ANALYSIS II (Michaelmas 2011): EXAMPLES 3

The questions are not equally difficult. The questions marked with * may be harder, but merit some attention even if you do not write out solutions. Comments, corrections are welcome at any time and may be sent to `a.j.scholl@dpmms.cam.ac.uk`.

1. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

2. At which points of $\mathbb{R}^2$ are the following functions $\mathbb{R}^2 \to \mathbb{R}$ differentiable?

(i) $f(x,y) = \begin{cases} x/y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$

(ii) $f(x,y) = |x|\,|y|$.

(iii) $f(x,y) = xy\,|x-y|$.

(iv) $f(x,y) = \begin{cases} xy\,/\sqrt{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(v) $f(x,y) = \begin{cases} xy\,\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

3. Let $f(x,y) = x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$. Show that f is continuous at $(0,0)$ and that it has directional derivatives in all directions there. Is f differentiable at $(0,0)$?

4. We work in $\mathbb{R}^3$ with the usual inner product. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable except at 0 and

$$Df(a)\colon u \mapsto \frac{u}{\|a\|} - \frac{a(a\cdot u)}{\|a\|^3}, \qquad a \neq 0,\ u \in \mathbb{R}^3$$

Verify that for any $u \in \mathbb{R}^3$, $Df(a)(u)$ is orthogonal to a. Explain geometrically why this is the case.

5. (i) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $\partial f/\partial x$ is continuous in some open ball around (a,b), and $\partial f/\partial y$ exists at (a,b). Show that f is differentiable at (a,b).

(ii) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $\partial f/\partial x$ exists and is bounded near (a,b), and that for a fixed, $f(a,y)$ is continuous as a function of y. Show that f is continuous at (a,b).

6. Let $M_n = M_n(\mathbb{R})$ be the space of $n \times n$ real matrices (it can be identified with $\mathbb{R}^{n^2}$). Show that the function $f : M_n \to M_n$ defined by $f(X) = X^2$ is differentiable everywhere in M_n. Is it true that $Df(A) = 2A$? If not, what is the derivative of f?

7. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Recall that the *operator norm* of A is

$$\|A\| = \sup\{\|Ax\| \mid x \in \mathbb{R}^n,\ \|x\| \leq 1\} = \sup\left\{\frac{\|Ax\|}{\|x\|} \mid 0 \neq x \in \mathbb{R}^n\right\}.$$

Complete the proof that this defines a norm on the vector space $L(\mathbb{R}^n, \mathbb{R}^m)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$.

Now assume $m = n$ and identify $L(\mathbb{R}^n, \mathbb{R}^n)$ with $M_n(\mathbb{R})$, the space of $n \times n$ real matrices. Show that if the operator norm of $A \in M_n$ satisfies $\|A\| < 1$, then the sequence $B_k = I + A + A^2 + \ldots + A^{k-1}$ converges (here I is the identity matrix), and deduce that $I - A$ is then invertible. Deduce that the set $GL_n(\mathbb{R})$ of all invertible $n \times n$ real matrices is an open subset of $M_n(\mathbb{R})$.

8. We regard $GL_n(\mathbb{R})$ as an open subset of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ (cf. the previous question). Define $g : GL_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $g(X) = X^{-1}$ for $X \in GL_n(\mathbb{R})$. Show that g is differentiable at the identity matrix $I \in GL_n(\mathbb{R})$, and that its derivative there satisfies $Dg(I)\colon H \mapsto -H$.

Let $A \in GL_n(\mathbb{R})$. By writing $(A+H)^{-1} = A^{-1}(I + HA^{-1})^{-1}$, or otherwise, show that g is differentiable at $X = A$. What is $Dg(A)$?

Show further that g is twice differentiable at I, and find $D^2g(I)$ as a bilinear map $M_n \times M_n \to M_n$.

9. (i) Define $f : M_n \to M_n$ by $f(X) = X^3$. Find the Taylor series of $f(A+H)$ about A.

(ii)* (This assumes that you did the previous question!) Let again $g : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be defined by $g(X) = X^{-1}$. Find the Taylor series of $g(I+H)$ about I.

10.* Show that $\det : M_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D\det(I)(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det(A)(H) = \det(A)\operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2\det(I)$ as a bilinear map $M_n \times M_n \to \mathbb{R}$.

11. Show that there is a continuous square-root function on some neighbourhood of I in M_n; that is, show that there is an open ball $B(I;r) \subset M_n$ for some $r > 0$ and a continuous function $g : B(I;r) \to M_n$ such that $g(X)^2 = X$ for all $X \in B(I;r)$.

Is it possible to define a square-root function on all of M_n? What about a cube-root function?

12. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x, x^3+y^3-3xy)$ and let $C = \{(x,y) \in \mathbb{R}^2 : x^3+y^3-3xy = 0\}$. Show that f is locally invertible around each point of C except $(0,0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0,y_0) \in C \setminus \{(0,0),(2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0,y_0) and V containing $f(x_0,y_0)$ such that f maps U bijectively to V. What is the derivative of the local inverse function? Deduce that for each point $(x_0,y_0) \in C$ other than $(0,0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$ there exist open intervals I containing x_0 and J containing y_0 such that for each $x \in I$ there is a unique $y \in J$ with $(x,y) \in C$.

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g(x) = f(x, c-x)$ where c is a constant. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable and find its derivative

(i) directly from the definition of differentiability

and also

(ii) by using the chain rule.

Deduce that if $\partial f/\partial x = \partial f/\partial y$ holds throughout $\mathbb{R}^2$, then $f(x,y) = h(x+y)$ for some differentiable function h.

14.* Let $U \subset \mathbb{R}^2$ be an open set that contains a rectangle $[a,b] \times [c,d]$. Suppose that $g : U \to \mathbb{R}$ is continuous and that the partial derivative $\partial g/\partial y$ exists and is continuous. Set $G(y) = \int_a^b g(x,y)dx$. Show that G is differentiable on (c,d) with derivative $G'(y) = \int_a^b (\partial g/\partial y)(x,y)dx$. Show further that $H(y) = \int_a^y g(x,y)dx$ is differentiable. What is its derivative $H'(y)$?
[Hint: consider a function $F(y,z) = \int_a^z g(x,y)dx$ before dealing with H.]