Algebraic Geometry IID (Lent 2013) — example sheet II

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Assume throughout that the base field k is algebraically closed. If it helps, feel free to assume that it has characteristic zero.

*Questions (or parts of questions) marked * might be harder.*

1. Consider the plane affine curves with defining equations

$$(a)\ y = x^3 \qquad (b)\ xy = x^6 + y^6$$
$$(c)\ x^3 = y^2 + x^4 + y^4 \qquad (d)\ x^2y + xy^2 = x^4 + y^4$$
$$(e)\ 2x^2y^2 = x^2 + y^2 \qquad (f)\ y^2 = f(x) \text{ with } f \in k[x], \deg f = n.$$

   Write down the projective closures of the curves, determine their points at infinity and find all their singular points.

2. Determine the singular points of the surface in $\mathbb{P}^3$ defined by the polynomial $X_1X_2^2 - X_3^3 \in k[X_0, \ldots, X_3]$. Find the dimension of the tangent space at all the singularities.

3. Consider $V = V(I) \subset \mathbb{A}^3$ where I is generated by $X_1^3 - X_3$ and $X_2^2 - X_3$. Verify that I is a prime ideal, and thus that $I(V) = I$. Determine the points at which V is singular and compute the dimensions of the tangent spaces there.

4. * (i) Repeat the previous question for $V = V(I)$ where I is generated by

$$X_1^4 - X_2X_3, \quad X_1^3X_2 - X_3^2, \quad X_2^2 - X_1X_3$$

   (ii) If you assumed $I = I(V)$ in (i), justify it. (Hint: consider a suitable morphism $\mathbb{A}^1 \to \mathbb{A}^3$ of the form $Y \mapsto (Y^a, Y^b, Y^c)$.)

5. Consider the birational map $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ given by $(X_1X_2 : X_0X_2 : X_0X_1)$, and let $P_0 = (1 : 0 : 0)$, $P_1 = (0 : 1 : 0)$ and $P_2 = (0 : 0 : 1)$ be the basis points, at which ϕ is not regular. Let $L \subset \mathbb{P}^2$ be a line which is not one of the lines $\{X_i = 0\}$. Show that ϕ defines a *morphism* $L \to \mathbb{P}^2$ such that:

   (i) if $L \cap \{P_i\} = \emptyset$ then ϕ is an isomorphism of L with a conic in $\mathbb{P}^2$ which passes through all of the $\{P_i\}$;

   (ii) if L contains just one P_i then ϕ is an isomorphism of L with another line in $\mathbb{P}^2$

   * Determine the effect of ϕ on the cubic C with defining polynomial $X_0(X_1^2 + X_2^2) + X_1^2X_2 + X_1X_2^2$. (Assume $\mathrm{char}(k) \neq 2$.) What happens to the singularity of C? Draw appropriate pictures.

6. * Let $C = V(f) \subset \mathbb{A}^2$ be an affine plane curve such that $P = (0, 0)$ is a smooth point of C, and let the tangent at P to C be $ax + by = 0$. By considering the polynomial f, show that if $b \neq 0$ then the rational function y/x is regular at P.

   Deduce that if $V \subset \mathbb{P}^2$ is a plane curve which contains P_0 as a smooth point, then the rational map $\phi\colon V \dashrightarrow \mathbb{P}^2$ of question 5 is regular at P_0.

   Show conversely that if $V \subset \mathbb{P}^2$ is a plane curve of which P_0 is a singular point, then the rational map $\phi\colon V \dashrightarrow \mathbb{P}^2$ is not regular at P_0.

[1]Comments and corrections to `a.j.scholl@dpmms.cam.ac.uk`

7. Let $V = V(F) \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Let $P = (a_i) \in \mathbb{P}^2 \setminus V$. Consider the polynomial

$$G_P = a_0 \partial F/\partial X_0 + a_1 \partial F/\partial X_1 + a_2 \partial F/\partial X_2 \in k[\underline{X}]$$

   Assume that G_P is not identically zero. The curve $V(G_P)$ is called the *polar* to V from P. Show that $Q \in V \cap V(G_P)$ if and only if the line PQ is the tangent to V at Q.

8. Recall that the *rational normal curve* of degree d is the image C_d of the morphism $\phi = (X_0^2 : X_0^{d-1}X_1 : \cdots : X_1^d) \colon \mathbb{P}^1 \to \mathbb{P}^d$.

   Let $P_0, P_1, \ldots, P_r$ be distinct points on C_d. Show that if $r < d$ then there is a unique linear subvariety of $\mathbb{P}^d$ of dimension r containing $\{P_i\}$.

9. Assume $\mathrm{char}(k) \neq 2$. Show that a homogeneous polynomial $F(X_0, \ldots, X_n)$ of degree 2 can be written uniquely in the form ${}^T X A X$, where A is a symmetric $(n+1) \times (n+1)$ matrix with entries in k and $X = {}^T(X_0, \ldots, X_n)$ (here ^{T} denotes transpose). Show that if $n = 2$ then F is irreducible if and only if A is nonsingular. Show that for a suitable choice of coordinates in $\mathbb{P}^n$ ($n \geq 2$) any quadric (i.e. variety defined by a single quadratic polynomial) may be written as $V(F)$ for $F = X_0^2 + \cdots + X_m^2$ for some $m \leq n$, and that V is smooth if and only if $m = n$.

10. (Conics in characteristic 2.) Work over a field k of characteristic 2. Show that the plane curve given by the polynomial $X_0X_1 + X_0X_2 + X_1X_2$ is everywhere nonsingular. Compute the tangent line at a point of C, and show that all tangent lines meet in the point $P = (1 : 1 : 1)$. Verify that the polar polynomial G_P (cf. question 7) is identically zero.

11. A point P of a smooth plane curve V is a *point of inflexion* (or *flex*) if the tangent line at P meets V there with multiplicity ≥ 3. Assume $\mathrm{char}(k) \neq 2$.

    (i) Let $V = V(f) \subset \mathbb{A}^2$ be affine. Let $v \in T_{V,P} \subset k^2$ be any nonzero tangent vector at P. Show that P is a flex if and only if ${}^T v J_P v = 0$ where $J_P = (\partial^2 f/\partial X_i \partial X_j(P))_{ij}$.

    (ii) Let $V = V(F) \subset \mathbb{P}^2$ be projective. Show that P is a point of inflection of V if and only if the determinant $H = |\partial^2 F/\partial X_i \partial X_j|$ vanishes at P. (The curve $H = 0$ is called the *Hessian* of V.) In particular, any smooth projective curve of degree ≥ 3 has points of inflexion. [Hint: assume $P = (1 : 0 : 0)$.]

    (iii) Let C be the plane cubic $V(X_0^3 + X_1^3 + X_2^3 - aX_0X_1X_2)$. Assume $\mathrm{char}(k) \neq 3$. Show that if $a^3 \neq 27$ then C is nonsingular. Find in this case the flexes of C.

12. Suppose $V = V(F)$ is a nonsingular plane cubic, P_0 a point of inflexion. Show that coordinates may be chosen in $\mathbb{P}^2$ for which $P_0 = (0 : 0 : 1)$ and $F = X_2^2X_0 - X_1(X_1 - X_0)(X_1 - \lambda X_0)$ for some $\lambda \in k \setminus \{0, 1\}$.