Modular forms (Lent 2011) — example sheet #2

Unless otherwise stated, Γ is a subgroup of finite index in $SL_2(\mathbb{Z})$.

1. (i) Show that Γ has a fundamental domain which is a *connected* union of translates of the standard fundamental domain for $SL_2(\mathbb{Z})$.

   (ii) Show that if Γ has no elements of finte order other than ± 1 then Γ is a free group. (Use the fact that any group acting without fixed points on a tree is free.)

2. (i) Show that $\Gamma(N)$ is torsionfree if $N \geq 3$ and that the only elements of finite order of $\Gamma(2)$ are $\{\pm 1\}$.

   (ii) Show that if $N \geq 4$ then $\Gamma_1(N)$ is torsionfree.

3. Let
$$\Gamma^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

   (i) Show that if $f \in M_k(\Gamma)$, then the function $f^*(z) = \overline{f(-\bar{z})}$ belongs to $M_k(\Gamma^*)$.

   (ii) Show that if $\Gamma = \Gamma^*$ (for example, any one of $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$) then $M_k(\Gamma)$ has a basis all of whose elements have real Fourier coefficients.

4. (i) Show that if every cusp of Γ has width one then Γ must be $\Gamma(1)$.

   (ii) (Somewhat hard, but useful to know!) Show that if Γ is a congruence subgroup containing -1, then $\Gamma \supset \Gamma(N)$ where N is the least common multiple of the widths of the cusps of Γ. (This gives a way to tell whether or not a given group is a congruence subgroup.)

   (iii) Assuming (ii), write down an infinite family of subgroups of $\Gamma_0(11)$ which are not congruence subgroups. (Recall that the compact Riemann surface $\widehat{\Gamma_0(11)\backslash\mathcal{H}}$ has genus 1.)

5. Show that a holomorphic function $f \colon \mathcal{H} \to \mathbb{C}$ belongs to $S_k(\Gamma)$ if and only if it is Γ-modular of weight k and the function $y^{k/2}\,|f(z)|$ is bounded for $z = x + iy \in \mathcal{H}$.

6. (i) If $N = DM$ with $(D, M) = 1$ define
$$W_D = \begin{pmatrix} Da & b \\ Nc & Dd \end{pmatrix}, \qquad adD - bcM = 1.$$

   If $f \in S_k(N)$ show that $f|_k W_D$ is independent of the choice of $a \ldots d$ and belongs to $S_k(N)$. Show that if $(D, E) = 1$ then $f|_k W_D W_E = f|_k W_{DE}$. Deduce that the operators W_p for $p|N$ commute and that their product is W_N.

   (ii) Prove that W_p commutes with T_ℓ (where $(\ell, N) = 1$) and with U_ℓ (where $\ell | N$ and $\ell \neq p$).

7. (i) Let p be prime and $r \geq 1$. Let $g \in S_k(N)$ be a nonzero cusp form. If$p \nmid N$ (respectively $p|N$) assume that g is an eigenfunction of T_p (resp. of both U_p and W_p). Consider the oldforms
$$f_i = g\Big|_k \begin{pmatrix} p^i & 0 \\ 0 & 1 \end{pmatrix} \in S_k(p^r N), \qquad 0 \leq i \leq r.$$

   Compute the action of the operators W_p and U_p on the space spanned by $\{f_i\}$. (Treat separately the cases when p does or does not divide N). Show that these spaces are simple modules for the action of the algebra generated by U_p and W_p.

   (ii) Let $\mathcal{H}_k(N) \subset \operatorname{End}_{\mathbb{C}} S_k(N)$ be the subalgebra generated over $\mathbb{C}$ by all T_p, U_p and W_p operators. Show that S_k is a direct sum of pairwise non-isomorphic simple $\mathcal{H}_k(N)$-modules, one corresponding to each newform in $S_k^{\text{new}}(M)$ for $M|N$. (This is easier if you use strong multiplicity one, but it is not strictly necessary.)

   (iii)* Show that $\mathcal{H}_k(N)$ equals the (a priori larger) subalgebra of $\operatorname{End}_{\mathbb{C}} S_k(N)$ generated by all the double coset operators $[\Gamma_0(N)\gamma\Gamma_0(N)]$, $\gamma \in GL_2^+(\mathbb{Q})$. (Assume strong multiplicity one.)

8. Assume $-1 \in \Gamma$. Write $\Gamma_\infty \subset \Gamma$ for the stabiliser of the cusp ∞. Show that $j(\gamma, z)^2$ depends only on the coset $\Gamma_\infty\gamma$, and that if $k > 2$ is even, the series

$$E_{\Gamma,k}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k}$$

is convergent. Show that $E_{\Gamma,k}$ is an element of $M_k(\Gamma)$ which is not a cusp form. Identify $E_{k,\Gamma}$ in the case $\Gamma = \Gamma(1)$.

9. Show that the constant term of $E_{k,\Gamma}$ at the cusp $x \in \mathbb{P}^1(\mathbb{Q})$ vanishes if and only if x is not Γ-equivalent to ∞. By considering modular forms of the shape $E_{k,\Gamma'}|_k\gamma$, deduce that for every even $k > 2$ one has

$$\dim M_k(\Gamma) - \dim S_k(\Gamma) = \nu_\infty.$$

10. (i) Let $\chi\colon (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ be a primitive Dirichlet character, and $f \in S_k(N)$ with $(D, N) = 1$. Show that $f_\chi = \sum \chi(n) a_n(f) q^n$ belongs to $S_k(\Gamma_1(ND^2))$.
(ii) Show that if $(D, N) \neq 1$ it can happen that $f \neq 0 = f_\chi$.

11. The cusp form Δ is known to have an infinite product expansion

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

Assuming this, let $E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$. Show that

$$E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i} \qquad (1)$$

Deduce that $E_2^*(\tau) = E_2(\tau) - \dfrac{3}{\pi y}$ is modular of weight 2.
(The infinite product for Δ follows from classical formulae in the theory of theta-functions. Conversely, any proof of (1) can be used to give a proof of the product formula for Δ — see for example the one is Serre's book. From a modern standpoint, the most natural proof of (1) uses the analytic continuation of the function $E(z, s)$ in the next question.)

12. (i) (Real analytic Eisenstein series). With the same notation as Q.8, define for $\mathrm{Re}(s) > 1$

$$E_\Gamma(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma(z))^s$$

Show that the series converges and represents a function on $\mathcal{H}$ which is invariant under Γ.
(ii) (Rankin-Selberg integral) Assume that ∞ is a cusp of width 1. Let f, $g \in S_k(\Gamma)$, with q-expansions $\sum a_n q^n$ and $\sum b_n q^n$ respectively. By writing the strip $\{x + iy \in \mathcal{H} \mid -1/2 < x \leq 1/2\}$ as a union of translates of the fundamental domain for Γ, show that

$$\begin{aligned}\int_{\Gamma\backslash\mathcal{H}} & E_\Gamma(z, s) f(z) \overline{g(z)} y^k \, \frac{dx\,dy}{y^2} \\ &= \int\limits_{\substack{-1/2 \leq x \leq 1/2 \\ 0 < y < \infty}} f(z)\overline{g(z)} y^{k+s-2} \, dx\,dy \\ &= (4\pi)^{1-k-s} \Gamma(s + k - 1) \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^{s+k-1}}\end{aligned}$$