Algebraic Geometry IID (Lent 2013) — example sheet IV

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Assume throughout that the base field k is algebraically closed. If it helps, feel free to assume throughout that it has characteristic zero.

*Questions marked * might be harder.*

1. Let V be a smooth irreducible projective curve of genus g, and let D be any divisor with $\ell(D) > 0$. Show that for all but a finite number of $P \in V$, $\ell(D - P) = \ell(D) - 1$.

2. Let V be the smooth plane cubic with equation $X_0 X_2^2 = X_1(X_1 - X_0)(X_1 - \lambda X_0)$, for some $\lambda \in k \setminus \{0, 1\}$. (Here $\mathrm{char}(k) \neq 2$). Let $P = (0 : 0 : 1)$ be the point at infinity on V. Writing $x = X_1/X_0$, $y = X_2/X_0$, show that x/y is a local parameter at P. [Hint: consider the affine piece $X_2 \neq 0$.] Show that for each $m \geq 1$, the space $L(mP)$ has a basis consisting of functions x^i, $x^j y$, for suitable i and j.

3. (i) Let $f \in k[x]$ a polynomial of degree $d > 1$ with distinct roots, and $V \subset \mathbb{P}^2$ the projective closure of the affine curve with equation $y^{d-1} = f(x)$. Assume that $\mathrm{char}(k)$ does not divide $d - 1$. Prove that V is smooth, and has a single point Q at infinity. Calculate $v_Q(x)$ and $v_Q(y)$.

   (ii) Deduce (without using Riemann–Roch) that if $n > d(d - 3)$, then $\ell((n + 1)Q) = \ell(nQ) + 1$.

   (ii) Suppose $d = 4$ and $f = x^4 + 1$. Let ω be the rational differential dx/y^2 on V. Show that $v_P(\omega) = 0$ for all $Q \in V_0$. prove that $v_Q(\omega) = 4$ and hence that ω, $x\omega$ and $y\omega$ are all regular on V.

4. Show that there is no non-constant morphism from a curve of genus 4 to a curve of genus 3.

5. Let V be a curve of genus 2. Recall that if K is an effective canonical divisor and $L(K) = k \oplus k.x$ then $\pi = (1 : x) \colon V \to \mathbb{P}^1$ has degree 2. Let $\phi \colon V \to \mathbb{P}^1$ be any other morphism of degree 2. By considering $\pi^*\omega$ for a rational differential ω on $\mathbb{P}^1$, show that there is an automorphism $\alpha \colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi = \alpha \circ \pi$.

6. (Assume $\mathrm{char}(k) \neq 2$.) Let V be a curve of genus $g \geq 2$ which is hyperelliptic, with degree 2 morphism $\pi \colon V \to \mathbb{P}^1$. Let $W_1, \ldots, W_{2g+2} \in V$ be the ramification points of π. Assume that $\pi(W_1) = \infty$, and let $D = 2W_1 = \pi^*(\infty)$. Let $y^2 = f(x)$ (where f has degree $2g + 1$) be the plane model of $V - \{W_1\}$.

   (i) Show that $K = \sum_{i=1}^{2g+2} W_i - 2D$ is a canonical divisor on V.

   (ii) Show also that $K \sim (g - 1)D$, and that $\{1, x, \ldots, x^{g-1}\}$ is a basis for $L((g - 1)D)$.

   (iii) Show that there is a unique nontrivial morphism $\sigma \colon V \to V$ such that $\pi \circ \sigma = \pi$, and that $\sigma \circ \sigma = id_V$, the identity morphism. (σ is called the *hyperelliptic involution* of V). What are the fixed points of σ?

7. (i) Let $\pi \colon V \to \mathbb{P}^1$ be a hyperelliptic cover, and Q, R distinct ramification points of π. Show that $Q - R \not\sim 0$ but $2(Q - R) \sim 0$.

   (ii) Let $g(V) = 2$. Show that every divisor of degree 2 on V is linearly equivalent to $P + P'$ for some P, $P' \in V$, and deduce that every divisor D of degree 0 is linearly equivalent to $P - P'$ for some P, $P' \in V$. Are the points P, P' uniquely determined by D?

   (iii) Show that if $g(V) = 2$ then the subgroup $\{ [D] \in \mathrm{Cl}^0(V) \mid 2[D] = 0 \}$ of the divisor class group of V has order 16.

8. * Let $V \subset \mathbb{P}^n$ be a smooth irreducible projective curve of degree d. By considering a suitable rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^1$, show that some hyperplane section of V consists of d distinct points.

9. Let V be a smooth irreducible projective curve and $P \in V$. Show that for some $n \geq 2$ there exists an embedding $\phi \colon V \hookrightarrow \mathbb{P}^n$ such that $\phi^{-1}(\{X_0 = 0\}) = \{P\}$. In particular, $V \setminus \{P\}$ is an affine curve.

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10. Let $V \subset \mathbb{P}^3$ be the intersection of the quadrics $V(F)$, $V(G)$ where $\mathrm{char}(k) = 0$ and

$$F = X_0X_1 + X_2^2, \quad G = \sum_{i=0}^{3} X_i^2$$

(i) Show that V is a smooth curve (possibly reducible).

(ii) Let $\phi = (X_0 : X_1 : X_2) \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from $(0:0:1)$. Show that $\phi(V)$ is a conic $C \subset \mathbb{P}^2$. By parametrising C, compute the ramification of ϕ and show that $\phi \colon V \to C$ has degree 2. Deduce that V is irreducible of genus 1.

11. * Let $V(F), V(G) \subset \mathbb{P}^3$ be smooth quadrics, whose intersection $V = V(\{F, G\})$ is also smooth. Show that V is an irreducible curve of genus 1.

Further examples for enthusiasts:

12. (i) Let V be a smooth irreducible projective curve of genus $g \geq 2$. Recall that for $P \in V$ the Riemann–Roch theorem implies that $\ell(mP) \geq 1 - g + m$. We say that P is a *Weierstrass point* of V is $\ell(gP) \geq 2$. Show that if $g = 2$, the Weierstrass points of V are the ramification points of the degree 2 morphism $\pi \colon V \to \mathbb{P}^1$.

(ii) Prove that for any hyperelliptic curve $\pi \colon V \to \mathbb{P}^1$ the ramification points of π are Weierstrass points.

(iii) Let V be a smooth plane quartic. Show that $P \in V$ is a Weiersrtrass point if and only if it is a point of inflexion.

13. Let V be a smooth irreducible projective curve, and let π, $\pi' \colon V \to \mathbb{P}^1$ be morphisms of degree 2, such that there is no automorphism $\psi \colon \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ with $\pi' = \psi \circ \pi$.

(i) Show that coordinates on the two copies of $\mathbb{P}^1$ may be chosen such that $\pi^{-1}(\infty) = \{P, Q\}$ and $\pi'^{-1}(\infty) = \{P, Q'\}$ for distinct P, Q, Q'.

(ii) Show that if $x = \pi^*(X_1/X_0)$, $x' = \pi'^*(X_1/X_0)$ then $k(x, x') = k(V)$, and that there exists an irreducible polynomial $f = \sum_{i,j \leq 2} a_{ij} X^i Y^j \in k[X, Y]$ with $f(x, x') = 0$. Show also that $a_{22} = 0$.

(iii) Show that V is birational to a plane cubic curve (possibly singular), and that $g(V) \leq 1$. Deduce that on any curve of genus ≥ 2 there exists at most one divisor class $[D]$ of degree 2 with $\ell(D) = 2$.

(iv) Let V be a hyperelliptic curve, $\mathrm{char}(k) \neq 2$. By considering $\pi' = \pi \circ \alpha$ if α is an automorphism of V, show that V has only finitely many automorphisms. [This holds for all curves of genus ≥ 2 but the proof is harder.]

14. Suppose V is a smooth irreducible projective curve of genus $g > 0$ and $P \in V$ a point. Show that $\ell(P) = 1$, $\ell((2g-1)P) = g$, and that there exist exactly $(g-1)$ integers $\{n_i\}$ with $1 < n_1 < \cdots < n_{g-1} \leq 2g - 1$ with the property that, for each i, there exists a rational function f_i regular outside P with $v_P(f_i) = -n_i$. If V is hyperelliptic of genus ≥ 2 and P is a ramification point of the degree 2 morphism $V \to \mathbb{P}^1$, determine the integers $\{n_i\}$.

15. Let $V \subset \mathbb{P}^2$ be a smooth cubic defined by a polynomial whose coefficients are real. Show that V has either one or three real points of inflection.

16. By considering canonical divisors, show that a nonsingular plane curve of degree > 3 can never be hyperelliptic.

17. (i) Let $\phi \colon V \to W$ be a finite morphism of smooth projective curves, such that $k(V)/\phi^* k(W)$ is inseparable. Show that $e_P \neq 1$ for every $P \in V$. Deduce that if π_P is a local parameter at $P \in V$ then $k(V)/k(\pi_P)$ is separable.

(ii) The hypotheses being as in (i), show that for all $f \in k(V)$, $d(\phi^* f) = 0$.

18. Let V be a curve of genus 2, E a divisor of degree 2 which is not canonical. Show that $D = K + E$ satisfies $\ell(D - P - Q) = \ell(D) - 2$ if P and Q are distinct, but not in general. What can you say about the image of the morphism $\phi_D \colon V \to \mathbb{P}^2$?