D is a *fundamental domain* for the action of G on the half plane H. More precisely:

Theorem 1.—(1) *For every $z \in H$, there exists $g \in G$ such that $gz \in D$.*

(2) *Suppose that two distinct points z, z' of D are congruent modulo G. Then, $R(z) = \pm\frac{1}{2}$ and $z = z' \pm 1$, or $|z| = 1$ and $z' = -1/z$.*

(3) *Let $z \in D$ and let $I(z) = \{g | g \in G,\ gz = z\}$ the stabilizer of z in G. One has $I(z) = \{1\}$ except in the following three cases:*

$z = i$, in which case $I(z)$ is the group of order 2 generated by S;

$z = \rho = e^{2\pi i/3}$, in which case $I(z)$ is the group of order 3 generated by ST;

$z = -\bar{\rho} = e^{\pi i/3}$, in which case $I(z)$ is the group of order 3 generated by TS.

Assertions (1) and (2) imply:

Corollary.—*The canonical map $D \to H/G$ is surjective and its restriction to the interior of D is injective.*

Theorem 2.—*The group G is generated by S and T.*

Proof of theorems 1 and 2.—*Let G'* be the subgroup of G generated by S and T, and let $z \in H$. We are going to show that *there exists $g' \in G'$ such that $g'z \in D$*, and this will prove assertion (1) of theorem 1. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of G', then

$$(1) \qquad Im(gz) = \frac{Im(z)}{|cz+d|^2}.$$

Since c and d are integers, the numbers of pairs (c, d) such that $|cz+d|$ is less than a given number is *finite*. This shows that there exists $g \in G'$ such that $Im(gz)$ is maximum. Choose now an integer n such that $T^n gz$ has real part between $-\frac{1}{2}$ and $+\frac{1}{2}$. The element $z' = T^n gz$ *belongs to* D; indeed, it suffices to see that $|z'| \geqq 1$, but if $|z'| < 1$, the element $-1/z'$ would have an imaginary part strictly larger than $Im(z')$, which is impossible. Thus the element $g' = T^n g$ has the desired property.

We now prove assertions (2) and (3) of theorem 1. Let $z \in D$ and let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ such that $gz \in D$. Being free to replace (z, g) by (gz, g^{-1}), we may suppose that $Im(gz) \geqq Im(z)$, i.e. that $|cz+d|$ is $\leqq 1$. This is clearly impossible if $|c| \geqq 2$, leaving then the cases $c = 0, 1, -1$. If $c = 0$, we have $d = \pm 1$ and g is the translation by $\pm b$. Since $R(z)$ and $R(gz)$ are both between $-\frac{1}{2}$ and $\frac{1}{2}$, this implies either $b = 0$ and $g = 1$ or $b = \pm 1$ in which case one of the numbers $R(z)$ and $R(gz)$ must be equal to $-\frac{1}{2}$ and the other to $\frac{1}{2}$. If $c = 1$, the fact that $|z+d|$ is $\leqq 1$ implies $d = 0$ except if $z = \rho$ (resp. $-\bar{\rho}$) in which case we can have $d = 0, 1$ (resp, $d = 0, -1$). The case $d = 0$ gives $|z| \leq 1$ hence $|z| = 1$; on the other hand, $ad-bc = 1$ implies $b = -1$, hence $gz = a-1/z$ and the first part of the discussion proves that $a = 0$ except if $R(z) = \pm\frac{1}{2}$, i.e. if $z = \rho$ or $-\bar{\rho}$ in which case we have $a = 0, -1$ or $a = 0, 1$. The case $z = \rho$, $d = 1$ gives $a-b = 1$ and $g\rho = a-1/(1+\rho) = a+\rho$, hence $a = 0, 1$; we argue similarly when $z = -\bar{\rho}$, $d = -1$. Finally the case $c = -1$ leads to the case $c = 1$ by changing the signs of a, b, c, d (which does not change g, viewed as an element of G). This completes the verification of assertions (2) and (3).

It remains to prove that $G' = G$. Let g be an element of G. Choose a point z_0 *interior* to D (for example $z_0 = 2i$), and let $z = gz_0$. We have seen above that there exists $g' \in G'$ such that $g'z \in D$. The points z_0 and $g'z = g'gz_0$ of D are congruent modulo G, and one of them is interior to D. By (2) and (3), it follows that these points coincide and that $g'g = 1$. Hence we have $g \in G'$, which completes the proof.

Remark.—One can show that $\langle S, T; S^2, (ST)^3 \rangle$ is a *presentation* of G, or, equivalently, that G is the *free product of the cyclic group of order* 2 *generated by* S *and the cyclic group of order* 3 *generated by* ST.

§2. *Modular functions*

2.1. *Definitions*

Definition 2.—*Let k be an integer. We say a function f is weakly modular of weight $2k$*[1] *iff f is meromorphic on the half plane H and verifies the relation*

$$(2) \qquad f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \textit{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}).$$

[1] Some authors say that f is "of weight $-2k$", others that f is "of weight k".