# ANALYSIS II (Michaelmas 2011): EXAMPLES 4

The questions are not equally difficult. Those marked with * are intended as 'additional'; attempt them if you have time after the first eleven questions. Comments, corrections are welcome at any time and may be sent to `a.j.scholl@dpmms.cam.ac.uk`.

**1.** (i) For each of the following metric spaces $Y$

(a) $Y = \mathbb{R}$, (b) $Y = [0, 2]$, (c) $Y = (1, 3)$, (d) $Y = (1, 2] \cup (3, 4]$,

with metric $d(x, y) = |x - y|$, is the set $(1, 2]$ an open subset of $Y$? Is it closed?

(ii) Suppose that $X$ is a metric space and $A_1, A_2$ are two closed balls in $X$ with radii respectively $r_1, r_2$, such that $r_1 > r_2 > 0$. Can $A_1$ be a proper subset of $A_2$ (i.e. $A_1 \subset A_2$ and $A_1 \neq A_2$)?

**2.** For each of the following sets $X$, determine whether or not the given function $d$ defines a metric on $X$. In each case where the function does define a metric, describe the open ball $B(x; \varepsilon)$ for each $x \in X$ and $\varepsilon > 0$ small.
(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.
(ii) $X = \mathbb{Z}$; $d(x, x) = 0$ and for $x \neq y$, $d(x, y) = 2^n$, where $x - y = 2^n a$ with $n$ a non-negative integer and $a$ an odd integer.
(iii) $X = \mathbb{Q}$; $d(x, x) = 0$ and for $x \neq y$, $d(x, y) = e^{-n}$, where $x - y = 3^n a/b$ for $n$, $a$, $b \in \mathbb{Z}$ with both $a$ and $b$ not divisible by 3.
(iv) $X$ is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$ and for $f \neq g$, $d(f, g) = 2^{-n}$ for the least $n$ such that $f(n) \neq g(n)$.
(v) $X = \mathbb{C}$; $d(z, z) = 0$ and for $z \neq w$, $d(z, w) = |z| + |w|$.
(vi) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if $z$ and $w$ lie on the same straight line through the origin, $d(z, w) = |z| + |w|$ otherwise.

**3.** Let $d$ and $d'$ denote the usual and discrete metrics respectively on $\mathbb{R}$. Show that all functions $f$ from $\mathbb{R}$ with metric $d'$ to $\mathbb{R}$ with metric $d$ are continuous. What are the continuous functions from $\mathbb{R}$ with metric $d$ to $\mathbb{R}$ with metric $d'$?

**4.** (a) Show that if $Y$ is a subset of a metric space $X$, there is a unique closed subset $Z$ of $X$ such that $Z$ contains $Y$ and any closed subset of $X$ containing $Y$ also contains $Z$. The set $Z$ is called the *closure* of $Y$ in $X$, denoted $\overline{Y}$ or $cl(Y)$.
(b) Show that

$$cl(Y) = \{x \in X : x_n \to x \text{ for some sequence } (x_n) \text{ in } Y\}.$$

**5.** Let $V$ be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B(x; r)$ is the closed ball $\{y \in V : \|x - y\| \leq r\}$. Give an example to show that, in a general metric space $(X, d)$, the closure of the open ball $B(x; r)$ need not be the closed ball $\{y \in X : d(x, y) \leq r\}$.

**6.** Show that the space of real sequences $a = (a_n)$, such that all but finitely many of the $a_n$ are zero, is not complete in the norm defined by $\|a\|_1 = \sum_{n=1}^{\infty} |a_n|$. Is there an obvious 'completion'?

**7.** Use the Contraction Mapping Theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using a pocket calculator or the calculator on your computer (remember to work in radians!), and justify the claimed accuracy of your approximation.

**8.** Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on $I$. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x \in I} |f(x)e^{-\alpha x}|$, and that the norm $\|\cdot\|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x \in I} |f(x)|$.

Now suppose that $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable $|\varphi(t, x) - \varphi(t, y)| \leq K|x - y|$, for all $t, x, y \in \mathbb{R}$. Consider the map $T$ from $C(I)$ to itself sending $f$ to $y_0 + \int_0^x \varphi(t, f(t))dt$. Give an example to show that $T$ need not be a contraction under the uniform norm. Show, however, that $T$ is a contraction under the norm $\|\cdot\|_\alpha$ for some $\alpha$, and deduce that the differential equation $f' = \varphi(x, f(x))$ has a unique solution on $I$ satisfying $f(0) = y_0$.

**9.** Let $(X, d)$ be a non-empty complete metric space. Suppose $f : X \to X$ is a contraction and $g : X \to X$ is a function which commutes with $f$, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that $g$ has a fixed point. Must this fixed point be unique?

**10.** Give an example of a non-empty complete metric space $(X, d)$ and a function $f : X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, but such that $f$ has no fixed point. Suppose now that $X$ is a non-empty closed bounded subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case $f$ must have a fixed point. If $g : X \to X$ satisfies $d(g(x), g(y)) \leq d(x, y)$ for all $x, y \in X$, must $g$ have a fixed point?

**11.** (i) Suppose that $(X, d)$ is a non-empty complete metric space, and $f : X \to X$ a continuous map such that, for any $x, y \in X$, the sum $\sum_{n=1}^{\infty} d(f^n(x), f^n(y))$ converges. ($f^n$ denotes the function $f$ applied $n$ times.) Show that $f$ has a unique fixed point.
(ii) By considering the function $x \mapsto \max\{x - 1, 0\}$ on the interval $[0, \infty) \subset \mathbb{R}$, show that a function satisfying the hypotheses of (i) need not be a contraction mapping.
(iii) Give an example to show that the result of (i) need not be true if $f$ is not assumed to be continuous.

**12.*** (i) Let $(X, d)$ be a metric space. For a nonempty subset $Y \subset X$ and $x \in X$ define

$$d(x, Y) = \inf_{y \in Y} d(x, y).$$

Show that for fixed $Y$, the function $x \mapsto d(x, Y)$ defines a continuous function on $X$, and determine the subset of $X$ on which it vanishes.
(ii) For $Y$, $Z \subset X$ nonempty, define

$$d(Y, Z) = \inf_{y \in Y} d(y, Z).$$

Show that if $Y$ and $Z$ are closed subsets of $\mathbb{R}^n$, and at least one of $Y$, $Z$ is bounded, then $d(Y, Z) > 0$ iff $Y$ and $Z$ are disjoint. Show that this conclusion can fail if the boundedness condition is removed.

**13.*** A metric $d$ on a set $X$ is called an *ultrametric* if it satisfies the following stronger form of the triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

Which of the metrics in question 2 are ultrametrics? Show that in an ultrametric space 'every triangle is isosceles' (that is, at least two of $d(x, z)$, $d(y, z)$ and $d(x, y)$ must be equal), and deduce that every open ball in an ultrametric space is a closed set. Does it follow that every open set must be closed?

**14.*** There is (rumoured to be) a persistent 'urban myth' about the mathematics research student who spent three years writing a thesis about properties of 'antimetric spaces', where an *antimetric* on a set $X$ is a function $d : X \times X \to \mathbb{R}$ satisfying the same axioms as a metric except that the triangle inequality is reversed (i.e. $d(x, z) \geq d(x, y) + d(y, z)$ for all $x, y, z$). Why would such a thesis be unlikely to be considered worth a Ph.D.?

**15.*** Let $X$ be the space of bounded real sequences. Is there a metric on $X$ such that a sequence $(x^{(n)})$ in $X$ converges to $x$ in this metric if and only if $(x^{(n)})$ converges to $x$ in every coordinate (i.e. $x_k^{(n)} \to x_k$ in $\mathbb{R}$ for every $k$)? Is there a norm with this property?

**16.*** Metrics $d$, $d'$ on $X$ are said to be *uniformly equivalent* if the identity maps $(X, d) \to (X, d')$ and $(X, d') \to (X, d)$ are both uniformly continuous. Give an example of a pair of metrics on $\mathbb{R}$ which are uniformly equivalent but not Lipschitz equivalent. Show that for every metric space $d$ on a set $X$ there exists a metric $d'$ which is uniformly equivalent to $d$ and which is bounded.

**17.*** Let $(X, d)$ be a non-empty complete metric space and let $f : X \to X$ be a function such that for each positive integer $n$ we have
(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.
Must $f$ have a fixed point?