1 Use the residue theorem to give a proof of Cauchy's derivative formula: if $f$ is holomorphic on $D(a, R)$, and $|w - a| < r < R$, then
$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

2 Let $g(z) = p(z)/q(z)$ be a rational function with $\deg(q) \geq \deg(p) + 2$. Show that the sum of the residues of $f$ at all its poles equals zero.

3 Evaluate the following integrals:

$$(a) \quad \int_0^{\pi} \frac{d\theta}{4 + \sin^2\theta} \qquad\qquad (b) \quad \int_0^{\infty} \sin x^2\, dx$$
$$(c) \quad \int_0^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\, dx \qquad\qquad (d) \quad \int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1}\, dx$$

4 For $\alpha \in (-1, 1)$ with $\alpha \neq 0$, compute
$$\int_0^{\infty} \frac{x^{\alpha}}{x^2 + x + 1}\, dx$$

5 Use Rouché's Theorem to give another proof of the Fundamental Theorem of Algebra.

6 Let $p(z) = z^5 + z$. Find all $z$ such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for such $z$. Hence sketch the curve $p \circ \gamma$, where $\gamma(t) = e^{2\pi i t}$ and use your sketch to determine the number of $z$ (counted with multiplicity) such that $|z| < 1$ and $p(z) = x$ for each real number $x$.

7 (i) For a positive integer $N$, let $\gamma_N$ be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every $N$, $|\cot \pi z| < C$ on $\gamma_N$.

(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2 + 1}$ around $\gamma_N$, show that
$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1 + \pi \coth\pi}{2}.$$

(iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n/(n^2+1)$.

8 (i) Show that $z^4 + 26z + 2 = 0$ has exactly three zeroes with $5/2 < |z| < 3$.

(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \mid \operatorname{Re}(z) < 0 \,\}$.

(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{\, z \mid |z| = 3/2 \,\}$.

9 Show that the equation $z \sin z = 1$ has only real solutions.

*[Hint: Find the number of real roots in the interval* $[-(n+1/2)\pi, (n+1/2)\pi]$ *and compare with the number of zeros of* $z \sin z - 1$ *is a square box* $\{|\operatorname{Re} z|, |\operatorname{Im} z| < (n+1/2)\pi\}$. *]*

10 (i) Let $w \in \mathbb{C}$, and let $\gamma$, $\delta\colon [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve $\sigma(t) = \dfrac{\delta(t) - w}{\gamma(t) - w}$ about the origin, show that $I(\gamma; w) = I(\delta; w)$.

(ii) If $w \in \mathbb{C}$, $r > 0$, and $\gamma$ is a closed curve which does not meet $D(w, r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w, r)$.

(iii) Deduce that if $\gamma$ is a closed curve in $\mathbb{C}$ and $U$ is the complement of (the image of) $\gamma$, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on $U$.

*Supplementary examples — these are not part of the examples sheet, but are provided as a starting point for revision, or for the addicted. Myriad examples of integrals may be found in past tripos questions.*

S1 Evaluate the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(a^2+x^2)}\,dx$ where $a,\ m \in \mathbb{R}^+$

(b) $\displaystyle\int_{0}^{2\pi} \frac{\cos^3 3t}{1-2a\cos t+a^2}\,dt$ where $a \in (0,1)$

(c) $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx$ ("dog-bone" contour)

(d) $\displaystyle\int_{-\infty}^{\infty} e^{-ax^2}e^{-itx}\,dx$ where $a>0,\ t\in\mathbb{R}$

(e) $\displaystyle\int_{0}^{\infty} \frac{\cosh ax}{\cosh x}\,dx$ where $a \in (-1,1)$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}e^{-itx}\,dx$ where $t\in\mathbb{R}$

S2 By integrating $z/(a-e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi+iR$, prove that

$$\int_0^{\pi} \frac{x\sin x}{1-2a\cos x+a^2}\,dx = \frac{\pi}{a}\log(1+a)$$

for every $a \in (0,1)$.

S3 Assuming $\alpha \geq 0$ and $\beta \geq 0$ prove that

$$\int_0^{\infty} \frac{\cos\alpha x-\cos\beta x}{x^2}\,dx = \frac{\pi}{2}(\beta-\alpha),$$

and deduce the value of

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$