1 Let $T\colon \mathbb{C} = \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ be a real linear map. Show that there exist unique complex numbers A, B such that for every $z \in \mathbb{C}$, $T(z) = Az + B\bar{z}$. Show that T is complex differentiable if and only if $B = 0$.

2 (i) Let $f : D \to \mathbb{C}$ be an holomorphic function defined on a domain D. Show that f is constant if any one of its real part, imaginary part, modulus or argument is constant.

(ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$ where u and v are both real valued.

(iii) Find all holomorphic functions on $\mathbb{C}$ with real part $x^3 - 3xy^2$.

3 Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f satisfies the Cauchy-Riemann equations at 0 but is not differentiable there.

4 (i) Verify directly that e^z, $\cos z$ and $\sin z$ satisfy the Cauchy-Riemann equations everywhere.

(ii) Find the set of complex numbers z for which $\left|e^{iz}\right| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

(iii) Find the zeros of $1 + e^z$ and of $\cosh z$.

5 (i) Denote by Log the principal branch of the logarithm. If $z \in \mathbb{C}$, show that $n \operatorname{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce that for any $z \in \mathbb{C}$,

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z.$$

(ii) Defining $z^\alpha = \exp(\alpha \operatorname{Log} z)$, where Log is the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $d/dz(z^\alpha) = \alpha z^{\alpha-1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold?

6 Prove that each of the following series converges uniformly on the corresponding subset of $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty} \sqrt{n} e^{-nz} \quad \text{on } \{\, z \mid 0 < r \leq \operatorname{Re}(z) \,\}; \qquad (b)\ \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}} \quad \text{on } \{\, z \mid |z| \leq r < \frac{1}{2} \}.$$

7 Find conformal equivalences between the following pairs of domains:

(i) the sector $\{z \in \mathbb{C} \mid -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc $D(0, 1)$;

(ii) the lune $\{z \in \mathbb{C} \mid |z - 1| < \sqrt{2}$ and $|z + 1| < \sqrt{2}\}$ and $D(0, 1)$;

(iii) the strip $S = \{z \in \mathbb{C} \mid 0 < \operatorname{Im}(z) < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0\}$.

By considering a suitable bounded solution of Laplace's equation $u_{xx} + u_{yy} = 0$ on S, find a non-constant harmonic function on Q which is constant of its boundary axes.

8 (i) Show that the most general Möbius transformation which maps the unit disk onto itself has the form $z \mapsto \lambda \dfrac{z - a}{\bar{a}z - 1}$, with $|a| < 1$ and $|\lambda| = 1$. [*Hint: first show that these maps form a group.*]

(ii) Find a Möbius transformation taking the region between the circles $\{|z| = 1\}$ and $\{|z - 1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: a circle can be described by an equation of the shape* $|z - a|/|z - b| = \ell$.]

(iii) Find a conformal map from an infinite strip onto an annulus. Can such a map ever be a Möbius transformation?

9 Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

10 Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivatives) on the domains indicated:

$$(a)\ \frac{1}{z} - \frac{1}{z-1} \quad (0 < |z| < 1)\,; \qquad (b)\ \frac{z}{1+z^2} \quad (1 < |z| < \infty).$$

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