Algebraic Geometry IID (Lent 2013) — example sheet I

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Assume throughout that the base field k is algebraically closed.
*Questions marked * might be harder.*

1. Let $P = (a_1, \dots, a_n) \in \mathbb{A}^n$. Show that P is the affine variety defined by the ideal $(X_1 - a_1, \dots, X_n - a_n)$ and that this is a maximal ideal.

2. Find a finite set $S \subset k[X, Y]$ of polynomials such that $V(S) = \{(0,0), (1,1), (1,2)\} \subset \mathbb{A}^2$.

3. Find the irreducible components of the affine variety $V(S) \subset \mathbb{A}^2$ defined by $S = \{X_1X_2, X_1^2 - X_1\}$.

4. Let $f \in k[\underline{X}]$ be a nonconstant polynomial. Show that if f is irreducible, then $V(f)$ is an irreducible variety. Show that if f, g are irreducible polynomials, then $V(\{f, g\})$ needn't be irreducible.

5. Show that the radical $\sqrt{I} = \{f \in R \mid f^d \in I \text{ for some } d \geq 1\}$ is an ideal. (R can be any commutative ring here.) Show that if $I = (f) \subset k[\underline{X}]$ where f is a polynomial with prime factorisation $f = \prod f_i^{e(i)}$, $e(i) \geq 1$, then $\sqrt{I} = (g)$ where $g = \prod f_i$.

6. Let $V \subset \mathbb{A}^n$ be an affine variety. Show that V is a point iff $k[V] = k$. (You do not need the Nullstellensatz to prove this). Prove also that V is finite iff $k[V]$ is finite-dimensional as a vector space over k, and *that the dimension of $k[V]$ then equals $\#V$.

7. Let $f, g \in k[X, Y]$ be polynomials in two variables with no common factor. Show that there exists u, $v \in k[X, Y]$ such that $uf + vg$ is a non-zero polynomial in X alone.

   Now let $f \in k[X, Y]$ be irreducible. The variety $V(f)$ is called a *plane affine curve*. Show that any proper subvariety W of $V(f)$ is finite. [Hint: first show that $W \subset V(\{f, g\})$ for some g not divisible by f].

8. Show that any two irreducible plane affine curves (as in the previous question) are homeomorphic (in many ways!) as topological spaces (with the Zariski topology). So the Zariski topology on its own doesn't tell us very much about a variety.

9. * Prove that if $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ are affine varieties, then $\phi \mapsto \phi^*$ is a bijection between the set of morphisms $V \to W$ and the set of k-algebra homomorphisms $k[W] \to k[V]$ (ring homomorphisms which are the identity on k).

10. Given points $P_0, \dots, P_{n+1}$ in $\mathbb{P}^n = \mathbb{P}(W)$, no $(n+1)$ of which are contained in a hyperplane, show that homogeneous coordinates may be chosen so that $P_0 = (1 : 0 : \dots : 0)$,..., $P_n = (0 : \dots : 0 : 1)$ and $P_{n+1} = (1 : 1 : \dots : 1)$.

11. Given hyperplanes $H_0, \dots, H_1$ in $\mathbb{P}^n = \mathbb{P}(W)$ such that $H_0 \cap \dots \cap H_n = \emptyset$, show that homogeneous coordinates may be chosen such that H_i has equation $X_i = 0$.

12. Show that the set of hyperplanes in $\mathbb{P}(W)$ is parameterised by $\mathbb{P}(W^*)$, where W^* is the vector space dual to W. If P_1,...,P_n are points of $\mathbb{P}(W)$, describe the set in $\mathbb{P}(W^*)$ whose points correspond to hyperplanes not containing any of the P_i. Deduce that there are infinitely many such hyperplanes.

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13. Let $V \subset \mathbb{P}^n$ be a hypersurface defined by a non-constant homogeneous polynomial F, and L a (projective) line in $\mathbb{P}^n$. Show that $V \cap L \neq \emptyset$.

14. Prove that the decomposition of a (projective or affine) variety into irreducible components is unique up to order. Decompose the variety $V \subset \mathbb{P}^3$ with equations $X_2^2 = X_1X_3$, $X_0X_3^2 = X_2^3$ into irreducible components.

15. Let $V \subset \mathbb{P}^2$ be defined by $X_1X_2^2 = X_0^3$.

    (a) Show that the formula $(u : v) \mapsto (u^2v : u^3 : v^3)$ defines an morphism $\phi\colon \mathbb{P}^1 \to V$.

    (b) Write down a rational map $\psi\colon V \dashrightarrow \mathbb{P}^1$, regular on $U = V \setminus \{(0 : 0 : 1)\}$ which coincides with ϕ^{-1} on U. What is the geometric interpretation of ψ?

    (c) Show that ψ is not regular at $(0 : 0 : 1)$.

16. Let $V \subset \mathbb{P}^2$ be defined by $X_1^2X_2 = X_0^2(X_0 + X_2)$. Find a surjective morphism $\phi\colon \mathbb{P}^1 \to V$ such that, for $P \in V$,

$$\#\phi^{-1}(P) = \begin{cases} 2 & \text{if } P = (0:0:1) \\ 1 & \text{otherwise} \end{cases}$$

    Is there a rational map $\psi\colon V \dashrightarrow \mathbb{P}^1$, regular on $U = V \setminus \{(0 : 0 : 1)\}$, which coincides with ϕ^{-1} on U?

17. Let V be an irreducible projective variety and $\phi\colon V \to \mathbb{P}^m$ a morphism. Assume that $W = f(V) \subset \mathbb{P}^m$ is a subvariety [this is in fact always true]. Show that W is irreducible.

18. * Let V be the quadric $V(X_0X_3 = X_1X_2) \subset \mathbb{P}^3$, and H the plane $X_0 = 0$. Let $P = (1 : 0 : 0 : 0)$. Show that $\phi = (0 : X_1 : X_2 : X_3)$ defines a rational map $\phi\colon V \dashrightarrow H$ such that for $Q \in V$, the line PQ meets H is $\phi(Q)$ whenever this is defined.

    Show that ϕ is not a morphism.

    Let $V_1 = V \cap \{X_1 = X_2\}$ and $L = H \cap \{X_1 = X_2\}$. Verify explicitly that ϕ induces an isomorphism $V_1 \xrightarrow{\sim} L$.

19. * Show that if V is an irreducible plane curve with equation $X_0X_2^2 = X_1^3 + aX_0^2X_1 + bX_0^3$, then V is isomorphic to the variety $W \subset \mathbb{P}^3$ given by $X_0X_3 = X_1^2$, $X_2^2 = X_1X_3 + aX_0X_1 + bX_0^2$ via the map $\phi = (X_0^2 : X_0X_1 : X_0X_2 : X_1^2)$. [You may find question 15 helpful.]