Complex analysis IB 2007 — lecture notes

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These are the notes I used to give the course — the lectures may have deviated from these in a few places (in particular, there may be corrections I made in the course which haven't made it into these notes).

1 Basic notions

1.1 Introduction

Course builds on notions from real analysis. Particularly important: uniform convergence. Also will use various notions from metric spaces at times (mainly to do with compactness). If you haven't done the metric space course yet, you'll have to take some things on trust and fill in the gaps next term.

1.2 Complex differentiation

Recall notions:

- $D(a, r)$ = open ball (disc) of radius $r > 0$, centre $a \in \mathbb{C}$.
- An open set in $\mathbb{C}$ is a subset $U \subset \mathbb{C}$ such that, for every $a \in U$, there exists $\epsilon > 0$ such that $D(a, r) \subset U$
- Curve is a continuous map from a closed interval $\gamma\colon [a, b] \to \mathbb{C}$. It is continuously differentiable (or C^1) if γ' exists and is continuous on $[a, b]$ (at endpoints a, b this means one-sided derivative).
- An open set $U \subset \mathbb{C}$ is *path-connected* if for every z, $w \in U$ there exists a curve $\gamma\colon [0, 1] \to U$ with endpoints z, w. If such a γ exists then one can find another curve in U with the same endpoints which is polygonal (a finite sequence of line seqments).

Definition. A *domain* is a non-empty path-connected open subset of $\mathbb{C}$.

This course is for the most part about complex-valued functions $f\colon U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is an open subset or domain. Given such a function f we may write $f(x + iy) = u(x, y) + iv(x, y)$ where $u, v\colon U \to \mathbb{R}$ are the real and imaginary parts of f (we identify U with a subset of $\mathbb{R}^2$ via $\mathbb{C} \simeq \mathbb{R}^2$).

[1]Comments and corrections to `a.j.scholl@dpmms.cam.ac.uk`

Definition. (i) $f\colon U \to \mathbb{C}$ is *differentiable* at $w \in U$ if the limit

$$f'(w) := \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists (the derivative of f at w).
(ii) $f\colon U \to \mathbb{C}$ is *holomorphic*[2] at $w \in U$ if there exists $r > 0$ such that f is differentiable at all points of $D(w, r)$. f is *holomorphic on* U if it is differentiable at all $w \in U$ (this is equivalent to f being holomorphic at all $w \in U$).

Complex differentiation satisfies the same formal rules (for derivatives of sum, product, quotient, chain rule, and inverse functions) as differentiation of functions of one real variable (and the proofs are identical).

Definition. An *entire function* is a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$.

Example: polynomials.
If $p(z)$, $q(z)$ are polynomials, with q not identically zero, then p/q is a holomorphic function on the complement in $\mathbb{C}$ of the zero-set of q.
Let's compare this with differentiability for functions of 2 variables. Recall that if $U \subset \mathbb{R}^2$ is open and $u\colon U \to \mathbb{R}$ then u is said to be differentiable at $(c, d) \in U$ if there exists $(\lambda, \mu) \in \mathbb{R}^2$ such that

$$\frac{u(x, y) - u(c, d) - (\lambda(x - c) + \mu(y - d))}{\sqrt{(x - c)^2 + (y - d)^2}} \to 0 \quad \text{as } (x, y) \to (c, d)$$

and then $Du(c, d) = (\lambda, \mu)$ is the derivative of u at (c, d). If this holds then $\lambda = u_x(c, d)$ and $\mu = u_y(c, d)$ are equal to the partial derivatives of u at (c, d).

Theorem 1.2.1 (Cauchy-Riemann equations)**.** *$f\colon U \to \mathbb{C}$ is differentiable at $w = c + id \in U$ iff the functions u, v are differentiable at (c, d) and*

$$u_x(c, d) = v_y(c, d), \quad u_y(c, d) = -v_x(c, d). \tag{1}$$

If this holds then $f'(w) = u_x(c, d) + iv_x(c, d)$.

Proof. From the definition, f will be differentiable at w with derivative $f'(w) = p + iq$ if and only if

$$\lim_{z \to w} \frac{f(z) - f(w) - f'(w)(z - w)}{|z - w|} = 0$$

[2] Some old (and not-so-old) texts use the term *regular*. The term *analytic* is also commonly employed — see Remark 2.5 below.

or equivalently, splitting into real and imaginary parts, if and only if

$$\lim_{(x,y)\to(c,d)} \frac{u(x,y) - u(c,d) - (p(x-c) - q(y-d))}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

and

$$\lim_{(x,y)\to(c,d)} \frac{v(x,y) - v(c,d) - (q(x-c) + p(y-d))}{\sqrt{(x-c)^2 + (y-d)^2}} = 0$$

since

$$f'(w)(z-w) = (p(x-c) - q(y-d)) + i(q(x-c) + p(y-d)).$$

So f is differentiable at w with derivative $f'(w) = p + iq$ if and only if u, v are differentiable at (c,d) with $Du(c,d) = (p,-q)$ and $Dv(c,d) = (q,p)$, whence the result. □

Remarks. (i) For example, applying to the function $f(z) = \bar{z}$, so that $u(x,y) = x$, $v(x,y) - y$, we see that $u_x = 1$, $v_y = -1$, and so $f(z)$ is nowhere complex differentiable.

(ii) If one just wants to show that the differentiability of f at w implies that the partial derivatives exist and satisfy (1), one can proceed more simply: Let h be real, and first letting $z = w + h$, we have

$$\begin{aligned} f'(w) &= \lim_{h\to 0} \frac{f(w+h) - f(w)}{h} \\ &= \lim_{h\to 0} \frac{u(c+h,d) - u(c,d)}{h} + i\frac{v(c+h,d) - v(c,d)}{h} \\ &= u_x(c,d) + iv_x(c,d). \end{aligned}$$

Next letting $z = w + ih$, we get

$$\begin{aligned} f'(w) &= \lim_{h\to 0} \frac{f(w+ih) - f(w)}{ih} \\ &= \lim_{h\to 0} \frac{v(c,d+h) - v(c,d)}{h} - i\frac{u(c,d+h) - u(c,d)}{h} \\ &= u_x(c,d) + iv_x(c,d). \end{aligned}$$

(iii) Leter we'll see that if f is holomorphic then so is f'. This being so, it follows that all the higher partial derivatives of u and v exists, and we may differentiate the Cauchy-Riemann equations again to get

$$\partial^2 u/\partial x^2 = \partial^2 v/\partial y \partial x, \qquad \partial^2 u/\partial y^2 = -\partial^2 v/\partial x \partial y,$$

and so (using the fact that $\partial^2 v/\partial x \partial y = \partial^2 v/\partial y \partial x$, since the 2nd partial derivatives are continuous)

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \tag{2}$$

which is *Laplace's equation* (we also say that u is a *harmonic function*). Similarly, v also satisfies Laplace's equation, in other words

> *The real and imaginary parts of a holomorphic function are harmonic functions.*

Corollary 1.2.2. *Let $f = u + iv\colon U \to \mathbb{C}$. Suppose the functions u, v have continuous partial derivatives everywhere on U and that they satisfy the Cauchy-Riemann equations* (1)*. Then f is holomorphic on U.*

Proof. Since the partial derivatives are continuous on U, u and v are differentiable on U (Analysis II). The result follows by 1.2.1. □

Remark. Later we shall show that the converse of Corollary 1.2.2 is true. In fact, if $f\colon U \to \mathbb{C}$ is holomorphic then Corollary 2.5.2 will show that its derivative is also holomorphic, hence in particular that the partial derivatives of u, v are continuous.

Corollary 1.2.3. *Let $f\colon D \to \mathbb{C}$ be holomorphic on a domain D, and suppose that $f'(z) = 0$ for all $z \in D$. Then f is constant on D.*

Proof. Follows from the analogous result for differentiable functions on a path-connected subset of $\mathbb{R}^2$. □

1.3 Power series

Recall:

Theorem 1.3.1 (Radius of convergence)**.** *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence*[3] *of complex numbers. Then there exists a unique $R \in [0, \infty]$, the* radius of convergence *of the series, such that the series*

$$\sum_{n=0}^{\infty} c_n(z-a)^n, \quad z, a \in \mathbb{C}$$

converges absolutely if $|z-a| < R$ and diverges if $|z-a| > R$. If $0 < r < R$ then the series converges uniformly on $\{|z-a| \le r\}$. The radius of convergence is given by

$$R = \sup\{r \ge 0 \mid |c_n|\, r^n \to 0\}.$$

Theorem 1.3.2. *Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ be a complex power series*[4] *with radius of convergence $R > 0$. Then:*

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[3]For me, $\mathbb{N} = \{0, 1, 2, \dots\}$.

[4]If one is pedantic one should write "let $\sum \dots$ be a power series with radius of convergence $R > 0$, and let $f\colon D(a, R) \to \mathbb{C}$ be the function it represents" .

(i) f is holomorphic on $D(a,R)$;

(ii) its derivative is given by the series

$$\sum_{n=1}^{\infty} nc_n(z-a)^{n-1},$$

which also has radius of convergence R;

(iii) f has derivatives of all orders on $D(a,R)$, and $f^{(n)}(a) = n!c_n$.

(iv) If f vanishes identically on some disc $D(a,\epsilon)$ then $c_n = 0$ for every n.

Proof. We can assume, making a change of variables, that $a = 0$.

First we show that the derived series has radius of convergence R. Since $|nc_n| \geq |c_n|$ its radius of convergence can be no greater than R. And if $|z| < R_1 < R$ then the derived series converges by comparison with $\sum |c_n| R_1^{n-1}$, since

$$\frac{|n|\, c_n z^{n-1}}{|c_n|\, R_1^{n-1}} = n\left(\frac{|z|}{R_1}\right)^{n-1} \to 0.$$

Next consider the series, for $|z|$, $|w| < R$

$$\sum_{n=1}^{\infty} c_n \Big(\sum_{j=0}^{n-1} z^j w^{n-1-j}\Big) \tag{3}$$

I claim that for every $\rho < R$ this series converges uniformly on the set $\{(z,w) \mid |z|, |w| \leq \rho\}$. In fact for the n-th term we have the bound

$$\Big|c_n \Big(\sum_{j=0}^{n-1} z^j w^{n-1-j}\Big)\Big| \leq n\,|c_n|\,\rho^{n-1} = M_n, \quad \text{say,}$$

and $\sum M_n$ converges since $\rho < R$, so by the Weierstrass M-test, the series converges uniformly. Hence the series converges on $\{(z,w) \mid |z|, |w| < R\}$ to a continuous function $g(z,w)$.

Next, if $z \neq w$ we can rewrite (3) as

$$g(z,w) = \sum_{n=1}^{\infty} c_n \frac{z^n - w^n}{z-w} = \frac{f(z) - f(w)}{z-w}$$

whereas if $z = w$ it reduces to

$$g(w,w) = \sum_{n=1}^{\infty} nc_n w^{n-1}.$$

So since g is continuous, fixing w and letting $z \to w$ we get

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w} = \sum_{n=1}^{\infty} n c_n w^{n-1}$$

so $f'(w)$ exists and equals $g(w, w)$, as required. This proves (i) and (ii). Then (iii) follows by induction on n. Finally, if f vanishes identically on a disc about $z = a$ then $f^{(n)}(a) = 0$ for all n, so by (iii) all the c_n are zero. □

Remark. We shall use the continuity of $g(z, w)$ later on (in Theorem 3.2.1).

Definition. The *complex exponential* function is defined by

$$e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Proposition 1.3.3. *(i)* $\exp(z)$ *is an entire function, and* $(d/dz) \exp(z) = \exp(z)$.
(ii) For all $z, w \in \mathbb{C}$, $\exp(z + w) = \exp(z) \exp(w)$ *and* $\exp(z) \neq 0$.
(iii) If $z = x + iy$ *then* $e^z = e^x(\cos y + i \sin y)$.
(iv) $\exp(z) = 1$ *if any only if* $z \in 2\pi i$.
(v) If $w \in \mathbb{C}$ *then there exists* $z \in \mathbb{C}$ *with* $\exp(z) = w$ *iff* $w \neq 0$.

Proof. (i) RoC=∞. Differentiate term-by-term.
(ii) Obviously $e^0 = 1$. Let $w \in \mathbb{C}$. Define $F(z) = \exp(z + w) \exp(-z)$. Then $F'(z) = 0$, so F is constant, hence $F(z) = F(0) = \exp(w)$. Taking $w = -z$ gives $\exp(z) \exp(-z) = 1$ so $\exp(z) \neq 0$.
(iii) $e^z = e^x e^{iy}$ by (ii), then use Maclaurin series for sin, cos.
(iv), (v) Follows from (iii). □

We now turn to the logarithm function. By definition, if $z \in \mathbb{C}$ we say that $w \in \mathbb{C}$ is a *logarithm* of z if $\exp(w) = z$. From Proposition 1.3.3(v), z has a logarithm iff $z \neq 0$; and by (iv) if $z \neq 0$ then z has an infinite number of logarithms, differing from one another by integer multiples of $2\pi i$.

Unlike for real numbers, there is no preferred logarithm of a given complex number; both πi and $-\pi i$ are logarithms of -1 and there is no mathematical reason to choose one over the other.

Definition. Let $U \subset \mathbb{C} \setminus \{0\}$ be an open set. We say that continuous function $\lambda \colon U \to \mathbb{C}$ is a *branch of the logarithm* if for exery $z \in U$, $\lambda(z)$ is a logarithm of z — equivalently, if $\exp(\lambda(z)) = z$.

Will see later that any branch of the logarithm is in fact automatically holomorphic. The following is often a useful choice:

Definition. Let $U = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$. The *principal branch of the logarithm* is the function $\operatorname{Log}\colon U \to \mathbb{C}$ given by

$$\operatorname{Log}(z) = \ln |z| + i \arg(z)$$

where $\arg(z)$ is the unique argument of z in the range $(-\pi, \pi)$.

As the name suggests, $\operatorname{Log}(z)$ is indeed a branch of the logarithm. In fact, Log is continuous on U, since the function $z \mapsto \arg(z)$ is continuous on U,[5] and for any $z \in U$,

$$\exp \operatorname{Log}(z) = e^{\ln|z|}(\cos \arg(z) + i \sin \arg(z)) = z.$$

Proposition 1.3.4. *(i)* log *is holomorphic on* U, *with derivative* $1/z$.
(ii) If $|z| < 1$ *then*

$$\log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}.$$

Proof. (i) Inverse of chain rule.
(ii) The power series has radius of convergence 1. Differentiating both sides, see that the difference is a constant, then put $z = 0$. □

Notice that there is no way to extend $\operatorname{Log}(z)$ to a holomorphic function on $\mathbb{C} \setminus \{0\}$, since

$$\lim_{\theta \to \pi-} \operatorname{Log}(e^{i\theta}) = \pi \quad \text{but} \quad \lim_{\theta \to \pi+} \operatorname{Log}(e^{i\theta}) = \lim_{\theta \to \pi+} \theta - 2\pi = -\pi$$

Later (Theorem 2.1.4) we'll see that there is no branch of the logarithm defined on $\mathbb{C} - \{0\}$.
Fractional/complex powers: $z^\alpha = \exp(\alpha \operatorname{Log} z)$.

$$d/dz(z^\alpha) = \alpha z^{\alpha - 1}, \quad \text{but} \quad (zw)^\alpha \neq z^\alpha w^\alpha \quad \text{in general.}$$

1.4 Conformal maps

Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open subset $U \subset \mathbb{C}$. Let $w \in U$ and suppose that $f'(w) \neq 0$. We investigate the properties of the mapping determined by f in a neighbourhood of w. For this consider a simple C^1-curve through w

$$\gamma\colon [-1, 1] \to U, \quad \gamma(0) = w$$

[5] Projection onto the unit circle $z \mapsto z/|z|$ is a continuous map $\mathbb{C} \setminus \{0\} \to \{|z| = 1\}$, which maps U to $\{|z| = 1\} \setminus \{-1\}$; and $\theta \mapsto e^{i\theta}$ is a homeomorphism from $(-\pi, \pi)$ to $\{|z| = 1\} \setminus \{-1\}$.

satisfying $\gamma'(0) \neq 0$. Writing $\gamma(t) = w + r(t)e^{i\theta(t)}$, one sees that $\arg(\gamma'(0)) = \theta(0)$ is the angle that γ makes with a line through w parallel to the real axis. Consider the image δ of γ by f, so that $\delta(t) = f(\gamma(t))$. Then

$$\delta'(t) = \gamma'(t)f'(\gamma(t)), \quad \arg\delta'(0) = \arg\gamma'(0) + \arg f'(w) + 2\pi n$$

for some $n \in \mathbb{Z}$, since $f'(w) \neq 0$. So the direction of δ at $f(w)$ is the direction of γ at w, rotated by a constant angle $\arg f'(w)$. In other words, the mapping f preserves angles at the point w. We say that f is *conformal* at w is this is the case.

A special but impostant case is when $f\colon D \to \mathbb{C}$ is holomorphic on a domain D with $f' \neq 0$, and is 1-to-1, to that $f\colon D \xrightarrow{\sim} f(D)$. In this case we say that f is a *conformal equivalence* between D and $f(D)$ — sometimes one just says that f is a *conformal mapping.*

An important example of conformal equivalence is given by the Moebius map

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad = bc \neq 0$$

which is a conformal equivalence from the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to itself. Other examples are given (for $n \geq 1$) by $z \mapsto z^n$, which is a conformal equivalence between $\{z \in \mathbb{C}\backslash\{0\} \mid 0 < \arg z < \pi/n\}$ and the upper half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, with inverse mapping the principal branch of $z^{1/n}$; and by the exponential

$$\exp\colon \{z \in \mathbb{C} \mid -\pi < \arg(z) < \pi\} \to \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$$

with inverse $\mathrm{Log}(z)$, the principal branch of the logarithm. Using a combination of functions of this kind one can construct many non-trivial examples of conformal mappings. Underpinning all of these is the *Riemann Mapping Theorem*, which implies in particular the following:

> *Let $D \subset \mathbb{C}$ be any domain bounded by a simple closed curve. Then there exists a conformal equivalence $D \xrightarrow{\sim} D(0,1)$ between D and the open unit disc.*

2 Complex integration I

2.1 Integrals along curves

Familiarity with basic notions of integration of real-valued functions of a real variable is assumed. If $f\colon [a,b] \to \mathbb{C}$ is complex function which is (say) continuous, then its integral is defined as

$$\int_a^b f(t)\,dt = \int_a^b \operatorname{Re}(f(t))\,dt + i\int_a^b \operatorname{Im}(f(t))\,dt.$$

The following estimate is basic:

Proposition 2.1.1.

$$\left|\int_a^b f(t)\,dt\right| \leq (b-a)\sup_{a\leq t\leq b}|f(t)| \tag{4}$$

with equality if and only if f is constant.

Proof. Let $\theta = \arg\left(\int_a^b f(t)\,dt\right)$, and set $M = \sup_{a\leq t\leq b}|f(t)|$. Then

$$\left|\int_a^b f(t)\,dt\right| = \int_a^b e^{-i\theta}f(t)\,dt = \int_a^b \operatorname{Re}(e^{-i\theta}f(t))\,dt \leq \int_a^b |f(t)|\,dt \leq M(b-a)$$

(the second equality because the left-hand side is real) giving (4). For the second part, we may assume that f is not identically zero. If we have equality in (4) then both the inequalities above must be equalities. The second is an equality iff $|f(t)| = M$, so that $|f|$ is constant; the first is an equality iff $e^{i\theta} = \arg(f(t))$, which means that $\arg(f)$ is constant. Taken together this means f is constant. □

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a C^1-curve, $\gamma(t) = x(t) + iy(t)$. Then as $|\gamma'(t)|^2 = (dx/dt)^2 + (dy/dt)^2$, it makes sense to define the *arc length* of γ as the integral

$$\operatorname{length}(\gamma) := \int_a^b |\gamma'(t)|\,dt.$$

We say that γ is *simple* if $\gamma(t_1) \neq \gamma(t_2)$ unless $t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. If γ is a simple curve then $\operatorname{length}(\gamma)$ is just the length of (the image of) γ.

Definition. Let $f\colon U \to \mathbb{C}$ be continuous, and let $\gamma\colon [a,b] \to \mathbb{C}$ be a C^1 curve. The integral of f along γ is

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Basic properties:

- linearity $\int_\gamma c_1 f_1(z) + c_2 f_2(z)\,dz = c_1\int_\gamma f_1(z)\,dz + c_2\int_\gamma f_2(z)\,dz$.

- additivity: if $a < a' < b$ and $\gamma_1\colon [a, a'] \to U$, $\gamma_2\colon [a', b] \to U$ are given by $\gamma_i(t) = \gamma(t)$ then

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

- inverse path: if $(-\gamma)\colon [-b, -a] \to U$ is the curve $(-\gamma)(t) = \gamma(-t)$ then $\int_{-\gamma} f\,dz = i \int_\gamma f\,dz$.

- reparameterisation: if $\phi\colon [a', b'] \to [a, b]$ is C^1 and $\phi(a') = a$, $\phi(b') = b$ then if $\delta = \gamma \circ \phi\colon [a', b'] \to U$, have $\int_\gamma f\,dz = \int_\delta f\,dz$.

  This means that we may (and often shall) restrict attention to curves $\gamma\colon [0, 1] \to \mathbb{C}$.

Let $\gamma\colon [a, b] \to \mathbb{C}$ be a (continuous) curve. Suppose we have $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ such that for each $0 \leq i < m$ the restriction γ_i of γ to $[a_{i-1}, a_i]$ is C^1. We then say that γ is a *piecewise continuously differentiable*, or *piecewise-C^1 curve*, and define

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n-1} \int_{\gamma_i} f(z)\,dz.$$

By additivity this does not depend on the decomposition.

Remarks. (i) We will often abuse notation by identifying γ with its image in $\mathbb{C}$ — for example, we may say "f is non-zero on γ" — although it must always be remembered that γ is a map from an interval to $\mathbb{C}$, and not a subset of $\mathbb{C}$. (For example, a curve and its inverse path have the same image, but the integrals along them are different.)[6]

(ii) Appearances can be deceptive: one should not read too much into the notion of C^1 curve. In particular, a C^1-curve need not have a tangent at every point, even if it is simple. For example, the curve $\gamma\colon [0, 1] \to \mathbb{C}$ given by

$$\gamma(t) = \begin{cases} 1 + i\sin \pi t & \text{if } 0 \leq t \leq 1/2 \\ \sin \pi t + i & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

is a C^1 curve which has no tangent at $t = 1/2$.

Precisely, it is easy to show that if $\gamma'(t_0) \neq 0$ then the curve $\gamma(t)$ has a tangent at $t = t_0$, (in fact a tangent vector is just $\gamma'(t_0)$). But a C^1-function $\gamma\colon [0, 1] \to \mathbb{C}$ can have zero derivative at infinitely many points.

[6]Some writers use the notation $[\gamma]$ or γ^* to denote the image of γ, but I find this pedantic.

By suitable reparameterisation one can replace any piecewise-C^1-curve by a C^1-curve — simply replace each C^1 segment γ_i with $\gamma_i \circ h_i$, for any monotonic C^1 bijection $h_i \colon [a_{i-1}, a_i] \to [a_{i-1}, a_i]$ with $h_i'(a_{i-1}) = h_i'(a_i) = 0$.

It's convenient to be able to combine curves as well. If $\gamma\colon [a,b] \to \mathbb{C}$, $\delta\colon [c,d] \to \mathbb{C}$ are curves with $\gamma(b) = \delta(c)$ then we can define their *sum* to be the curve $\gamma + \delta\colon [a, b+d-c] \to \mathbb{C}$ given by

$$t \mapsto \begin{cases} \gamma(t) & \text{if } a \le t \le b \\ \delta(t+c-a) & \text{if } b \le t \le b+d-c \end{cases}$$

Recall that in the last lecture we defined a *piecewise C^1-curve* to be a curve $\gamma\colon [a,b] \to \mathbb{C}$ for which there exists a decomposition $a = a_0 < a_1 < \cdots < a_n = b$ for which the restriction γ_i of γ to the subinterval $[a_{i-1}, a_i]$ is continuously differentiable, for each $1 \le i \le n$. So $\gamma = \gamma_1 + \cdots + \gamma_n$ is a sum of C^1-curves.

Unless otherwise stated, by "curve" we shall henceforth always mean "piecewise-C^1 curve".

Proposition 2.1.2. *For any continuous $f\colon U \to \mathbb{C}$ and any curve $\gamma\colon [a,b] \to U$,*

$$\left| \int_\gamma f(z)\,dz \right| \le \operatorname{length}(\gamma) \sup_\gamma |f|\,.$$

Proof. By additivity we may assume that γ is a C^1-curve. If $M = \sup_\gamma |f|$ then

$$\begin{aligned} \left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\,dt \right| &\le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \\ &\le M \int_a^b |\gamma'(t)|\,dt = M \operatorname{length}(\gamma). \end{aligned}$$

□

Proposition 2.1.3. *If $f_n\colon U \to \mathbb{C} (n \in \mathbb{N})$ and $f\colon U \to \mathbb{C}$ are continuous functions, and $\gamma\colon [a,b] \to U$ is a curve such that $f_n \to f$ uniformly on (the image of) γ, then*

$$\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$$

Proof. Let $M_n = \sup_\gamma |f - f_n|$. Then by definition of uniform convergence $M_n \to 0$, and by the previous result

$$\left| \int_\gamma f(z)\,dz - \int_\gamma f_n(z)\,dz \right| \le M_n \operatorname{length}(\gamma) \to 0.$$

□

Theorem 2.1.4 ("Fundamental Theorem of Calculus")**.** *If $F\colon U \to \mathbb{C}$ is holomorphic [and F' is continuous] and $\gamma\colon [a,b] \to U$ is a curve, then*

$$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

If moreover γ is closed, then $\int_\gamma F\,dz = 0$.

(We say $\gamma\colon [a,b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$.)

In particular, if f is the derivative of a holomorphic function on U, then the integral $\int_g f\,dz$ depends only on the endpoints of γ.

Later (2.5.2) we shall see that the condition *"F' is continuous"* is automatically satisfied.

Proof.

$$\int_\gamma F'(z)\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b (d/dt)(F(\gamma(t)))\,dt = [F(\gamma(t))]_a^b$$

□

Example: $\int_\gamma z^n\,dz$ where γ is the circular path $\gamma\colon [0,1] \to \mathbb{C}$, $\gamma(t) = Re^{2\pi it}$, $R > 0$.

If $n \neq -1$ then f is the derivative of $z^{n+1}/(n+1)$, which is holomorphic on $\mathbb{C} \setminus \{0\}$ (even on $\mathbb{C}$ if $n \geq 0$), so $\int_\gamma f(z)\,dz = 0$.

However if $n = -1$ we don't know a holomorphic function on any open subset of $\mathbb{C}$ containing γ with derivative $1/z$ — the natural candidate, $\mathrm{Log}(z)$, being only holomorphic on the "cut-plane". Instead compute from the definition: since $\gamma'(t) = 2\pi iRe^{2\pi it}$,

$$\int_\gamma z^{-1}\,dz = \int_0^1 R^{-1}e^{-2\pi it} \cdot 2\pi iRe^{2\pi it}\,dt = 2\pi i.$$

Since this is $\neq 0$, we can deduce from FTC that *there does not exists a holomorphic function on any open subset of $\mathbb{C}$ containing the circle $\{|z| = R\}$, whose derivative is* $1/z$. In particular, there is no branch of the logarithm defined on $\mathbb{C} \setminus \{0\}$.

Rather strikingly, it turns out that the converse to this is true:

Theorem 2.1.5 (converse of FTC)**.** *Let $f\colon D \to \mathbb{C}$ be continuous on a domain D. If $\int_\gamma f(z)\,dz = 0$ for all closed γ in D, then there exists a holomorphic $F\colon D \to \mathbb{C}$ with $F' = f$.*

Proof. Pick a point $a_0 \in D$. If $w \in D$, pick any curve $\gamma_w \colon [0,1] \to D$ with $\gamma_w(0 + a_0)$, $\gamma_w(1) = w$, and define

$$F(w) = \int_{\gamma_w} f(z)\,dz.$$

We'll show that F is holomorphic on D with derivative f.

Now fix some $w \in D$, choose $r > 0$ such that $D(w,r) \subset D$. If $|h| < r$, let $\delta_h \colon [0,1] \to D$ be the line segment $\delta_h(t) = w + th$ from w to $w + h$. Then the integral of f around the closed path $\gamma_w + \delta_h + (-\gamma_{w+h})$ is by hypothesis zero,[7] hence

$$F(w+h) = \int_{\gamma_w+\delta_h} f(z)\,dz = F(w) + \int_{\delta_h} f(z)\,dz = F(w) + hf(w) + \int_{\delta_h} f(z) - f(w)\,dz \tag{5}$$

and therefore

$$\begin{aligned}\left|\frac{F(w+h)-F(w)}{h} - f(w)\right| &= \left|\frac{1}{h}\int_{\delta_h} f(z) - f(w)\,dz\right| \\ &\le \operatorname{length}(\delta_h)\,|h|^{-1} \sup_{\delta_h} |f(z) - f(w)| \\ &\le \sup_{|z-w|\le|h|} |f(z) - f(w)| \to 0 \quad \text{as } |h| \to 0\end{aligned}$$

so F is differentiable at w with derivative $f(w)$. □

We shall use the following variant of this as an intermediate result:

Lemma 2.1.6. *Let D be a disc (or more generally, any convex or starlike domain), and let f be continuous on D. If $\int_\gamma f(z)\,dz = 0$ for every triangle γ in D, then there exists a holomorphic function F on D with $F' = f$.*

Proof. First we define the terms:

- an open subset $U \subset \mathbb{C}$ is *convex* if for every $a, b \in U$ the line segment $\{tb + (1-t)a \mid t \in [0,1]\}$ from a to b lies in U;
- U is *starlike* if there exists $a_0 \in U$ such that for every $b \in U$ the line segement from a_0 to b lies in U.

Obviously disc $\implies$ convex $\implies$ starlike $\implies$ domain. The domain U occurring in the definition of $\log(z)$ is starlike but not convex.

For the proof, define $F(w)$ as in the proof above, taking γ_w to be the straight-line path from a_0 to w (which is contained in D by the assumption on D). Then the closed curve $\gamma_w + \delta_h + (-\gamma_{w+h})$ is a triangle, so (5) still holds under the hypothesis on f. □

[7] The same argument shows that $F(w)$ is independent of the choice of curve γ_w.

2.2 Cauchy's theorem for a disc

Cauchy's theorem states that, if f is holomorphic on a domain D and $\gamma\colon [a,b] \to D$ is a closed curve, then under certain hypotheses on D and γ, $\int_\gamma f(z)\,dz = 0$. There are various versions of Cauchy's theorem, which differ only in their hypotheses. The basic version, from which all others are derived, applies to the simplest kind of domain — a disc.

Notation for a triangle.

Theorem 2.2.1. *Let $f\colon U \to \mathbb{C}$ be holomorphic, $\Delta \subset U$ a triangle. Then $\int_{\partial\Delta} f(z)\,dz = 0$.*

Proof. Let $L = \text{length}(\Delta)$, and let $I = \left|\int_{\partial\Delta} f(z)\,dz\right|$. Bisecting the sides of Δ gives 4 subtriangles $\Delta^{(i)}$ $(i = 1,2,3,4)$ and we have

$$\int_{\partial\Delta} f(z)\,gz = \sum_{i=1}^{4} \int_{\partial\Delta^{(i)}} f(z)\,gz$$

since the integrals along the lines joining the midpoints of the sides of Δ cancel in pairs. Therefore, for some $j \in \{1,2,3,4\}$ we have

$$\left|\int_{\partial\Delta^{(j)}} f(z)\,dz\right| \geq \frac{1}{4}I.$$

Denote $\Delta^{(j)}$ by Δ_1. Iterating, we obtain a sequence of nested triangles

$$\Delta = \Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \dots$$

such that

$$\text{length}(\partial\Delta_n) = 2^{-n}L \quad \text{and} \quad \left|\int_{\partial\Delta_n} f(z)\,dz\right| \geq 4^{-n}I.$$

Now $\cap_{n\in\mathbb{N}}\Delta_n = \{w\}$, a single point[8]. Then the function

$$g(z) = \frac{f(z) - f(w)}{z - w} - f'(w)$$

[8] Since the perimeter of Δ_n tends to 0, clearly the intersection can contain at most one point. Let $w_n \in \Delta_n$ be arbitrary; then $|w_n - w_{n+1}| \leq \text{length}(\Delta_n) \to 0$ so $w = \lim w_n$ exists; and for every $n \in \mathbb{N}$, $w \in \Delta_n$ since Δ_n is closed and $w_m \in \Delta_n$ for $m \geq n$.

is continuous on D with $g(w) = 0$. Therefore

$$\begin{aligned}
4^{-n}L &\le \left| \int_{\partial\Delta_n} f(z)\,dz \right| \\
&= \left| \int_{\partial\Delta_n} f(z) - f(w) - (z-w)f'(w)\,dz \right| \quad \text{since } \int_{\partial\Delta_n} dz = \int_{\partial\Delta_n} z\,dz = 0 \\
&\le 2^{-n}L \sup_{z\in\delta_n} |(z-w)g(z)| \\
&\le 2^{-n}L \cdot 2^{-n}L \sup_{z\in\Delta_n} |g(z)|
\end{aligned}$$

and so $I \le L^2 \sup_{\Delta_n} |g| \to 0$ as $n \to \infty$, since $g(w) = 0$. Hence $I = 0$. □

It is important for later use to know that the theorem holds under (apparently) weaker hypotheses.

Theorem 2.2.2. *Let $S \subset U$ be a finite subset, and assume that $f\colon U \to \mathbb{C}$ is continuous on U and holomorphic on $U \setminus S$. Let $\Delta \subset U$ be a triangle. Then $\int_{\partial\Delta} f(z)\,dz = 0$.*

Proof. By subdividing Δ we may assume that $S = \{a\}$ is a singleton, and $a \in \Delta$. Let $M = \sup_\Delta |f| < \infty$. If $\Delta' \subset \Delta$ is any smaller triangle containing the point a then subdivision and the previous result shows that

$$\left| \int_{\partial\Delta} f(z)\,dz \right| = \left| \int_{\partial\Delta'} f(z)\,dz \right| \le M \,\text{length}(\partial\Delta')$$

so by letting the length of Δ' tend to zero, we have the result. □

Corollary 2.2.3 (Cauchy's theorem for a disc)**.** *Let D be a disc (or convex or starlike domain) and $f\colon D \to \mathbb{C}$ a function which is holomorphic except possibly at a finite number of points, where it is continuous. Then for any closed γ in D, $\int_\gamma f(z), dz = 0$.*

Proof. Combine Lemma 2.1.6 and Theorem 2.2.2. □

2.3 Cauchy integral formula for a disc

Theorem 2.3.1. *Let $D = D(a, r)$ be a disc and $f\colon D \to \mathbb{C}$ holomorphic. For every $w \in D$ and ρ with $|w - a| < \rho < r$,*

$$f(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{z-w} dz.$$

Proof. Apply Corollary 2.2.3 to the function

$$g(z) = \begin{cases} (f(z) - f(w)/(z-w) & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

This gives

$$\int_{|z-a|=\rho} \frac{f(z)}{z-w}\,dz = \int_{|z-a|=\rho} \frac{f(w)}{z-w}\,dz = \sum_{n=0}^{\infty} \int_{|z-a|=r} f(w) \frac{(w-a)^n}{(z-a)^{n+1}}\,dz = 2\pi i f(w)$$

using the geometric series

$$\frac{1}{z-w} = \frac{1}{(z-a)(1-(w-a)/(z-a))} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} \tag{6}$$

which converges uniformly for $|z-a| = \rho$ (so that we may interchange integration and summation, by Proposition 2.1.3)). □

Corollary 2.3.2 (The Mean-Value Property)**.** *If $f\colon D(w,R) \to \mathbb{C}$ is holomorphic, then for every $0 < r < R$,*

$$f(w) = \int_0^1 f(w + re^{2\pi i t})\,dt.$$

Remark. The corollary can be restated as saying that $f(w)$ equals the average value of f on any circle with centre w.

Proof. Take $w = a$ in the Theorem, and parameterise the circle of integration as $\gamma(t) = w + re^{2\pi i t}$, $t \in [0,1]$. □

2.4 First applications of CIF

Theorem 2.4.1 (Liouville's Theorem)**.** *Every bounded entire function is constant.*

Proof. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f| < M$, and let $w \in \mathbb{C}$. Then if $R > |w|$

$$\begin{aligned} |f(w) - f(0)| &= \frac{1}{2\pi} \left| \int_{|z|=R} f(z) \left(\frac{1}{z-w} - \frac{1}{z} \right) dz \right| \\ &= \frac{1}{2\pi} \left| \int_{|z|=R} \frac{f(z)}{z(z-w)}\,dz \right| \\ &\leq \frac{1}{2\pi} \times 2\pi R \times \frac{M}{R(R-|w|)} \\ &\to 0 \quad \text{as } R \to \infty \end{aligned}$$

which shows that $f(w) = f(0)$. □

Theorem 2.4.2 (Fundamental Theorem of Algebra). *Every non-constant polynomial with complex coefficients has a complex root.*

Proof. Let $P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ be a polynomial of degree $n > 0$. Then $|P(z)| \to \infty$ as $|z| \to \infty$, so there exists R such that $|P(z)| > 1$ for all z with $|z| > R$. Consider $f(z) = 1/P(z)$. If P has no complex zeroes then f is entire, and so (being continuous) f is bounded on $\{|z| \leq R\}$. As $|f(z)| < 1$ when $|z| > R$, f is a bounded entire function, so by Liouville's Theorem f is constant; contradiction. □

Theorem 2.4.3 (Local maximum modulus principle). *Let $f\colon D(a,r) \to \mathbb{C}$ be holomorphic. If for every $z \in D(a,r)$, $|f(z)| \leq |f(a)|$, then f is constant.*

Proof. By the Mean-Value Property we have, for $0 < \rho < r$,

$$|f(a)| = \left| \int_0^1 f(a + \rho e^{2\pi i t})\, dt \right| \leq \sup_{|z-a|=\rho} |f(z)|$$

and by hypothesis equality holds, so by Proposition 2.1.1 $|f(z)| = |f(a)|$ for all z on the circle $|z - a| = \rho$. Since this holds for all ρ, $|f|$ is constant on $D(a,r)$. By the Cauchy–Riemann equations, this implies that f is constant[9].

□

2.5 Taylor expansion

Theorem 2.5.1. *Let $f\colon B(a,r) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation on $B(a,r)$:*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad \text{for any } 0 < \rho < r.$$

[9] $|f| = c$ constant $\implies f\bar{f} = c^2$, so (unless $c = 0$, in which case $f = 0$) $\bar{f} = c^2/f$ is also holomorphic. Then by Cauchy–Riemann equations for f and $\bar{f}$ we see at once that f is constant.

Proof. If $|w-a| < \rho < r$ then by the Cauchy Integral Formula

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(z)}{z-w}\,dz \\ &= \frac{1}{2\pi i}\int_{|z-a|=\rho} f(z)\sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}\,dz \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|z-a|=\rho} f(z)\frac{1}{(z-a)^{n+1}}\,dz\right)(w-a)^n \end{aligned}$$

where we have used (6) and the interchange of integration and summation is justified by the uniform convergence of the geometric progression. So f has a convergent power series representation on $B(a,\rho)$ for any $\rho < r$, and the rest of the theorem follows. □

Corollary 2.5.2. *If $f\colon U \to \mathbb{C}$ is holomorphic then its derivatives of all orders exist and are holomorphic.*

Remark. A function $f\colon U \to \mathbb{C}$ is said to be *analytic* if for every $a \in U$, f can be represented by a convergent power series on some $B(a,r) \subset U$. (By Theorem 1.3.2(iv), this power series is unique).

Theorem 1.3.2 shows that analytic functions are homolorphic. Theorem 2.5.1 shows that every holomorphic function is analytic. So in complex analysis the words "analytic" and "holomorphic" are interchangeable (and indeed many authors define analytic to be what we have termed holomorphic).

However in real analysis there is a big difference. We say by analogy that a function $f\colon (a,b) \to \mathbb{R}$ is analytic if for every $c \in (a,b)$ there exists an interval $(c-r, c-r)$ on which f can be represented as a convergent power series in $(x-c)$. There are many functions which are infinitely differentiable but which are not analytic. In particular, if $f\colon \mathbb{R} \to \mathbb{R}$ is C^∞ then it can happen that the Taylor series of f at the origin has RoC zero. Even if the RoC is positive, the function defined by the Taylor series may not be equal to f.

From now one, we shall use "holomorphic" and "analytic" interchangeably.

We next prove the converse to Cauchy's Theorem:

Corollary 2.5.3 (Morera's Theorem)**.** *Let D be a disc and $f\colon D \to \mathbb{C}$ be a continuous function such that $\int_\gamma f(z)\,dz = 0$ for every closed curve γ in D. The f is homolorphic.*

Proof. By Theorem 2.1.5, $f = F'$ for a holomorphic $F\colon D \to \mathbb{C}$. The previous corollary then implies that f is holomorphic. □

Here's an application of Morera's theorem.

Corollary 2.5.4. *Let $D \subset \mathbb{C}$ be open and $a, b \in \mathbb{R}$. Let $\phi\colon D \times [a, b] \to \mathbb{C}$ be continuous, such that for each $s \in [a, b]$ the function $z \mapsto \phi(z, s)$ is holomorphic on D. Then*

$$g(z) = \int_a^b \phi(z, s)\, ds$$

is holomorphic on D.

Remark. One can also show (example sheet — use the CIF representation for $\partial\phi/\partial z$) that $(\partial/\partial z)\phi(z, s)$ is continuous in s and that

$$g'(z) = \int_a^b \frac{\partial\phi}{\partial z}(z, s)\, ds$$

Lemma 2.5.5. *Let $f\colon [a, b]\times[c, d] \to \mathbb{R}$ be continuous. Then the functions $f_1\colon x \mapsto \int_c^d f(x, y)\, dy$ and $f_2\colon y \mapsto \int_a^b f(x, y)\, dx$ are also continuous, and satisfy*

$$\int_a^b \left(\int_c^d f(x, y)\, dy\right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx\right) dy. \qquad (7)$$

Proof. This is a simple form of Fubini's theorem in integration theory. For the proof, since f is continuous on a compact subset of $\mathbb{R}^2$, f is uniformly continuous. Therefore given $\epsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x, y) - f(x_0, y)| < \epsilon$, which in turn implies that $|f_1(x) - f_1(x_0)| \leq (d - c)\epsilon$. Hence f_1 is continuous.

For the second part, recall that a step function on a rectangle $R = [a, b] \times [c, d]$ is a finite linear function of characteristic functions of subrectangles $[a', b'] \times [c', d'] \subset R$. Every continuous function on R is a uniform limit of step functions, and for step functions the identity (7) is obvious. □

Proof of Corollary 2.5.4. We may assume that D is a disc, and will apply Morera's theorem. For any closed curve $\gamma\colon [0, 1] \to D$ we have

$$\begin{aligned}
\int_\gamma g(z)\, dz &= \int_0^1 \left(\int_a^b \phi(\gamma(t), s)\, ds\right) \gamma'(t)\, dt \\
&= \int_a^b \left(\int_0^1 \phi(\gamma(t), s)\gamma'(t)\, dt\right) ds \\
&= \int_a^b \left(\int_\gamma \phi(z, s)\, dz\right) ds = 0.
\end{aligned}$$

by the previous lemma[10] and Cauchy's Theorem 2.2.3 for a disc. □

[10] Since γ need only be pw-C^1 one needs to break the integral over γ up into integrals over C^1 curves before applying the lemma.

Let $f\colon D(w,R) \to \mathbb{C}$ be holomorphic, and write f as a power series $\sum c_n(z-w)^n$ converging on $D(w,R)$. If f is not identically zero on $D(w,R)$ then not all of the coefficients c_n can vanish; let $m = \min\{n \in \mathbb{N} \mid c_n \neq 0\}$. Then $f(z) = (z-w)^m g(z)$ where $g(z) = \sum_{n=m}^{\infty} c_m(z-w)^{n-m}$ is holomorphic on $D(w,R)$ and $g(w) \neq 0$. If $m > 0$ we say that f has a *zero of order m at $z = w$*. Clearly m is the least n such that $f^{(n)}(w) \neq 0$.

Theorem 2.5.6 (Principle of Isolated Zeroes)**.** *Let $f\colon D(w,R) \to \mathbb{C}$ be holomorphic and not identically zero. Then there exists $0 < r \leq R$ such that $f(z) \neq 0$ for $0 < |z-w| < r$.*

Proof. Suppose $f(w) \neq 0$. Then by continuity of f, there exists $r > 0$ such that $f(z) \neq 0$ for $z \in D(w,r)$.
Otherwise, f has a zero of order $m > 0$ at $z = w$, so $f(z) = (z-w)^m g(z)$ with g holomorphic and nonzero at $z = w$. So there exists $r > 0$ such that g is nonzero on $D(w,r)$, and then $f(z) \neq 0$ for $0 < |z-w| < r$. □

2.6 Analytic continuation

The fact that holomorphic functions are analytic has an interesting and important consequence — a holomorphic function on a domain D is determine by its restriction to a disc in D.

Theorem 2.6.1 (Uniqueness of analytic continuation)**.** *Let $D' \subset D$ be domains, and $f\colon D' \to \mathbb{C}$ be analytic. There is at most one analytic function $g\colon D \to \mathbb{C}$ such that $g(z) = f(z)$ for all $z \in D'$.*

Such a function g, if it exists, is said to be an *analytic continuation* of f to D.

Proof. Let $g_1, g_2\colon D \to \mathbb{C}$ be analytic continuations of f to D. Then $h = g_1 - g_2\colon D \to \mathbb{C}$ is analytic and $h(z) = 0$ on D'. It suffices to prove h is identically zero on D'. To do this, define

$$D_0 = \{w \in D \mid h \text{ is identically zero on some open disc } D(w,r)\}$$
$$D_1 = \{w \in D \mid h^{(n)}(w) \neq 0 \text{ for some } n \geq 0\}.$$

Then since h has a convergent power series expansion about each point $w \in D$, we see by Theorem 1.3.2 that $D = D_0 \cup D_1$ and $D_0 \cap D_1 = \emptyset$. Moreover both D_0 and D_1 are open subsets of $\mathbb{C}$. So as D is connected, one of D_i is empty, and as $D_0 \supset D' \neq \emptyset$ we must have $D_1 = \emptyset$, so that $D = D_0$ and $h = 0$ on all of D. □

Combining this with Theorem 2.5.6 we get:

Corollary 2.6.2 ("Identity Theorem")**.** *Let $f, g\colon D \to \mathbb{C}$ be analytic on a domain D. If $S = \{z \in D \mid f(z) = g(z)\}$ contains a non-isolated point, then $f = g$ on D.*

Proof. Let $w \in S$ be a non-isolated point[11]. The function $f - g$ is holomorphic on D and vanishes on S, so has a non-isolated zero at w. Therefore $f - g$ vanishes identically on an open disc with centre w by Theorem 2.5.6, hence by the previous result $f = g$ on D. □

Remark. Given an analytic function $f\colon D' \to \mathbb{C}$ and an overdomain $D \supset D'$ it is in general a hard problem to determine whether or not f can be analytically continued to D. Typically f may be given by a convergent power series. Contrast the following series, both of radius of convergence 1:

- $f(z) = \sum_{n=0}^{\infty} z^n$, which has an analytic continuation to $\mathbb{C} \setminus \{1\}$ given by $g(z) = 1/(1 - z)$.
- $f(z) = \sum_{n=0}^{\infty} z^{n^2}$. One can show (although not easily) that f cannot be analytically continued to any domain D properly containing $D(0, 1)$. (One says that the circle $\{|z| = 1\}$ is a *natural boundary* for the power series.)

Moreover one can show that for any domain D', there exists an analytic function $f\colon D \to \mathbb{C}$ which cannot be analytically continued to any strictly larger domain $D \supset D'$.

3 Complex integration II

3.1 Winding number

Suppose $\gamma\colon [a, b] \to \mathbb{C}$ is a closed curve, and that $w \in \mathbb{C}$ is not in the image of γ. We want to define mathematically "the number of times γ winds around w". There are two ways to do this, and it is important to understand both.

The "naive" method is the following: suppose that we have written

$$\gamma(t) = w + r(t)e^{i\theta(t)} \tag{8}$$

for *continuous* functions $r, \theta\colon [a, b] \to \mathbb{R}$, $r(t) > 0$. Obviously $r(t) = |\gamma(t) - w|$ is uniquely determined. Then the angle swept out by $\gamma(t)$ around w is the difference $\theta(b) - \theta(a)$, so we define the *winding number* or *index* of γ about w as

$$I(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi}.$$

[11] i.e. for every $\epsilon > 0$, there exists $z \in S$ with $0 < |z - w| < \epsilon$.

If θ and θ_1 both satisfy (8) then their difference is a continuous function with values in $2\pi\mathbb{Z}$, so is constant. Therefore if θ exists, then $I(\gamma; w)$ is well-defined. However the existence is not entirely trivial.

Theorem 3.1.1. *If $\gamma\colon [a,b] \to \mathbb{C} \setminus \{w\}$ is a continuous curve, then there exists a continuous function $\theta\colon [a,b] \to \mathbb{R}$ with*

$$\gamma(t) = w + r(t)e^{i\theta(t)}, \quad r(t) = |\gamma(t) - w| .$$

Proof. We can after translation assume that $w = 0$. First note that if the image of γ lies in the 1/2-plane $D = \{\mathrm{Re}(z) > 0\}$ then we may take $\theta(t) = \arg\gamma(t)$ where arg is the principal branch of the argument, which is continuous on D. Similarly if γ has image in the 1/2-plane $\{\mathrm{Re}(z/e^{i\alpha}) > 0\}$ then $\theta(t) = \alpha + \arg(\gamma(t)/e^{i\alpha}$ will do.

The natural guess would be to just add or subtract 2π from $\arg\gamma(t)$ each time that γ crosses the negative axis. But since it could cross infinitely many times (even uncountably many times), this will not work.

Note that replacing γ by $\gamma/|\gamma|$ doesn't change the problem, so we may assume that $|\gamma(t)| = 1$. Next, since γ is continuous on $[a,b]$ it is uniformly continuous, so there exists $\epsilon > 0$ such that if $s,t \in [a,b]$ with $|s-t| < \epsilon$, then $|\gamma(s) - \gamma(t)| < \sqrt{2}$. Now subdivide: consider $a = a_0 < a_1 < \cdots < a_N = b$ where $a_n - a_{n-1} < 2\epsilon$. Then if $t \in [a_{n-1}, a_n]$, we have $\left|\gamma(t) - \gamma(\frac{a_{n-1}+a_n}{2})\right| < \sqrt{2}$, i.e. the image of $[a_{n-1}, a_n]$ lies in a semicircle, so in a half-plane. So by the above, for each n there exists a continuous $\theta_n\colon [a_{n-1}, a_n] \to \mathbb{R}$ such that $\gamma(t) = e^{i\theta_n(t)}$ for all $t \in [a_{n-1}, a_n]$, and so $\theta_{n-1}(a_n) = \theta_(a_n) + 2\pi B_n$, $B_n \in \mathbb{Z}$. Adding suitable integer multiples of 2π to each θ_n we can assume that $B_n = 0$, and then the θ_n's fit together to define a continuous θ. □

The second approach is by integration. From now on we again only consider piecewise-C^1 curves.

Lemma 3.1.2. *Let $\gamma\colon [a,b] \to \mathbb{C} - \{w\}$ be a (piecewise-C^1) closed curve. Then*

$$I(\gamma; w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}. \tag{9}$$

Proof. Write $\gamma(t) = w + r(t)e^{i\theta(t)}$ as in the theorem. Then as γ is piecewise C^1 so are r and θ, and

$$\begin{aligned}\int_\gamma \frac{1}{z-w}dz = \int_a^b \frac{\gamma'(t)}{\gamma(t)-w}dt &= \int_a^b \frac{r'(t)}{r(t)} + i\theta'(t)\,dt \\ &= [\ln r(t) + i\theta(t)]_a^b = i(\theta(b) - \theta(a)) = 2\pi i I(\gamma; w).\end{aligned}$$

□

Remark. In fact some authors (e.g. Ahlfohrs) take (9) as the *definition* of winding number. Although elegant this seems a bit artificial (for example, it is then non-trivial to prove it is integer-valued).

Proposition 3.1.3. *If* $\gamma\colon [0,1] \to D(a,R)$ *is a closed curve and* $w \notin D(a,R)$ *then* $I(\gamma; w) = 0$.

Proof. The hypothesis implies that $D(a,R)$ is contained in the 1/2-plane $U = \{z \mid \operatorname{Re}(z-w)/(a-w) > 0\}$. So there is a branch of $\arg(z-w)$ which is continuous on U, and then $2\pi I(\gamma; w) = \arg(\gamma(1) - w) - \arg(\gamma(0) - w) = 0$. □

Remark. For piecewise-C^1 curves one could use Lemma 3.1.2 and appeal to Cauchy's theorem for a disc, since $1/(z-w)$ is holomorphic on $B(a,R)$ if $w \notin B(a,R)$. (This is a sledgehammer approach, though.)

Definition. Let $U \subset \mathbb{C}$ be open.

(i) A closed curve γ in U is *homologous to zero in* U if for every $w \notin U$, $I(\gamma; w) = 0$.

(ii) U is *simply connected* if every closed curve γ in U is homologous to zero.

Remark. This is not the same as the usual topologist's definition of simply-connected (which is that every closed curve is null-homotopic), but for open subsets of the plane it can be shown to be equivalent. See the example sheet for another equivalent definition (the complement of D in the Riemann sphere $\mathbb{CP}^1$ is connected). One can also prove that the definition remains the same if one considers all continuous curves, piecewise-C^1 curves or even just polygonal curves.

It is convenient sometimes to generalise the notion of closed curve. By a *cycle* in an open subset $U \subset \mathbb{C}$, we mean a formal sum of closed curves in U

$$\Gamma = \gamma_1 + \cdots + \gamma_n.$$

If $f\colon U \to \mathbb{C}$ is continuous, we then define

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,dz$$

and likewise $I(\Gamma; w) = \sum I(\gamma_i; w)$ if w does not lie on any of the curves γ_i. A cycle is said to be homologous to zero in U if $I(\gamma; w) = 0$ for all $w \notin U$. Notice that although this holds if each γ_i is homologous to zero, the converse is not true. For example, take $U = \mathbb{C} \setminus \{a\}$ and $\Gamma = \gamma_1 + \gamma_2$ where $\gamma_i\colon [0,1] \to U$ are the circles $\gamma_1(t) = a + r_1 e^{2\pi i t}$, $\gamma_2(t) = a + r_2 e^{-2\pi i t}$, $r_i > 0$.

3.2 Cauchy's integral formula (general case)

Theorem 3.2.1. *Let $f\colon D \to \mathbb{C}$ be holomorphic, and let γ be a closed curve (or cycle) in D which is homologous to zero. Then for all $w \in D \setminus \gamma$,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz = I(\gamma; w) f(w) \tag{i}$$

and

$$\int_\gamma f(z)\,dz = 0. \tag{ii}$$

Proof. (i) Consider the function $g\colon D \times D \to \mathbb{C}$ defined by

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Since f is analytic on D we can conclude from the proof of Theorem 1.3.2 that g is continuous, and for fixed z it is an analytic function of w. We want to show that if $w \in D \setminus \gamma$ then $\int_\gamma g(z,w)\,dz = 0$ — by the definition of winding number, this will prove (i). To do this, consider the function h defined by

$$h(w) = \begin{cases} \displaystyle\int_\gamma g(z,w)\,dz & \text{if } w \in D \\ \displaystyle\int_\gamma \frac{f(z)}{z-w}\,dz & \text{if } w \in E := \{w \in \mathbb{C} \setminus \gamma \mid I(\gamma; w) = 0\}. \end{cases}$$

Since γ is homologous to zero in D we have $D \cup E = \mathbb{C}$, and if $w \in D \cap E$ the given definitions of $h(w)$ coincide. So h is defined on all of $\mathbb{C}$, and is holomorphic by Corollary 2.5.4. Moreover if R is sufficiently large then $|w| > R$ implies that $I(\gamma; w) = 0$ (by Proposition 3.1.3) and so

$$|h(w)| \leq \frac{\text{length}(\gamma)\sup_\gamma |f|}{|w| - R} \to 0 \quad \text{as } |w| \to \infty.$$

so by Liouville's Theorem 2.4.1 h is identically zero.

For (ii), simply apply (i) to the function $(z-w)f(z)$, for any $w \in D \setminus \gamma$. □

Corollary 3.2.2 (Cauchy's theorem for simply-connected domains)**.** *Let f be holomorphic on a simply-connected domain D. Then for all closed curves γ in D, $\int_\gamma f(z)\,dz = 0$.*

3.3 Singularities and the Laurent expansion; the residue theorem

Just as a holomorphic function on a disc $D(a, r)$ can be expanded as a series in powers of $(z - a)$, we'll see that a function which is holomorphic on $D(a, r) \setminus \{a\}$ can be expanded as a series in positive and negative powers of $(z - a)$. In fact a rather stronger result holds.

Theorem 3.3.1. *Let f be holomorphic on an annulus $A = \{z \in \mathbb{C} \mid r < |z - a| < R\}$, where $0 \leq r < R \leq \infty$. Then:*
(i) f has a unique convergent series expansion on A

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n \tag{10}$$

(ii) For any $\rho \in (r, R)$ the coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z - a)^{n+1}} \, dz.$$

(iii) If $r < \rho' \leq \rho < R$ then the series converges uniformly on the set $\{z \in \mathbb{C} \mid \rho' \leq |z - a| \leq \rho\}$

Proof. Start with the CIF: given $w \in A$, choose $r < \rho_2 < |w - a| < \rho_1 < R$ and consider the cycle $\gamma = \gamma_1 - \gamma_2$, where γ_i is the circle $|z - a| = \rho_i$. Then γ is homologous to zero in A, hence $f(w) = f_1(w) + f_2(w)$ where

$$f_1(w) = \frac{1}{2\pi i} \int_{|z-a|=\rho_1} \frac{f(z)}{z - w} \, dz, \qquad f_2(w) = -\frac{1}{2\pi i} \int_{|z-a|=\rho_2} \frac{f(z)}{z - w} \, dz.$$

The integral for f_1 can be expanded just as in the proof of the Taylor series to get $f_1(w) = \sum_{n=0}^{\infty} c_n (w - a)^n$, where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho_1} \frac{f(z)}{(z - a)^{n+1}} \, dz \quad \text{for all } n \geq 0. \tag{11}$$

For the f_2 integral use the convergent geometric series

$$\frac{-1}{z - w} = \frac{1/(w - a)}{1 - (z - a)/(w - a)} = \sum_{m=1}^{\infty} \frac{(z - a)^{m-1}}{(w - a)^m}$$

which converges uniformly for $|z - a| = \rho_2$, giving $f_2(w) = \sum_{m=1}^{\infty} d_m (w - a)^{-m}$ where

$$d_m = \frac{1}{2\pi i} \int_{|z-a|=\rho_2} \frac{f(z)}{(z - a)^{-m+1}} \, dz \quad \text{for all } m \geq 1. \tag{12}$$

Writing $n = -1$ then gives the series expansion (i).

To show (ii) and (iii), suppose that we have any convergent series (10) on A, and let $r < \rho' \leq \rho < R$. Then the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ must have radius of convergence $\geq R$, so converges uniformly on $\{|z-a| \leq \rho\}$. Likewise, putting $u = 1/(z-a)$, the series $\sum_{n=1}^{\infty} c_{-n}u^n$ must have radius of convergence $\geq 1/\rho'$, so converges uniformly on $\{|u| \leq 1/\rho'\}$. So (10) converges uniformly on $\{\rho' \leq |z-a| \leq \rho\}$ and therefore in particular can be integrated term-by-term along any curve in this set; so

$$\int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{m+1}}\, dz = \sum_{n=-\infty}^{\infty} c_n \int_{|z-a|=\rho} (z-a)^{n-m-1}\, dz = 2\pi i c_m.$$

□

Remark. Note that the proof of this result shows in particular that if f is holomorphic on the annulus A, then $f = f_1 + f_2$ where f_1 is holomorphic for $|z-a| < R$ and f_2 is holomorphic for $|z-a| > r$.

The theorem in particular applies when a function has an *isolated singularity*, that is f is holomorphic on $D(a,R) \setminus \{a\}$ (a *punctured disc*). For such a function f, let $\sum c_n(z-a)^n$ be its Laurent expansion. There are three cases:

(i) $c_n = 0$ for all $n < 0$. In this case the Lauent expansion is just a power series, so converges on the (unpunctured) disc $D(a,R)$, and defines an analytic function on $D(a,R)$. We say f has a *removable singularity* at $z = a$. Typically this arises when f is given by some formula which is not well-defined at $z = a$; for example, take $a = 0$ and $f(z) = (e^z - 1)/z$.

(ii) There exists $k > 0$ such that $c_r \neq 0$ but $c_n = 0$ for all $n < -k$. In this case we say f has a *pole of order* k at $z = a$. Example: e^z/z^{11}.

(iii) None of the above: $c_n \neq 0$ for infinitely many negative n. We say f has an *essential singularity* at $z = a$.

Let's explore this trichotomy further. We assume $f\colon D(a,R) \setminus \{a\} \to \mathbb{C}$ is holomorphic.

Proposition 3.3.2. *f has an isolated singularity at $z = a$ iff $(z-a)f(z) \to 0$ as $z \to a$.*

Proof. If it has an isolated singularity then the Laurent series for $(z-a)f(z)$ is a power series with zero constant term, so vanishes at $z = a$. In the other direction, if $(z-a)f(z) \to 0$, consider the function g on $D(a,R)$ given by

$$g(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a. \end{cases}$$

Then g is holomorphic on $D(a,R)$ with $g'(a) = \lim_{z\to a}(z-a)f(z) = 0$ and $g(a) = 0$, so has a power series expansion $\sum_{n\geq 2} c_n(z-a)^n$. Dividing by $(z-a)^2$ shows that the Laurent series for f is a power series. □

Proposition 3.3.3. *f has a pole at $z = a$ iff $|f(z)| \to \infty$ as $z \to a$. Moreover, TFAE:*

(i) f has a pole of order k at $z = a$.

(ii) $f = (z-a)^{-k}g(z)$ where $g\colon D(a,R) \to \mathbb{C}$ is homolomorphic and $g(a) \neq 0$.

(iii) $f(z) = 1/h(z)$ where h is holomorphic at $z = a$ with a zero of order k.

Proof. First prove (i) $\iff$ (ii). Given f with a pole, multiplying the Laurent series by $(z-a)^k$ gives a power series with non-zero constant term, defining g, and the converse is clear. series for f with $(z-a)^{-k}$ times the Taylor series for g.

Next, (ii) $\iff$ (iii), since g is holomorphic at $z = a$ with $g(a) \neq 0$ iff $1/g$ is holomorphic at $z = a$.

Finally, suppose f has a pole at $z = a$. Then by (ii), $|f| \to \infty$ at $z = a$. Conversely, if $|f| \to \infty$ at $z = a$, then for some $r > 0$, f is non-zero for $0 < |z-a| < r$. Therefore $1/f$ is holomorphic for $0 < |z-a| < r$ and $1/f \to 0$ at $z = a$. By the previous proposition, $1/f$ has a removable singularity at $z = a$. So there is a function h, holomorphic on $D(a,r)$, with $1/f(z) = h(z)$ for $0 < |z-a| < r$. As $1/f \to 0$ at $z = a$, h has a zero at $z = a$. □

Combining these gives:

Proposition 3.3.4. *f has an essential singularity at $z = a$ iff $|f|$ has no limit (in $\mathbb{R} \cup \{\infty\}$) as $z \to a$.*

In fact even more is true:

Theorem 3.3.5 (Casorati-Weierstrass). *Let $f\colon D(a,R) \to \mathbb{C}$ have an essential singularity at $z = a$. Then for any $w \in \mathbb{C}$ and any $r, \epsilon > 0$, there exist z with $0 < |z-a| < r$ and $|f(z) - w| < \epsilon$.*

This is easy to prove (example sheet). Much harder is the "big Picard theorem":

Theorem 3.3.6. *Let f have an essential singularity at $z = a$. Then there exists $b \in \mathbb{C}$ such that, for any $w \neq b$ and $r > 0$ there exists z with $0 < |z-a| < r$ and $f(z) = w$.*

In other words, in any neighbourhood of an essential singularity, an analytic function misses at most one value. (To see why one value can be skipped, consider the function $\exp(1/z)$ which is never 0.)

For one point of view, a pole is not a singularity at all. Consider the *Riemann sphere* $\mathbb{C}\cup\{\infty\}$, usually denoted $\mathbb{CP}^1$ or $\widehat{\mathbb{C}}$. A holomorphic function f on $D(a,R)\setminus\{a\}$ with a pole at $z=a$ defines a function $\hat{f}\colon D(a,R)\to\mathbb{CP}^1$, which is said to be holomorphic. So the only "genuine" singularities are essential singularities (hence the name).

If D is a domain and $S\subset D$ is set of isolated points in D, then function $f\colon D\setminus S\to\mathbb{C}$ with at worst poles[12] at the points in S is said to be *meromorphic.*

Definition. Let $f\colon D(a,R)\setminus\{a\}\to\mathbb{C}$ be holomorphic with Laurent expansion $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$. The *residue* of f at $z=a$ is the number $\operatorname{Res}_{z=a} f(z)=c_{-1}\in\mathbb{C}$. The *principal part* of f at $z=a$ is the series $\sum_{n=-\infty}^{-1} c_n(z-a)^n$.

Proposition 3.3.7. *If γ ia a closed curve in $D(a,R)\setminus\{a\}$ then*

$$\int_\gamma f(z)\,dz = 2\pi i I(\gamma;a)\operatorname{Res}_{z=a} f(z).$$

In particular, $\int_{|z-a|=r} f(z)\,dz = 2\pi i\operatorname{Res}_{z=a} f(z)$.

Proof. Using uniform convergence of the Laurent expansion this reduces to the computation of $\int_\gamma (z-a)^n\,dz$, which equals $2\pi i I(\gamma;a)$ if $n=-1$, and equals zero otherwise (since then $(z-a)^{n+1}/(n+1)$ is an antiderivative). □

The situation is simplest when f has a pole of order k at $z=a$. Then its principal part $P(z)$ at $z=a$ is just a *polynomial* in $1/(z-a)$ of degree k with no constant term, so defines an analytic function on $\mathbb{C}\setminus\{a\}$ which vanishes at infinity.[13] Moreover, the difference $f-P$ has a removable singularity at $z=a$.

Remark. (Can be omitted at first reading.) For a general singularity, if $P_a f$ is the principal part of f at $z=a$ then $P_a f = h(u)$ where h is a power series in $u=1/(z-a)$ with no constant term, and which converges for all u with $|u|>1/R$. So it converges for all $u\in\mathbb{C}$, and vanishes at $u=0$. In particular, the series for $P_a f$ defines an analytic function on $\mathbb{C}\setminus\{a\}$ which vanishes at infinity.

Now suppose now that f is meromorphic on D, and that $\{a_i,\ldots,a_m\}$ are poles of f in D (not necessarily all of them). Let f_i be the principal part of f at $z=a_i$. Then $g=f-\sum_i f_i$ is meromorphic on D with removable singularities at $z=a_i$. This is important because we can then prove:

Theorem 3.3.8 (Residue Theorem)**.** *Let f be meromorphic on D. Let γ be a closed curve in D, which is homologous to zero in D. Assume that f has no*

[12] i.e. poles or removable singularities

[13] informal language for "tends to zero as $|z|\to\infty$".

poles on γ, and has a finite number of poles in D with $\{a_1, \ldots, a_m\}$ for which $I(\gamma; a_i) \neq 0$. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{m} \operatorname{Res}_{z=a_i} f(z).$$

Notice that this includes Cauchy's theorem and the Cauchy Integral Formula as special cases.

Proof. Let $g = f - \sum f_i$ as above. Then by Cauchy's Theorem, $\int_\gamma g(z)\,dz = 0$, and from Proposition 3.3.7, $\int_\gamma f_i(z)\,dz = 2\pi i I(\gamma; a_i) \operatorname{Res}_{z=a_i} f_i(z) = 2\pi i I(\gamma; a_i) \operatorname{Res}_{z=a_i} f(z)$. $\square$

Remark. One can show that the set of poles $w \in D$ with $I(\gamma; w) \neq 0$ is always finite, if γ is homologous to zero.
In fact, let $V = \{w \in \mathbb{C} \mid I(\gamma; w) = 0\}$. Then $V \subset \mathbb{C}$ is open (by continuity of winding number, see Ex.II.15, and contains a set of the form $\{|z| > R\}$ by 3.1.3. Also since γ is homologous to zero in D, $V \cup D = \mathbb{C}$. So the complement $K = \mathbb{C} \backslash V$ of V is a compact (closed and bounded) subset of D. Since f only has isolated singularities in D, by Bolzano–Weierstrass only finitely many or them lie in K.

This result is useful as a theoretical tool and also for calculation (examples will come later). For the latter, it is convenient to have another formulation which does not involve winding number. The "traditional" formulation of Cauchy's theorem is: *let f be holomorphic on and within a simple closed curve γ; then $\int_\gamma f(z)\,dz = 0$.* This begs the question of what the phrase "inside" γ means. The answer is supplied by the *Jordan curve theorem*: if γ is a simple closed curve, then its complement is the disjoint union of two domains, exactly one of which is bounded. The bounded domain determined by γ (the "inside" of γ) is moreover simply-connected. It's not necessary to prove this (and the general proof is quite hard) since we can use winding number to finesse it:

Definition. A cycle γ bounds a domain D if $I(\gamma; w) = 1$ for every $w \in D$ and $I(\gamma; w) = 0$ for all $w \notin D \cup \gamma$.

Suppose γ is a closed curve or cycle which bounds a domain D. Let f be a function which is holomorphic on $D \cup \gamma$, meaning that there is an open set U containing $D \cup \gamma$ on which f is defined and homorphic. Then γ is homologous to zero in U (by definition). So from the earlier results we get:

Theorem 3.3.9. *Let γ bound a domain D.*

(i) [Cauchy's theorem and integral formula] Let f be holomorphic on $D \cup \gamma$. Then $\int_\gamma f(z)\,dz = 0$, and for every $w \in D$,

$$\int_\gamma \frac{f(z)}{z-w}\,dz = 2\pi i f(w).$$

(ii) [Residue theorem] Let f be meromorphic on $D \cup \gamma$, with no poles on γ. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_w \operatorname{Res}_{z=w}(f)$$

where the sum is over all poles of f in D.

3.4 Evaluation of definite integrals

As a break from the string of theoretical results, will use the residue theorem to compute some integrals.

Typical examples:

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\,d\theta; \qquad \int_0^{2\pi} \frac{\cos 11\theta}{(5+4\cos\theta)^2}\,d\theta$$

For computing definite integrals, need to be able to compute residues effectively. Summarise:

(i) If f has a simple pole at $z = a$ then the Laurent expansion is $c_{-1}(z-a)^{-1} + c_0 + \ldots$, so $\operatorname{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)f(z)$.

(ii) If $f = g/h$ where g, h are holomorphic at $z = a$, $g(a) \neq 0$ and h has a simple zero, then $\operatorname{Res}_{z=a} f(z) = g(a)/h'(a)$.

(iii) If $f = (z-a)^{-k}g(z)$ with g holomorphic, then $\operatorname{Res}_{z=a} f(z) =$ the coefficient of $(z-a)^{k-1}$ in the Taylor series of g, which is $f^{(k-1)}(a)/(k-1)!$.

Unfortunately there is no easy analogue of (ii) for poles of higher order.

The other main class of integrals which can be evaluated using complex integration:

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx \quad m \in \mathbb{R}; \qquad \int_0^{\infty} \frac{\sin x}{x}\,dx$$

Lemma 3.4.1. *Let f be holomorphic on $D(a,R) - \{a\}$ with a simple pole at $z = a$. for $0 < r < R$ let $\gamma_r \colon [\alpha, \beta] \to \mathbb{C}$ be the path $t \mapsto a + re^{it}$. Then*

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\,dz = (\beta - \alpha)i.$$

Proof. Write $f(z) = c/(z-a) + g(z)$ where g is holomorphic on $D(a, R)$. Then

$$\left|\int_{\gamma_r} g(z)\, dz\right| \leq (\beta - \alpha) r \sup |g| \to 0 \quad \text{as } r \to 0$$

and so

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\, dz = \int_{\gamma_r} \frac{c}{(z-a)}\, dz = (\beta - \alpha) i.$$

□

Lemma 3.4.2 (Jordan's Lemma)**.** *If for some $r > 0$, f is holomorphic on $\{|z| > r\}$ and $zf(z)$ is bounded, then for any $\alpha > 0$,*

$$\int_{\gamma_R} f(z) e^{i\alpha z}\, dz \to 0 \text{ as } R \to \infty$$

where $\gamma_R :: [0, \pi] \to \mathbb{C}$, $\gamma_R(t) = Re^{it}$.

Proof. By hypothesis, there exists C such that $|f(z)| \leq C/|z|$ for $|z|$ sufficiently large. Recall $\sin t \geq 2t/\pi$ for $t \in [0, \pi/2]$. Then if $z = Re^{it}$, $0 \leq t \leq \pi$,

$$\left|e^{i\alpha z}\right| = e^{-\alpha R \sin t} \leq \begin{cases} e^{-\alpha R \pi t/2} & \text{for } 0 \leq t \leq \pi/2 \\ e^{-\alpha R \pi t'/2} & \text{for } 0 \leq t' = \pi - t \leq \pi/2 \end{cases}$$

The absolute value of the part of the integral for $t \in [0, \pi/2]$ is then

$$\left|\int_0^{\pi/2} e^{i\alpha Re^{it}} f(Re^{it})\, iRe^{it} dt\right| \leq \int_0^{\pi/2} e^{-\alpha R t} C\, dt = \frac{1}{\alpha R}(1 - e^{-\alpha R \pi/2}) \to 0$$

A similar calculation bounds the other part of the integral. □

Remark. In practice one can always avoid Jordan's lemma just by performing a simple integration by parts.

Another example:

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\, dx, \quad 0 < \alpha < 1$$

3.5 The argument principle; Rouché's theorem

Proposition 3.5.1. *Let f have a zero (pole) of order $k > 0$ at $z = a$. Then $f'(z)/f(z)$ has a simple pole (order 1) at $z = a$, with residue k (respectively $-k$).*

Proof. If f has a zero of order k at $z = a$ then $f(z) = (z - a)^k g(z)$ where g is holomorphic and non-zero at $z = a$. Then

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$$

whence the result. For a pole we have $f(z) = (z - a)^{-k} g(z)$ and proceed in the same way. □

Theorem 3.5.2 (Argument principle)**.** *Let γ be a closed curve (or cycle) bounding a domain D, and let f by meromorphic on $D \cup \gamma$. Assume that f has no zeroes and poles on γ, and N zeroes and P poles in D (counted with multiplicity). Then*

$$N - P = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = I(\Gamma; 0)$$

where $\Gamma = f \circ \gamma$ is the image of γ under the mapping f.

Proof. Notice that Γ lies in $\mathbb{C} \setminus \{0\}$ since f has no zero or pole on γ. So writing $w = f(z)$, we have

$$I(\Gamma; 0) = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

Now apply the residue theorem to $f'(z)/f(z)$ and the previous proposition. □

The name of this theorem can be explained as follows. We suppose that $\gamma\colon [0,1] \to \mathbb{C}$ is a closed curve. Then the theorem says that $2\pi(N - P)$ is the change in argument of $f(z)$ as z traces γ.

This has an important consequence. If f is nonconstant and holomorphic at $z = a$ and $f(a) = b$, we say that the *local degree* of f at $z = a$ is the order of the zero of $f(z) - b$ at $z = a$, and denote it $\deg_{z=a} f(z)$; it is a positive integer.

Proposition 3.5.3. *The local degree of f at $z = a$ equals the winding number $I(f \circ \gamma, f(a))$ for any circle $\gamma(t) = a + re^{2\pi i t}$, $t \in [0, 1]$ of sufficiently small radius.*

Proof. Apply the argument principle to $f(z) - f(a)$. As it has isolated zeroes, it is nonzero for $0 < |z - a| \le r$ if r is sufficiently small. □

Theorem 3.5.4 (Local mapping degree)**.** *Let $f\colon D(a, R) \to \mathbb{C}$ be holomorphic and nonconstant, with local degree $\deg_{z=a} f(z) = d > 0$. Then if $r > 0$ is sufficiently small, there exists $\epsilon > 0$ such that, for every w with $|w - f(a)| \le \epsilon$, the equation $f(z) = w$ has exactly d roots in $D(a, r)$.*

Proof. Let $b = f(a)$, and let $r > 0$ be such that $f(z) - b$ and $f'(z)$ are both nonzero for $0 < |z - a| \leq r$. Let γ be the circle with centre $z = a$ and radius r. Then $\Gamma = f \circ \gamma$ is a closed curve not containing b. Choose $\epsilon > 0$ such that Γ does not meet $D(b, \epsilon)$. Then if $w \in D(b, \epsilon)$, the number of zeroes (counted according with multiplicity) of $f(z) - w$ in $D(a, r)$ equals $I(\Gamma; w)$ by the argument. But $I(\Gamma; w) = I(\Gamma; b) = d$. Since r was chosen such that f' is nonzero on $D^0(a, r)$, the zeroes are all simple. □

Corollary 3.5.5 (Open mapping theorem)**.** *A non-constant holomorphic function $f \colon D \to \mathbb{C}$ maps open sets to open sets.*

Proof. It's enough to show that for every $a \in D$ and every sufficiently small $r > 0$, there exists $\epsilon > 0$ such that $f(D(a, r)) \supset D(f(a), \epsilon)$, which follows at once from the previous result. □

Theorem 3.5.6 (Rouché's Theorem)**.** *Let γ bound a domain D, and let f and g be holomorphic on $D \cup \gamma$. If $|f| > |g|$ on γ then f and $f + g$ have the same number of zeroes in D.*

Note that as $|f| < |g|$ on γ the f and $f + g$ are never zero on γ.

Proof. It suffices to show that $h = 1 + (g/f)$ has the same numbers of zeroes as it has poles in D. By the Argument Principle, that holds iff $I(h \circ \gamma; 0) = 0$. But the hypothesis implies that $h(z) \in D(1, 1)$ for all z on γ, hence $h \circ \gamma$ is contained in $D(1, 1)$ and so by proposition 3.1.3 $I(h \circ \gamma; 0) = 0$. □

Typical practical application is to determine the approximate location of the zeroes of (say) a polynomial $P(z)$ of degree $d \geq 1$. By the Fundamental Theorem of Algebra we know P has d zeroes in $\mathbb{C}$, and by simple estimates we can get an upper bound for their size; with Rouché's Theorem can often do better.

Ex: $P(z) = z^4 + 6z + 3$. Then on the circle $|z| = 2$ we have $16 = |z^4| > 15 = 6|z| + 3 \geq |6z + 3|$, so by Rouché's Theorem, z^5 and $P(z)$ have the same number of zeroes with $|z| < 2$. So all the zeroes of $P(z)$ satisfy $|z| < 2$. But also if $|z| =$ then $6 = |6z| > 4 \geq |z^4 + 3|$, so $P(z)$ and $6z$ have the same number of zeroes with $|z| < 1$. So $P(z)$ has one zero with $|z| < 1$ and three zeroes with $1 < |z| < 2$.

We can also count zeroes in a half-plane by using a semicircular path. See the example sheet for examples of this and other kinds.

3.6 Uniform limits of analytic functions

Definition. Let $U \subset \mathbb{C}$ be an open set, and $f_n \colon U \to \mathbb{C}$ a sequence of functions. We say $\{f_n\}$ is *locally uniformly convergent on U* if for every $a \in U$ there exists an [open] neighbourhood $B(a, r) \subset U$ on which $\{f_n\}$ is uniformly convergent.

Example: The sequence of functions $f_n = 1/(1-z^n)$ is locally uniformly on $B(0,1)$ but is not uniformly convergent on $D(0,1)$. (It is uniformly convergent on every $D(a,r)$ with $r < 1$.)

Theorem 3.6.1. *A sequence of functions $f_n \colon U \to \mathbb{C}$ is locally uniformly convergent if and only if it converges uniformly on all compact subsets of U.*

Proof. Suppose $f_n \to f$ uniformly on compact subsets. Then in particular if $a \in U$ the sequence $\{f_n\}$ converges uniformly on any closed neighbourhood $\overline{B}(a,r) \subset U$ of a in U, so $\{f_n\} \to f$ locally uniformly.
Conversely, suppose $\{f_n\}$ converges locally uniformly on U, and let $K \subset U$ be a compact subset. For each $a \in K$ there exists an open neighborhood $B(a, r_a) \subset U$ on which $\{f_n\}$ converges uniformly. As K is compact, there is a finite subset $S \subset U$ such that $\bigcup_{a \in S} B(a, r_a) \supset K$. Hence $\{f_n\}$ converges uniformly on K. □

We now have a generalisation of Theorem 1.3.2:

Theorem 3.6.2. *Let $\{f_n\}$ be a sequence of analytic functions on U which is locally uniformly convergent. Then the limit function f is analytic, and the sequence $\{f_n'\}$ converges locally uniformly to f'.*

Proof. Let $D = B(a,r) \subset U$ be any disc. Then by Cauchy's theorem, for any closed curve γ in D, $\int_\gamma f_n(z)\,dz = 0$. Now $f_n \to f$ uniformly on γ (since γ is compact) so f is continuous and

$$\int_g f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0$$

(by Proposition 2.1.3) and so by Morera's theorem f is holomorphic on D.
Next, by Cauchy's formula, for any $w \in B(a, r/2)$

$$\begin{aligned} |f'(w) - f_n'(w)| &= \frac{1}{2\pi} \left| \int_{|z-a|=r} \frac{f(z) - f_n(z)}{(z-w)^2}\,dz \right| \\ &\le \frac{r \sup_{|z-a|=r} |f(z) - f_n(z)|}{r^2/4}. \end{aligned}$$

By the first part and Theorem 3.6.1 $f_n \to f$ uniformly on $\{|z-a| = r\}$, and therefore $f_n' \to f'$ uniformly on $B(a, r/2)$. □

Example 1: consider the series ($z \in \mathbb{C} \setminus \mathbb{Z}$)

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

which converges by comparison with $\sum 1/n^2$. If $w \subset \mathbb{C} \setminus \mathbb{Z}$ choose $r > 0$ such that $|w - n| \geq 2r$ for every $n \in \mathbb{Z}$. Then for all $z \in D(w, r)$, $|z - n| \geq \max\{r, |n| - |w| - r\}$ and so

$$\left|\frac{1}{(z-n)^2}\right| \leq \min\left\{\frac{1}{r^2}, \frac{1}{(n-|w|-r)^2}\right\} = M_n \quad \text{say}$$

and as $\sum_{n=1}^{\infty} M_n$ converges, the series defining f is uniformly convergent for $|z - w| \leq r$. So it is locally uniformly convergent, so $f(z)$ is analytic on $\mathbb{C} \setminus \mathbb{Z}$. At $z = n \in \mathbb{Z}$ is clearly has a double pole with principal part $(z - n)^{-2}$. Equally clearly, $f(z + 1) = f(z)$.

Consider now the function $g(z) = \pi^2 \operatorname{cosec}^2 \pi z = (\pi/\sin \pi z)^2$, which is analytic on $\mathbb{C} \setminus \mathbb{Z}$. At $z = n \in \mathbb{Z}$ it clearly has a double pole. We have $\lim_{z\to 0} z^2 g(z) = (\lim_{z\to 0} \sin(\pi z)/\pi z)^{-2} = 1$, and g is even, so its principal part at the origin is z^{-2}. Since $\sin(z + \pi) = -\sin z$, $g(z + 1) = g(z)$ so the principal part at $z = n$ is also $(z - n)^{-2}$.

We show that $f(z) = g(z)$. Since both functions have the same principal parts at every $z = n \in \mathbb{Z}$, we know that $f = g + h$ for an entire function h. We'll show $h = 0$, using Liouville's theorem.[14]

If $z = x \pm iy$ with $|x| \leq 1/2$ and $y > 0$ then

$$|g(z)| = \frac{4\pi^2}{|e^{\pi i x} e^{\pm \pi y} - e^{-\pi i x} e^{\mp \pi y}|^2} \leq \frac{4\pi^2}{|e^{\pi y} - e^{-\pi y}|^2} \to 0 \quad \text{as } y \to \infty$$

and

$$|f(z)| \leq \sum_{n=-\infty}^{\infty} \frac{1}{|x \pm iy - n|^2} \leq \frac{1}{y} + 2\sum_{n=1}^{\infty} \frac{1}{(n-1/2)^2 + y^2} \to 0 \quad \text{as } y \to \infty$$

[14]Here is another way to show $h = 0$. First notice that

$$f\Big(\frac{z}{2}\Big) + f\Big(\frac{z+1}{2}\Big) = \sum_{m\in\mathbb{Z}} \frac{4}{(z-2m)^2} + \frac{4}{(z-2m+1)^2} = \sum_{n\in\mathbb{Z}} \frac{4}{(z-n)^2} = 4f(z)$$

$$g\Big(\frac{z}{2}\Big) + g\Big(\frac{z+1}{2}\Big) = \frac{\pi^2}{\sin^2 \pi z/2} + \frac{\pi^2}{\cos^2 \pi z/2} = \frac{\pi^2}{\sin^2(\pi z/2)\cos^2(\pi z/2)} = 4g(z)$$

and therefore the entire function h satisfies $4h(z) = h(z/2) + h((z + 1)/2)$. Consider any $R > 1$, and let the maximum of $|h|$ on $\{|z| \leq R\}$ be $M = |h(w)|$. Then $|w/2| \leq R$ and $|(w + 1)/2| \leq R$, so

$$4M = |4h(w)| = \left|h\Big(\frac{w}{2}\Big) + h\Big(\frac{w+1}{2}\Big)\right| \leq \left|h\Big(\frac{w}{2}\Big)\right| + \left|h\Big(\frac{w+1}{2}\Big)\right| \leq M + M = 2M$$

which implies that $M = 0$. So for every $R > 1$, h is identically zero on $\{|z| \leq R\}$, hence $h = 0$. This ingenious argument is the **Herglotz trick**.

Since $f(z) = f(z+1)$ and $g(z+1) = g(z)$ the same estimates hold for any $x \in \mathbb{R}$. This implies that $h(x+iy) \to 0$ as $|y| \to \infty$, uniformly in x. Now for any $Y > 0$, $h(z)$ is bounded on the rectangle $\{z = x + iy \mid |x| \leq 1/2, |y| \leq Y\}$, and so as $h(z+1) = h(z)$ is also bounded on the strip $\{z = x+iy \mid |y| \leq Y\}$. So this means that h is bounded on $\mathbb{C}$, hence is constant, and as $h(z) \to 0$ for $\mathrm{Im}(z) \to \infty$ the constant is zero. We have therefore obtained the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

From this further series expansions may be obtained. For example, we have $(d/dz)(\pi \cot \pi z) = -\pi^2 \operatorname{cosec}^2 \pi z$. On the other hand we can also consider the series

$$f_1(z) = \frac{1}{z} + \sum_{0 \neq n=-\infty}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

(the extra terms $1/n$ are required since $\sum 1/(z-n)$ is divergent). which converges by comparison with $\sum 1/n^2$. A similar calculation as for $f(z)$ shows that the series is locally uniformly convergent, hence $f_1(z)$ is analytic on $\mathbb{C} \backslash \mathbb{Z}$, and differentiating term-by-term gives $df_1/dz = -f = -\pi^2 \operatorname{cosec}^2 \pi z$. Therefore $f_1(z) - \pi \cot \pi z$ is constant, and since both functions are odd, the constant must be 0. Therefore

$$\pi \cot \pi z = \frac{1}{z} + \sum_{0 \neq n=-\infty}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

Consequence:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{2} \lim_{z \to 0} \left(\frac{\pi^2}{\sin^2 \pi z} - \frac{1}{z^2} \right) = \frac{1}{2} \lim_{z \to 0} \left(\frac{\pi^2 z^2 - \sin^2 \pi z}{z^2 \sin^2 \pi z} \right) \\ &= \frac{1}{2} \lim_{z \to 0} \left(\frac{\pi^2 z^2 - (\pi^2 z^2 - \pi^4 z^4/3 + \dots)}{\pi^2 z^4 - \dots} \right) = \frac{\pi^2}{6} \end{aligned}$$

We can also obtain the infinite product for the sine function:

$$\sin \pi z = \pi z \prod_{n-1}^{\infty} \left(1 - \frac{z}{n}\right) \left(1 + \frac{z}{n}\right) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

(first written by Euler) by taking logarithmic derivatives of each side.

Example 2: the Gamma function.

Consider the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \, dt = \int_0^{\infty} \phi(z,t) \, dt. \tag{13}$$

Theorem 3.6.3. *The integral (*) defines a holomorphic function on the right half-plane* $\{\mathrm{Re}(z) > 0\}$*, which can be analytically continued to a meromorphic function* Γ *on* $\mathbb{C}$ *whose only poles are simple poles at* $z = -m = 0, -1, -2, \ldots$ *with residue* $(-1)^m/m!$*. It satisfies* $\Gamma(z) = (z-1)\Gamma(z-1)$ *and for every* $n \geq 1$, $\Gamma(n) = (n-1)!$.

Proof. For every $z \in \mathbb{C}$, the integral $\int_1^\infty \phi(z,t)\,dt$ converges. If $z = x + iy$ then for $0 < t \leq 1$, then $|e^{-t}t^{x-1}| \leq t^{x-1}$ and $\int_0^1 t^{x-1}\,dt$ converges for $x > 0$. So the integral defining $\Gamma(z)$ converges on the right half-plane $\{z \mid \mathrm{Re}(z) > 0\}$.

Claim $\Gamma(z)$ is holomorphic on the right half-plane. To see this, let for $N \in \mathbb{Z}$, $N \geq 1$

$$f_N(z) = \int_{1/N}^{N} e^{-t}t^{z-1}\,dt.$$

Then since $t^{z-1} = \exp(\ln t)(z-1)$ is holomorphic for any $t > 0$, each f_N is holomorphic on $\mathbb{C}$. Let $\delta > 0$. I claim that $f_N \to \Gamma$ uniformly on $\{z \mid \mathrm{Re}(z) \geq \delta\}$. To see this, it's enough to check that

$$\int_0^{1/N} e^{-t}t^{z-1}\,dt \to 0, \quad \int_N^\infty e^{-t}t^{z-1}\,dt \to 0 \quad \text{uniformly as } N \to \infty.$$

So $\Gamma(z)$ is holomorphic fo $\mathrm{Re}(z) > 0$.

Integrating by parts gives $\Gamma(z) = (z-1)\Gamma(z-1)$ if $\mathrm{Re}(z) > 1$. This shows that $\Gamma(n) = (n-1)!$ since clearly $\Gamma(1) = 1$. It also shows that for any $N \geq 0$, the function

$$\frac{\Gamma(s+N+1)}{s(s+1)\cdots(s+N)} \tag{$*$}$$

equals $\Gamma(s)$. But $(*)$ defines a meromorphic function on $\{\mathrm{Res}(s) > -N\}$, with simple poles at $s = -m = 0, -1, \ldots, -N$. So (by uniqueness of analytic continuation) $\Gamma(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, and $(*)$ shows that

$$\mathrm{Res}_{s=-N}\,\Gamma(s) = \frac{\Gamma(1)}{(-1)(-2)\cdots(-N)} = \frac{(-1)^N}{N!}$$

□

Remark. One can also show that $\Gamma(s)$ is never zero, and that its inverse has an infinite product expansion

$$1/\Gamma(s) = e^{\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

where γ is *Euler's constant*

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n)\right).$$

Carefully comparing this with the infinite product for $\sin \pi z$ gives Euler's identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi z}$$

Example: The Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Since $|n^s| = n^{\mathrm{Re}(s)}$, the series converges absolutely for $\mathrm{Re}(s) > 1$ and uniformly for $\mathrm{Re}(s) \geq \sigma$, for any $\sigma > 1$ (by comparison with $\sum n^{-\sigma}$). It is therefore an analytic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

Now since $\lim_{x\to 1+} \sum_{n=1}^{\infty} n^{-x} = \infty$, it's clear that $\zeta(s)$ cannot be analytically continued to any neighbourhood of $s = 1$. Rather surprisingly, however, it can be analytically continued to the rest of $\mathbb{C}$.

Theorem 3.6.4. *The function $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$. Moreover for every integer $k \geq 1$*

$$\zeta(1-k) = (-1)^{k-1}\frac{B_k}{k} \in \mathbb{Q}$$

where the Bernoulli numbers B_k *are defined by*

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

One has $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = B_5 = B_{2r+1} = 0$, $B_{12} = -691/2370$. Bernoulli numbers crop up all over mathematics — particularly in number theory and topology.

Proof. We first compute the following integral representation of $\zeta(s)$. Let $\mathrm{Re}(s) \geq 2$ say. Then (the second equality by substituting nt for t)

$$\begin{aligned}
\Gamma(s)\zeta(s) &= \sum_{n=1}^{\infty} \int_0^{\infty} \frac{t^{s-1}e^{-t}}{n^s}\,dt \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1}e^{-nt}\,dt \\
&= \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\,dt
\end{aligned}$$

Split the integral into $\int_0^1$ and $\int_0^\infty$. Then the integral

$$\int_1^\infty \frac{t^{s-1}}{e^t - 1}\, dt$$

defines an entire function of s (same proof as for the Γ-function integral). For the other piece, let N be a positive integer and expand

$$\frac{t}{e^t - 1} = \sum_{k=0}^{N} B_k \frac{t^k}{k!}. + t^{N+1} F_N(t).$$

Then

$$\begin{aligned}\int_0^1 \frac{t^{s-1}}{e^t - 1}\, dt &= \sum_{k=0}^{N} \int_0^1 B_k \frac{t^{s+k-2}}{k!}\, dt + \int_0^1 t^{s+N-1} F_N(t)\, dt \\ &= \sum_{k=0}^{N} \frac{B_k}{k!(s+k-1)} + (\text{holomorphic for } \mathrm{Re}(s) > 1 - N)\end{aligned}$$

which shows that $\Gamma(s)\zeta(s)$ has a meromorphic continuation to the entire plane with at worst simple poles as $s = 1 - k$, $k = 0, 1, 2, \ldots$, at which its residue is $B_k/k!$. Now since $\Gamma(s)$ has simple poles at $0, -1, -2, \ldots$ and is no-zero elsewhere, we deduce that $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$, and that for all $k \geq 1$,

$$\zeta(1-k) = (B_k/k!)/\operatorname{Res}_{s=1-k} \Gamma(s) = (-1)^{k-1} \frac{B_k}{k}.$$

□

The function $\zeta(s)$ is important because of its connection with number theory. The Fundamental Theorem of Arithmetic show that

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p (1 + p^{-s} + p^{-2s} + \ldots) = \prod_p \frac{1}{1 - p^{-s}}$$

where the product is taken over all primes p. This proves at once that the number of primes is infinite (if not, the product in the above equation would be a finite product and $\zeta(s)$ would then be analytic at $s = 1$.). It is the beginning of a long story of number theory, one of the high point of which is the *Prime Number Theorem*, which says that

$$\#\{\text{primes } p \leq X\} \sim \frac{X}{\log X}.$$