7 Divisors on curves

For the rest of this course, **curve** will mean smooth, projective, irreducible curve, unless explicitly stated to the contrary.

A **divisor** on a curve V is a finite formal sum $\sum_{P\in V} n_P P$ with $n_P \in \mathbb{Z}$ (finite means that for all but finitely many P, $n_P = 0$). Sometimes the points are put in brackets (P) to make the notation clearer. The set of divisors on V forms an abelian group under obvious addition, denoted $\mathrm{Div}(V)$. If $D = \sum n_P P$ is a divisor, define $\deg(D) = \sum n_P \in \mathbb{Z}$. The map $D \mapsto \deg(D)$ is obviously a homomorphism, whose kernel is denoted $\mathrm{Div}^0(V)$ (divisors of degree 0). We sometimes write $v_P(D)$ for the coefficient n_P of P in D.

Let $f \in k(V)^*$ be a nonzero rational function. Define the divisor of f to be

$$\mathrm{div}(f) = (f) := \sum_{P\in V} v_P(f)P.$$

Corol.6.8(ii) says that $\mathrm{div}(f) \in \mathrm{Div}^0(V)$. Divisors of the form $\mathrm{div}(f)$ are called **principal divisors**, and form a subgroup $\mathrm{div}(k(V)^*) \subset \mathrm{Div}^0(V)$.

If you're doing Number Fields you will probably notice the similarity between this and ideal theory for number fields. In particular, we can also define the **divisor class group** of V for be the quotient

$$\mathrm{Cl}(V) = \mathrm{Div}(V)/\mathrm{div}(k(V)^*).$$

and for $D \in \mathrm{Div}(V)$ write $[D]$ for the class of D in $\mathrm{Cl}(V)$. Divisors in the same divisor class are said to be **linearly equivalent**, written $D \sim E$. So $D \sim E$ iff $D - E$ is a principal divisor. If so then $\deg(D) = \deg(E)$.

Proposition 7.1. *Every divisor of degree* 0 *on* $\mathbb{P}^1$ *is principal.*

Proof. Write the divisor as $D = \sum_{a\in k} n_a(a) + n_\infty(\infty)$. As $\deg(D) = 0$, $n_\infty = -\sum n_a$. Let $f = \prod_{a\in k}(t-a)^{n_a}$. Then since $(t-a)$ is a local parameter at a and a unit at $b \neq a$, $v_a(f) = n_a$, and since $1/(t-a)$ is a local parameter at ∞ for any a, $v_\infty(f) = -\sum n_a = n_\infty$. □

For a general curve, $\deg\colon \mathrm{Div}(V) \to \mathbb{Z}$ induces a homomorphism $\mathrm{Cl}(V) \to \mathbb{Z}$, obviously surjective, and (by 7.1) an isomorphism for $V = \mathbb{P}^1$. Later will see this is a characteristic property of $\mathbb{P}^1$.

Other ways divisors arise:

Hyperplane sections $\mathrm{div}(L)$. Let $V \subset \mathbb{P}^n$ and consider a hyperplane $H = V(L) \subset \mathbb{P}^n$ not containing V, some linear form L. Define

$$\mathrm{div}(L) = \sum n_P P, \qquad \text{where if } X_i(P) \neq 0,\ n_P = v_P(L/X_i)$$

Note that this is independent of i, and that the only P occuring in the sum are $P \in V \cap H$.

If L' is another linear form then $\mathrm{div}(L') - \mathrm{div}(L) = \mathrm{div}(L'/L)$ which is a principal divisor, so $\mathrm{div}(L)$ and $\mathrm{div}(L')$ are linearly equivalent and in particular have the

same degree, called the **degree** of V. For an irreducible plane curve $V = V(F)$, $v_P(L/X_i)$ is just the multiplicity $m_P(V, H)$ (see proof of 5.1) and the degree of V is just the degree of F.

Likewise, any homogeneous $G \in k[\underline{X}]$ of degree m such that $V(G) \not\supset V$ determines a divisor $\mathrm{div}(G)$ which is linearly equivalent to $m \times \mathrm{div}(L)$, and therefore has degree md.

Special case: $V = V(F) \subset \mathbb{P}^2$, F irreducible of degree n. see that

$$\#V(F) \cap V(G)\} \leq mn$$

i.e. (cf. the special case Prop.5.3):

Theorem 7.2 (Bezout's Theorem, basic version)**.** *Two distinct irreducible plane curves of degrees m, n intersect in at most mn points.*

A divisor $D = \sum n_P P$ is **effective** if $n_P \geq 0$ for all P — notation $D \geq 0$. (Some authors confusingly use the term **positive**.) Let D be any divisor. Then associated to D are two important invariants: the first is

$$\begin{aligned} L(D) = \mathcal{L}(D) &= \{f \in k(V) \mid f = 0 \text{ or } \mathrm{div}(f) + D \geq 0\} \\ &= \{f \in k(V) \mid \forall P \in V,\ v_P(f) + n_P \geq 0\} \quad \text{if } D = \sum_P n_P P. \end{aligned}$$

Noting that $v_P(f+g) \geq v_P(f)$ we see that $L(D)$ is a vector space. Its dimension is written $\ell(D)$, which is **finite**. For example, let $\infty = (0:1) \in V = \mathbb{P}^1$, $D = m(\infty)$. Writing $x = X_1/X_0$ we see that $L(D)$ is spanned by $1, x, \ldots, x^m$ so $\ell(D) = m+1$.

In general we have:

Proposition 7.3. *Let $D \in \mathrm{Div}(V)$. Then:*

(i) $\deg(D) < 0 \implies L(D) = 0$.

(ii) $\deg(D) \geq 0 \implies \ell(D) \leq \deg(D) + 1$.

(iii) For any $P \in V$, $\ell(D) \leq \ell(D - P) + 1$.

Proof. (i) If $L(D) \neq 0$ then for $0 \neq f \in L(D)$, $\mathrm{div}(f) + D = E \geq 0$. But then $\deg(D) = \deg(E) \geq 0$ (as $\deg \mathrm{div}(f) = 0$).

(iii) Let $n = v_P(D)$. Define $\alpha\colon L(D) \to k$ by $\alpha(f) = (\pi_P^n f)(P)$. The kernel of this homomorphism is then $L(D-P)$ so $\ell(D-P) \geq \ell(D) - 1$.

(ii) now follows: if $d = \deg(D) \geq 0$ we see $\ell(D) \leq \ell(D - (d+1)P) + d + 1 = d + 1$ since $\deg(D - (d+1)P) = 0$. □

If $D \sim E$, so that $D - E = \mathrm{div}(g)$ then $L(D)$ and $L(E)$ are isomorphic by the map $f \mapsto fg$. So $\ell(D)$ depends only on the class of D.

8 Differentials

Differentials are a way of doing calculus on varieties, in a coordinate-free way.

K/k field extension. Informally a **differential** is a finite sum of formal expressions $x\,dy$ with x, $y \in K$, subject to the usual rules of calculus. Precisely:

Definition The space of **Kähler differentials** $\Omega_{K/k}$ is the quotient M/N where

$$M = \big(K\text{-vector space generated by symbols } \delta x,\ x \in K\big)$$
$$N = \left(\begin{array}{l}\text{subspace generated by } \delta(x+y) - \delta x - \delta y,\\ \delta(xy) - x\,\delta y - y\,\delta x,\ \delta a \text{ for } x,\ y \in K,\ a \in k.\end{array}\right)$$

and define $dx = \delta x + N \in \Omega_{K/k}$. (Think of K as functions, k as constants.)

The map $d\colon K \to \Omega_{K/k}$ is the **exterior derivative**. It is k-linear since if $a \in k$ then $d(ax) = a\,dx$

Any k-linear map $D\colon K \to U$ to a K-vector space U satisfying the product rule $D(xy) = xDy + yDx$ is called a **derivation** (more precisely, a k-**derivation**). So d is a derivation. Another example of a derivation is the formal differentiation map $d/dX\colon k(X) \to k(X)$. (We make the same definition of K is a ring containing k and U is a K-module.)

Lemma (/tautology). *A map $D : K \to U$ is a derivation iff there is a K-linear map $\lambda : \Omega_{K/k} \to U$ such that $\lambda(dx) = D(x)$ for all $x \in K$.*

Proof. If λ is such a K-linear map then obviously $D = \lambda \circ d$ is k-linear and $D(xy) = \lambda(d(xy)) = x\lambda(dy) + y\lambda(dx)$, so D is a derivation. Conversely, given a derivation $D : K \to U$, write $\Omega_{K/k} = M/N$ as in the definition, and define a K-linear map $\hat{\lambda} : M \to U$ by $\delta y \mapsto D(y)$ for all $y \in K$. Then as D is a derivation it follows that $\hat{\lambda}(N) = 0$ so we get a K-linear map λ with the desired properties. □

For any derivation (in particular d), if $y \neq 0$ then $Dx = D(y(x/y)) = yD(x/y) + (x/y)Dy$ giving the quotient formula $D(x/y) = y^{-2}(yDx - xDy)$.

Lemma 8.1. *(i) If $f = g/h \in k(X_1, \ldots, X_n)$ and $y = f(x_1, \ldots, x_n) \in K$, then $dy = \sum_i (\partial f/\partial X_i)(x_1, \ldots, x_n)\,dx_i$.*
(ii) If $K = k(x_1, \ldots, x_n)$ for $x_i \in K$ then $\{dx_i\}$ spans $\Omega_{K/k}$.

Proof. (i) follows from the rules for $d(xy)$, $d(x/y)$ and k-linearity. (ii) is an immediate consequence. □

Theorem 8.2. *Let $K/k(t)$ be finite and separable, t transcendental over k. Then $\Omega_{K/k}$ is one-dimensional, spanned by dt.*

Proof. First suppose $K = k(t)$. Then by 8.1(ii), $\Omega_{K/k}$ is generated by dt so has dimension ≤ 1. Enough to show it is nonzero. By Lemma-Tautology, enough to show there is a non-zero derivation $K \to K$, and d/dt is one.

For the general case, write $K_0 = k(t)$ so that $K = K(\alpha) = k(t, \alpha)$ by the primitive element theorem. Let $h \in K_0[X]$ be the minimal polynomial of α. Then $h'(\alpha) \neq 0$ by separability. By 8.1(ii), $\Omega_{K/k}$ is spanned by dt and $d\alpha$. If for $f \in K_0[X]$ we write $D_t f = \partial f/\partial t$ (i.e. apply d/dt to the coefficients of f), then 8.1(i) gives

$$0 = d(h(\alpha)) = (D_t h)(\alpha)dt + h'(\alpha)d\alpha$$

so $\Omega_{K/k}$ is spanned by dt. It therefore is enough to show $\Omega_{K/k} \neq 0$, or equivalently to write down a none-zero derivation $K \to K$.

Define a derviation $D : K_0[X] \to K$ (which is isomorphic to $K_0[X]/(h)$, hence is a $K_0[X]$-module) by

$$D(f) = D_t(f) \text{ if } f \in K_0, \quad D(X) = -\frac{(D_t h)(\alpha)}{h'(\alpha)}, \quad D(X^n) = n\alpha^{n-1}D(X).$$

Then $D(h) = D_t(h)(\alpha) + h'(\alpha)D(X) = 0$, so for any $f \in K_0[X]$, $D(fh) = f(\alpha)D(h) + h(\alpha)D(f) = 0$. So D vanishes on the ideal $hK_0[X] \subset K_0[X]$, hence defines a derivation $\bar{D}: K = K_0[X]/(h) \to K$, whose restriction to K_0 is D_t, hence is non-zero. □

Remark. We have $d(x^p) = px^{p-1}dx$, so if K has characteristic $p > 0$, then $d(x^p) = 0$ for all $x \in K$.In what follows I will generally stick to the case of characteristic zero, but point on when there are issues in the finite characteristic chase.

Our situation; V a curve (smooth, projective & irreducible), $K = k(V)$. An element of $\Omega_{k(V)/k}$ is called a **rational differential** on V. As k is fixed I will usually drop the "$/k$". Differentials are usually denoted ω, η, $\xi \ldots$.

Let $P \in V$, We say $\omega \in \Omega_{k(V)}$ is **regular at** P if it can be expressed as $\omega = \sum f_i dg_i$ with f_i, $g_i \in \mathcal{O}_{V,P}$. We let $\Omega_{V,P}$ or Ω_P denote the set of differentials regular at P. It is obviously an $\mathcal{O}_{V,P}$-module.

Theorem 8.3. *$\Omega_{V,P}$ is the free $\mathcal{O}_{V,P}$ module generated by $d\pi_P$ for any local parameter π_P at P.*

So $\Omega_{V,P} = \{fd\pi_P \mid f \in \mathcal{O}_{V,P}\}$. In particular, if π'_P is another local parameter, the $d\pi'_P = ud\pi_P$ where $u \in \mathcal{O}^*_{V,P}$ is regular an non-zero at P.

(It is not hard to show that Ω_P is just the module of differentials $\Omega_{\mathcal{O}_P/k}$.)

Definition. If $\omega \in \Omega_{k(V)}$ and $P \in V$, let $v_P(\omega) = v_P(f)$ where $\omega = fd\pi_P$.

By the last remark this doesn't depend on the choice of local parameter, and $v_P(\omega) \geq 0$ iff ω is regular at P.

Proof. Obviously $\mathcal{O}_P d\pi_P \subset \Omega_P$. Let $f = f(P) + \pi_P g \in \mathcal{O}_P = k + \mathfrak{m}_P$. Then $df = gd\pi_P + \pi_P dg \in \mathcal{O}_P d\pi_P + \pi_P \Omega_P$. Therefore

$$\mathcal{O}_P d\pi_P \subset \Omega_P \subset \mathcal{O}_P d\pi_P + \pi_P \Omega_P$$

and then applying Nakayama's Lemma with $R = \mathcal{O}_P$, $J = \mathfrak{m}_P$, $M = \Omega_P \supset N = \mathcal{O}_P d\pi_P$, we get $\Omega_P = \mathcal{O}_P d\pi_P$. The only thing we need to check is that Ω_P is finitely generated. Choose an affine piece $V_0 \subset \mathbb{A}^n$ of V containing P, so that $k[V_0] = k[x_1, \ldots, x_n]$ say. If $f \in \mathcal{O}_P$ then $f = g(\underline{x})/h(\underline{x})$ for polynomials g, h with $g(P) \neq 0$, and then

$$df = \sum \frac{h\partial g/\partial X_i - g\partial h/\partial X_i}{h^2}(\underline{x})dx_i$$

so $\{x_i\}$ generate Ω_P. □

We define the divisor of a non-zero differential $\omega \in \Omega_{k(V)}$ to be $(\omega) = \sum_P v_P(\omega)P$. If $0 \neq \omega' \in \Omega_{k(V)}$ then $\omega' = f\omega$ for some $f \in k(V)^*$, so $\mathrm{div}(\omega') = \mathrm{div}(f) + \mathrm{div}(\omega)$. Therefore the divisor class of $\mathrm{div}(\omega)$ doesn't depend on ω. It is called the **canonical class** of V. e write K_V for any element of the canonical class, and call it a **canonical divisor**. (Note the non-canonical use of the word "canonical"...)

$V = \mathbb{P}^1$. Compute: $v_P(dt) = 0$ if $P = a \in \mathbb{A}^1$ (since $t - a$ is a local parameter). At ∞, $\pi_\infty = t^{-1}$ is a local parameter and $dt = -t^2 d(1/t)$, so $v_\infty(dt) = v_\infty(t^2) = -2$. So $(dt) = -2(\infty)$ is a canonical divisor.

Lemma 8.4. *Let $0 \neq \omega \in \Omega_{k(V)/k}$. Then $v_P(\omega) = 0$ for all but finitely many P.*

Proof. As $v_P(f\,dg) = v_P(f) + v_P(dg)$ and $v_P(f) = 0$ for all but finitely many P, it's enough to consider $\omega = dg$ with $k(V)/k(g)$ finite and separable. Consider $\phi = (1:g)\colon V \to \mathbb{P}^1$. By the finiteness theorem, there are only finitely many $P \in V$ with $g(P) = \infty$ or $e_P > 1$. For all other P, $g - g(P) = \phi^*(t - g(P))$ is a local parameter at P, and therefore by 8.3(ii), $v_P(dg) = 0$. □

Define the divisor of $\omega \neq 0$ to be

$$(\omega) = \sum_P v_P(\omega)P.$$

As any other nonzero $\omega' \in \Omega_{k(V)/k}$ is of the form $f\omega$, $f \in k(V)^*$, the divisors of ω and ω' are linearly equivalent.

Define the **canonical class** of V to be the class of (ω). Denote by K_V any divisor in the canonical class.

Fix $\omega \in \Omega_{k(V)/k}$ and let $K_V = (\omega)$. Then $f\omega$ is regular iff $(f) + K_V \geq 0$, i.e.

$$L(K_V) \xrightarrow{\sim} \Omega(V), \quad f \mapsto f\omega.$$

In particular, $\Omega(V)$ is finite-dimensional. Major definition:

Definition: $g(V) = \dim \Omega(V) = \ell(K_V)$ is the **genus** of V.

Remark. The genus of V depends only on the isomorphism class of V, not on how V is embedded into projective space (unlike degree).

Ex: $V = \mathbb{P}^1$. We saw $K_{\mathbb{P}^1} = -2(\infty)$ and therefore $g(\mathbb{P}^1) = \ell(K_V) = 0$.

$V = V(F)$ plane cubic, $F = X_0X_2^2 - \prod_{i=1}^3(X_1 - \lambda_i X_0)$, with $\lambda_i \neq \lambda_j$ if $i \neq j$. We assume $ch(k) \neq 2$. Then V is nonsingular (cf. Q1 on example sheet #2). Affine equation is $f(x,y) = y^2 - \prod(x - \lambda_i) = y^2 - g(x)$ say.

Observe $2y\,dy = g'(x)\,dx$ in $\Omega_{k(V)/k}$. Let $\omega = dx/y$.

Claim: $v_P(\omega) = 0$ for all $P \in V$.

Assuming this is true then $K_V = 0$, so $g(V) = \ell(0) = 1$ and $\Omega(V) = k\omega$. Various cases:

- $P \in V_0$, $y(P) \neq 0$. Then $(\partial f/\partial y)(P) \neq 0$ so $x - x(P)$ is a local parameter at P, hence $v_P(\omega) = v_P(dx) = v_P(d(x - x(P))) = 0$.
- $P \in V_0$, $y(P) = 0$, $x(P) = \lambda_i$. Then $(\partial f)/(\partial x)(P) = -g'(\lambda_i) \neq 0$ (simple root), so y is a local parameter at P. Then $v_P(\omega) = v_P(2dy/g'(x)) = 0$.
- $P = P_0 = (0:0:1)$ point at infinity. Then as $v_{P_0}(x) = -2$ and $v_{P_0}(y) = -3$, have $v_{P_0}(dx/y) = (-2-1) - (-3) = 0$ by 8.5(?) (below).

[Alternative calculation at infinity: in the affine patch $\{X_2 \neq 0\}$, use coordinates $(z,t) = (X_0/X_2, X_1, X_2)$, $P = (0,0)$. Equation of V becomes $z = \prod(t - \lambda_i z)$, and $v_P(z) = 3$, $v_P(t) = 1$. Therefore $dx/y = d(1/t)/(z/t) = -(t^3/z)dt$ and $v_P(\omega) = 0$.]

In particular, this proves that V is not isomorphic to $\mathbb{P}^1$.

Proposition 8.5. *(i) Suppose* $\mathrm{char}(k) = 0$. *Let* $0 \neq f \in k(V)$, *and assume* $v_P(f) \neq 0$. *Then* $v_P(df) = v_P(f) - 1$.

(ii) Suppose $\mathrm{char}(k) = p \neq 0$, *and* $n = v_P(f)$. *Then* $v_P(df) \geq n - 1$, *with equality if* $(p,n) = 1$.

Proof. Let $n = v_P(f)$, so $f = \pi_P^n u$ with $u \in \mathcal{O}_P^*$. Write $du = g\, d\pi_P$. Then $df = \pi_P^{n-1}(nu + \pi_P g) d\pi_P$. So $v_P(df) = (n-1) + v_P(nu + \pi_P g)$. Both results follow. □

Proposition 8.6. *Let $V = V(F) \subset \mathbb{P}^2$ be a plane curve (irreducible projective nonsingular) of degree $d \geq 1$. Then $K_V = (d-3)H$, where H is the divisor of a hyperplane (i.e. line) section.*

Proof. Choose coordinates so that $(0:1:0) \notin V$. Let $x = X_1/X_0$, $y = X_2/X_0$ viewed as rational functions on V. Then $f(x,y) = 0$ where $f(X,Y) = F(1,X,Y)$ is the affine equation of V, so $(\partial f/\partial X)(x,y)\, dx + (\partial f/\partial Y)(x,y)\, dy = 0$ in $\Omega_{V/k}$. So let

$$\omega = \frac{dx}{(\partial f/\partial Y)(x,y)} = -\frac{dy}{(\partial f/\partial X)(x,y)}$$

Claim $(w) = (d-3)H$ with $H =$ hyperplane at infinity.

Let $P \in V \cap \mathbb{A}^2$. As in the previous example, if $(\partial f)/\partial Y)(P) \neq 0$, then $x - x(P)$ is a local parameter at P and so $v_P(\omega) = v_P(1/(\partial f)/\partial Y)(P)) = 0$. Otheriwise, $(\partial f)/\partial Y)(P) \neq 0$, in which case $y - y(P)$ is a local parameter and $v_P(\omega) = 0$.

It remains to consider points at infinity. Since $(0:1:0) \notin V$, any point at infinity is contained in the affine piece $\{X_2 \neq 0\}$, on which V has equation $g = 0$ with $z = X_0/X_2 = 1/y$, $t = X_1/X_2 = x/y$ and $g(Z,T) = F(Z,T,1) \in k[Z,T]$. Let $\eta = dz/(\partial g/\partial T)(z,t) = -dt/(\partial g/\partial Z)(z,t)$. The preceding argument shows that $v_P(\eta) = 0$ for any P in this the affine piece $\{X_2 \neq 0\}$. But $f(X,Y) = Y^d g(1/Y, X/Y)$ so $\partial f/\partial X = Y^{d-1}(\partial g/\partial V)(1/Y, X/Y)$ and so

$$\omega = -\frac{dy}{(\partial f/\partial X)(x,y)} = \frac{z^{-2}dz}{y^{d-1}(\partial g/\partial T)(z,t)} = z^{d-3}\eta$$

and so if $X_2(P) \neq 0$, $v_P(\omega) = (d-3)v_P(z) + v_P(\eta) = (d-3)v_P(z)$. Since $z = X_0/X_2$, this means $(\omega) = (d-3)\mathrm{div}(X_0) = (d-3)H$. □

Mention: topological nature of genus. Curvature.

9 Riemann-Roch

Let C be a (smooth, projective) curve. We have already seen the space $L(D) = \{f \mid (f) + D \geq 0\}$, where D is a divisor on C, and its dimension $\ell(D) = \dim L(D)$. By definition, $\ell(D) > 0$ iff D is linearly equivalent to an effective divisor.

The *Riemann-Roch problem* is to determine $\ell(D)$.

Recall (7.3) that $\ell(D) \leq \deg(D) + 1$. When $V = \mathbb{P}^1$ we have seen that for all D, $\ell(D) = \max(0, \deg(D) + 1)$.

Theorem 9.1 (Riemann-Roch)**.** *Let g be the genus of V, and $K = K_V$ a canonical divisor. For any divisor D,*

$$\ell(D) - \ell(K - D) = 1 - g + \deg(D).$$

This is a hard theorem, and the proof is beyond the course. The simplest proof uses *sheaf cohomology* — see chapter 2 of Serre, *Algebraic Groups and Class Fields* for a readable proof, or Hartshorne chapter 5 for a shorter but much fancier one. We will content outselves to discovering how powerful this result is.

Corollary 9.2. $\deg(K) = 2g - 2$.

Proof. Take $D = K$ so that $\ell(D) = \ell(K) = g$ and $\ell(K - D) = \ell(0) = 1$. □

Corollary 9.3. *A plane (smooth, projective) curve of degree d has genus $(d - 1)(d - 2)/2$.*

Proof. By 8.5 $K = (d-3)H$ and $\deg(H) = d$ so $\deg(K) = (d-3)d = 2g-2 \implies g = (d^2 - 3d + 2)/2$. □

So $d = 1$ or $2 \implies g = 0$ (line or conic, which we already know to be $\simeq \mathbb{P}^1$). For $d = 3$ we get $g = 1$, and for plane quartics, $g = 3$. In particular, no (smooth) plane curve has genus 2. (There are plenty of curves of genus 2 in $\mathbb{P}^3$ however.)

In particular we see that if nonsingular curves V, V' of degrees $d \neq d'$ are isomorphic, then $\{d, d'\} = \{1, 2\}$. (As they must have the same genus, $d(d-3) = d'(d'-3)$ i.e. $(d' - d)(d' + d - 3) = 0$.) The converse is far from true: if $d > 2$ there are infinitely many isomorphism classes of plane curves of degree d (we'll do the case $d = 3$ later).

Corollary 9.4. $\deg(D) > 2g - 2 \implies \ell(D) = 1 - g + \deg(D)$.

Proof. $\ell(K - D) = 0$ in this case because $\deg(K - D) = 2g - 2 - \deg(D) < 0$. □

Curves of genus 1.

Corollary 9.5. *Suppose $g(V) = 1$. Then $K_V \sim 0$, and $\deg(D) > 0 \implies \ell(D) = \deg(D)$.*

Proof. As $\ell(K_V) = g = 1$ there exists an effective divisor in the class of K_V, which must therefore be 0 as $\deg(K_V) = 2g - 2 = 0$. Second part follows from 9.4. □

Fix $P_0 \in V$. The pair (V, P_0) (or, less correctly, just V itself) is called an **elliptic curve**. Traditionally we write E instead of V (actually it is also more common to use C for curves...).

Let P, $Q \in E$. Then $\ell(P + Q - P_0) = 1$ so there exists a unique effective divisor of degree 1 (i.e. a point) R such that $P + Q - P_0 \sim R$. We define:

$$\boxed{P +_E Q = R}$$

(It would perhaps be more correct, but over-pedantic, to write $P +_{(E,P_0)} Q$.)

Theorem 9.6. *The operation $+_E$ makes E into an abelian group, with identity element P_0. Moreover the map $P \mapsto [P - P_0] \in \mathrm{Cl}(E)$ is an isomorphism of groups between E and $\mathrm{Cl}^0(E)$, the groups of divisor classes of degree 0 on E.*

Proof. Let $\beta(P) = [P - P_0] \in \mathrm{Cl}^0(E)$. First show that β is a bijection. Have $\beta(P) = \beta(Q) \iff P - P_0 \sim Q - P_0 \iff P \sim Q \iff P = Q$ since $\ell(P) = 1$. So β is injective. Also if D is a divisor of degree 0 then as $\ell(D + P_0) = 1$ there exists P with $D + P_0 \sim P$, so $[D] = \beta(P)$. Therefore β is a bijection (of sets). Finally, if $P +_E Q = R$ then $\beta(P +_E Q) = [R - P_0] = [P + Q - P_0 - P_0] = [P - P_0] + [Q - P_0] = \beta(P) + \beta(Q)$. So β transforms $+_E$ into addition in $\mathrm{Cl}^0(E)$, and therefore $(E, +_E)$ is a group and β is an isomorphism. □

We'll often write 0_E for the identity point P_0 in the group law. A smooth plane cubic has genus 1. Let's look at the special case we considered in the last lecture.

Theorem 9.7. *Assume* $\mathrm{char}(k) \neq 2$, *and let* $E = V(F) \subset \mathbb{P}^2$ *be the nonsingular plane cubic:*

$$F(X_0, X_1, X_2) = X_0X_2^2 - \prod_{i=1}^{3}(X_1 - \lambda_i X_0), \qquad \lambda_i \neq \lambda_j \quad \text{if } i \neq j.$$

Let $O_E = P_0 = (0:0:1) \in E$. *Then in the group law on* E

$$P +_E Q +_E R = 0_E \iff P,\ Q,\ R \text{ are collinear}$$

(We'll see soon that any curve of genus 1 is isomorphic to such a plane cubic.)

By *collinear* here we mean that there is a line $L \subset \mathbb{P}^2$ for which the line section on E is the divisor $P + Q + R$ (if P, Q, R are distinct this just means that they line on L.)

Proof. $P +_E Q +_E R = 0_E \iff P + Q + R \sim 3P_0$ (by definition of the group law) which holds iff $\exists f$ with $(f) = P + Q + R - 3P_0$. As $L(3P_0) = \langle 1, x, y\rangle = \langle 1, X_1/X_0, X_2/X_0\rangle$, this holds iff $f = G/X_0$ for a linear form G with $(G) = P + Q + R$. □

Before getting on to curves of higher genus, we'll first obtain the **Riemann-Hurwitz formula**.

Let $\phi\colon V \to W$ be a finite morphism of curves. Assume $\mathrm{char}(k) = 0$ here. Let $\omega = f\,dt \in \Omega_{k(W)/k}$, $k(W)/k(t)$ finite. Then $k(V)/\phi^*(k(t))$ is also finite so $\Omega_{k(V)/k}$ is generated by $d\phi^*(t)$. Define

$$\phi^*(\omega) = \phi^*(f)\,d\phi^*(t).$$

Let $P \in V$, $Q = \phi(P)$. We will compare $v_P(\phi^*(\omega)$ and $v_Q(\omega)$. Let e_P be the ramification degree of ϕ at P, and π_P, π_Q local parameters.

Lemma 9.8. *Assume* $\mathrm{char}(k) = 0$. *Then* $v_P(\phi^*\omega) = e_P v_Q(\omega) + e - 1$. *In particular,* $v_P(\phi^*(d\pi_Q)) = e - 1$.

Proof. Write $\omega = u\phi_Q^n\,d\pi_Q$, so that $v_Q(\omega) = v_Q(f) = n \in \mathbb{Z}$. Then $v_P * \phi^*(\omega) = v_P(\phi^*u) + nv_P(\phi^*\pi_P) + v_P(d\phi^*\pi_P) = ne_P + v_P(d\phi^*\pi_P)$. Now $\phi^*(\pi_Q) = y\pi_P^e$ for some $y \in \mathcal{O}_{V,P}^*$ with $dy = z\,d\pi_P$ say and so

$$d(\phi^*\pi_Q) = (ey + \pi_P z)\pi_P^{e-1}d\pi_P$$

so $v_P(d(\phi^*\pi_Q)) = e - 1$ since $\mathrm{char}(k) = 0$. □

Theorem 9.9 (Riemann-Hurwitz formula)**.** *Let* $\phi\colon V \to W$ *be a finite morphism of curves in characteristic zero. Let* $n = \deg(\phi)$. *Then*

$$2g(V) - 2 = n(2g(W) - 2) + \sum_{P \in V}(e_P - 1).$$

Proof. Let $0 \neq \omega \in \Omega_{k(W)/k}$. Then

$$\begin{aligned} 2g(V) - 2 = \deg \operatorname{div}(\phi^*\omega) &= \sum_{P \in V} v_P(\phi^*\omega) \\ &= \sum_{Q \in W} \sum_{P \mapsto Q} v_P(\phi^*\omega) \\ &= \sum_{Q \in W} \sum_{P \mapsto Q} (e_P v_Q(\omega) + e_P - 1) \\ &= \sum_{Q \in W} \Big(n v_Q(\omega) + \sum_{P \mapsto Q} (e_P - 1) \Big) \\ &= n \deg \operatorname{div}(\omega) + \sum_{P \in V} (e_P - 1) \end{aligned}$$

□

Remark. (not from lectures) In characteristic p, things change a bit:

- We must assume that $k(V)/k(W)$ is *separable* (otherwise $\phi^* \colon \Omega_{k(W)/k} \to \Omega_{k(V)/k}$ is identically zero).
- Assuming separability, let $\delta_P = v_P(\phi^* d\pi_Q)$. The proof of the lemma shows that $\delta_P = e_P - 1$ if $p \nmid e_P$, and is $\geq e_P$ if $p \mid e_P$. One says that ϕ is **wildly ramified** at P if $p \mid e_P$, **tamely ramified** otherwise.
- The Riemann-Hurwitz formula for a finite separable morphism $\phi \colon V \to W$ (in any characteristic) is then:

$$2g(V) - 2 = n(2g(W) - 2) + \sum_{P \in V} \delta_P.$$

Examples Say $\pi \colon V \to \mathbb{P}^1$ has degree 2. Then $e_P = 1$ or 2. R-H formula $\implies 2g - 2 = 2(0 - 2) + \sum(e_P - 1)$, i.e.

$$g = \frac{n}{2} - 1, \quad n = \#\{P \in V \mid e_P = 2\} = 2g + 2$$

(thus n is the number of ramificiation points of π). Specifically:

$g = 0 \implies n = 2$.

$g = 1 \implies n = 4$. In fact, if $V = E$ has Legendre equation $y^2 = x(x-1)(x-\lambda)$ and P_0 is the point at infinity then $\pi = \phi_{2P_0} = (1 : x) \colon E \to \mathbb{P}^1$ has degree 2 and is ramified precisely at $\{P_0, (0,0), (1,0), (\lambda, 0)\}$ (the points of order dividing 2 in the group of points of E), and $\pi(P) = \pi(Q) \iff P = \pm_E Q$.

Now consider $g > 1$.

Definition A curve V of genus $g > 1$ is **hyperelliptic** is there exists $\pi \colon V \to \mathbb{P}^1$ of degree 2. If so, then consider $D = \pi^*(\infty)$. Have 1, $\pi^*(X_1/X_0) \in L(D)$ and so $\ell(D) \geq 2$. Moreover if $\ell(D) = 3$ then $D = P + Q$ say and $\ell(P) = 2$, hence $V = \mathbb{P}^1$ which is impossible. So $\ell(D) = 2$.

Theorem 9.10. *(i) Let $g(V) > 1$. If there exists a divisor $D \geq 0$ of degree 2 on V with $\ell(D) = 2$ then $\pi = \phi_D \colon V \to \mathbb{P}^1$ has degree 2, $\pi^*(\infty) = D$ and V is hyperelliptic.*

(ii) Every curve of genus 2 is hyperelliptic.

Proof. (i) Say $D = P + Q$ and $\pi = \phi_D = (1 : x) \colon V \to \mathbb{P}^1$ where $L(D) = \langle 1, x \rangle$. Then $(x) = D' - D$, some $D' = P' + Q' \geq 0$. We must have $\{P, Q\} \cap \{P', Q'\} = \emptyset$ since if say $Q = Q'$ then $(x) = P' - P$ so $\ell(P) = 2$ and $V \simeq \mathbb{P}^1$.

Therefore $v_P(x) = -1 = v_Q(x)$ if $P \neq Q$, or $v_P(x) = -2$ if $P = Q$. In either case, $\pi^*(\infty) = P + Q$.

(ii) If $g = 2$ then $\ell(K) = 2 = \deg(K)$. □

We can write hyperelliptic curves explicitly as follows. Suppose $\pi \colon V \to \mathbb{P}^1$, $D = \pi^*(\infty)$, $L(D) = \langle 1, x \rangle$. Then $\deg(\pi) = 2 \implies k(V)/k(x)$ is an extension of degree 2, so (as we are assuming $\mathrm{char}(k) \neq 2$!) $k(V) = k(x, y)$ where $y^2 = r(x) \in k(x)$, $r(x)$ not a square. As $k[x]$ is a UFD, we can write $r(x) = h(x)(p(x)/q(x))^2$ for $p, q, h \in k[x]$, $h = \prod_{i=1}^{m}(x - \lambda_i)$ squarefree.

Then V is (by Theorem 2.5) birational to the plane curve V' with affine equation $f(x, y) = y^2 - h(x)$. The affine part $V' \cap \mathbb{A}^2$ is smooth, since if $P = (x_P, y_P) \in V' \cap \mathbb{A}^2$ then if $(\partial f/\partial y)(P) = 2y_P = 0$, we have $h(x) = -f(P) = 0$. But $(\partial f/\partial x)(P) = -h'(x_P) \neq 0$ and h is squarefree. The intersection $V' \cap \{X_0 = 0\}$ is one point $(0 : 0 : 1)$ which in fact is singular. In any case, we get a birational morphism

$$(1 : x : y) \colon V \to V' \subset \mathbb{P}^2$$

and a rational map

$$(X_0 : X_1) \colon V' \dashrightarrow \mathbb{P}^1$$

whose composite is $\pi = \phi_D \colon V \to \mathbb{P}^1$, and therefore π is ramified over $x = \lambda_1, \ldots, \lambda_m$ and possibly also infinity. Therefore since the number of ramification points is $2g + 2$ which is even, either

- $m = 2g + 2$ is even, π is unramified over ∞; or
- $m = 2g + 1$ is odd, π is ramified over ∞.

10 Projective embeddings

Let $V \subset \mathbb{P}^n$ be a curve of degree d, not contained in any hyperplane. Then $D = (X_0)$ is an effective divisor of degree d. A given curve V can occur in projective space in different ways (for example, a curve of genus 0 is isomorphic to $\mathbb{P}^1$, but also to a conic in $\mathbb{P}^2$, which has degree 2, and to a twisted cubic in $\mathbb{P}^3$, etc.) For a fixed curve V, we can ask: as we consider all ways of embedding V into projective space (or varying dimension) what such divisors D can arise?

If $F = \sum \lambda_i X_i \neq 0$ is any linear form, then $(F) \sim D$ and $F/X_0 \in L(D)$. So have

$$\beta \colon \{\text{linear forms } F = \sum \lambda_i X_i\} \hookrightarrow L(D), \quad F \mapsto F/X_0.$$

(Injective linear map, since V doesn't lie on a hyperplane.)

2 observations: let P, Q be distinct points of V, not lying on $\{X_0 = 0\}$. (We can always change coordinates so that this holds; this amounts to replacing D by a linearly equivalent divisor).

(1) There exist linear forms F, G with $F(P) \neq 0$ and $G(P) = 0 \neq G(Q)$. So $\beta(F) \in L(D) \setminus L(D - P)$ and $\beta(G) \in L(D - P) \setminus L(D - P - Q)$. Therefore $\ell(D - P - Q) \leq \ell(D) - 2$, and so by 7.3(iii), $\ell(D - P - Q) = \ell(D) - 2$.

(2) As P is a smooth point, it has a tangent line $L = T_P^{proj}$. There exists a linear form F with $F(P) = 0$ but not vanishing identically on L. Therefore the multiplicity of P in (F) is exactly 1, hence $\beta(F) \in L(D-P) \setminus L(D-2P)$

Se we deduce that D satisfies:

(∗) For every P, $Q \in V$ (not necessarily distinct), $\ell(D-P-Q) = \ell(D) - 2$.

Now start with a curve V and a divisor D with $\ell(D) = n+1 \geq 2$. Pick a basis $\{f_0, \ldots, f_n\}$ for $L(D)$. It defines a morphism

$$\phi_D = (f_0 : f_1 : \cdots : f_n) \colon V \to \mathbb{P}^n.$$

We say ϕ_D is an *embedding* if ϕ_D is an isomorphism between V and a (necessarily smooth, irreducible) curve in $\mathbb{P}^n$.

Note that choosing another basis changes ϕ_D by a linear transformation of $\mathbb{P}^n$. Also, if $D' = D - (g)$ is an equivalent divisor, then $\{gf_i\}$ is a basis for $L(D')$, hence $\phi_D = \phi_{D'}$ depends only on the equivalence class of D.

Theorem 10.1 (Embedding criterion)**.** *ϕ_D is an embedding iff (∗) holds.*

The above discussion shows that condition (∗) is necessary. The meat of the theorem is therefore that it is a sufficient condition.

I won't prove the theorem here — see for example Proposition 6.56 in Hulek (although he finesses some of the difficulties by defining "embedding" in a slightly different way). I will show however that (∗) implies that ϕ_D is injective. Let P, $Q \in V$ be distinct points. There exist functions p, $q \in k(V)$ with $v_P(p) = v_Q(q) = 1$, $v_P(q) = v_Q(p) = 0$ (take ratios of suitable linear forms on the projective space containing V). Replacing D with $D + (p^a q^b)$ for suitable a, $b \in \mathbb{Z}$, we may assume $v_P(D) = v_Q(D) = 0$. We have $\ell(D-P-Q) = \ell(D) - 2$, by 7.3(iii) we have $\ell(D-P) = \ell(D) - 1$ as well. Choose a basis $\{f_i\}$ for $L(D)$ such that $\{f_0, \ldots, f_{m-2}\}$ spans $L(D-P-Q)$ and $\{f_0, \ldots, f_{m-1}\}$ spans $L(D-P)$. Then all f_i are regular at P and Q and $f_{m-1}(P) = 0 \neq f_m(P)$, $f_{m-1}(Q) \neq 0$. Therefore $\phi_D(P) \neq \phi_D(Q)$.

This shows that if (∗) holds, then ϕ_D is injective. The idea of the rest of the proof is: by general theory, the image $\phi_D(V)$ is a possibly singular curve $V' \subset \mathbb{P}^m$. The condition with $P = Q$ is then used to show that V' is smooth and that $k(V) = k(V')$, which then implies that $\phi \colon V \xrightarrow{\sim} V'$.

Corollary 10.2. *If $\deg(D) > 2g$ then ϕ_D is an embedding.*

Proof. Apply Riemann-Roch: as $\deg(D) > \deg(D-P-Q) > 2g-2 = \deg(K)$, we have $\ell(K_D) = \ell(K_(D-P-Q)) = 0$ and so

$$\ell(D) = 1 - g + \deg(D), \quad \ell(D-P-Q) = 1 - g + \deg(D-P-Q) = \ell(D) - 2.$$

□

Examples:

First consider the case $g = 0$. Then $\deg(D) = n > 0$ implies $\ell(D) = n+1$ and $D \sim nP$ for any $P \in V$. Therefore ϕ_D is always an embedding. Taking $V = \mathbb{P}^1$ and $D = n(\infty)$ we get $L(D) = k \oplus k.x \oplus \cdots \oplus k.x^n$, hence

$$\phi_{n(\infty)} = (1 : x : \cdots : x^n) \colon \mathbb{P}^1 \to \mathbb{P}^n$$

is the n-tuple embedding.

Next consider $g = 1$. Corollary says that if $\deg(D) \geq 3$ then ϕ_D is an embedding. Pick $P_0 \in V$ and consider the case $D = 3P_0$. As $\ell(nP) = n$ by Riemann-Roch, we have:

$$\begin{aligned} L(P_0) = k \subsetneq L(2P_0) = \operatorname{span}\{1, x\} \qquad & \text{for some } x \text{ with } v_{P_0}(x) = -2 \\ \subsetneq L(3P_0) = \operatorname{span}\{1, x, y\} \qquad & \text{for some } y \text{ with } v_{P_0}(x) = -3 \end{aligned}$$

Then $L(4P_0) = L(3P_0) \oplus k.x^2$ and $L(5P_0) = L(4P_0) \oplus k.xy$, and x^3, y^2 both have $v_{P_0} = -6$, hence lie in $L(6P_0) \setminus L(5P_0)$. Therefore there must be a linear dependence between $1, x, x^2, x^3, y, xy, y^2$ in which the coefficients of x^3 and y^2 are nonzero. Replacing y by cy for suitable $c \neq 0$ this takes the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for suitable $a_i \in k$.

Theorem 10.3. *Let E, P_0) be an elliptic curve. Then $\exists a_1, a_2, a_3, a_4, a_6 \in k$ and an isomorphism $E \xrightarrow{\sim} V = V(F) \subset \mathbb{P}^2$ where V is a smooth cubic with affine defining polynomial*

$$f(x, y) = F(1, x, y) = y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6) \qquad \text{(W)}$$

and $P_0 \mapsto (0:0:1)$. Moreover if $\operatorname{char}(k) \neq 2$ coordinates may be chosen to that in addition, $a_1 = a_3 = 0$ and

$$f(x, y) = y^2 - x(x-1)(x-\lambda), \quad \lambda \in k,\ \lambda \notin \{0, 1\} \qquad \text{(L)}$$

The cubic (W) is called a (generalised) **Weierstrass equation** for E, and the form (L) is **Legendre normal form**. The indices are written in such a way that is the variables x, y are assigned weight 2,3 and a_i is assigned weight i then each term in f has weight 6.

Proof. From the above, $\phi_{3P_0} \colon V \to \mathbb{P}^2$ is an embedding, and its image lies in $V(F)$ for some F as in (W). As V is a curve of genus 1 this can only happen if the image equals $V(F)$ and if $V(F)$ is nonsingular.

If $\operatorname{char}(k) \neq 2$ then by completing the square,

$$\left(y + \frac{a_1}{2}x + \frac{a_3}{3}\right)^2 = (\text{cubic})(x) = \prod_{i-1}^{3}(x - \lambda_i)$$

and $\lambda_i \neq \lambda_j$ as V is smooth. Writing

$$x' = \frac{x - \lambda_1}{\lambda_2 - \lambda_1}, \quad y' = \frac{y + a_1x/2 + a_3/3}{(\lambda_2 - \lambda_1)^{3/2}}, \quad \lambda = \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1} \neq 0,\ 1,\ \infty$$

gives $(y')^2 = x'(x'-1)(x'-\lambda)$. □

Consider now Legendre normal form with $\operatorname{char}(k) \neq 2$. Then if $P = (1:a:b) = (a, b) \in E$, the $P' = (a, -b) \in E$ also, and the line $x = a$ cuts out the divisor $P + P' + P_0$. In other words, $P' = -P$ in the group law.

For $n \in \mathbb{Z}$, write $[n]P$ for n times P in the group law. Then $[2]P = 0_E$ iff $P = -P$, so we see that in the Legendre model,

$$\{P \in E \mid [2]P = 0_E\} = \{0_E, (0,0), (1,0), (\lambda, 0)$$

which is therefore isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

What about $[3]P = 0_E$? This holds iff the tangent at P has 3-fold intersection with E at P, i.e. iff P is a point of inflection. Using the Hessian one can show that if $\mathrm{char}(k) \neq 3$ then there are exactly 9 points of inflection on E (P_0 being one of them) and so

$$\{P \in E \mid [3]P = 0_E\} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \quad \text{if } \mathrm{char}(k) \neq 3$$

More generally one can show that

$$\{P \in E \mid [n]P = 0_E\} \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \quad \text{if } \mathrm{char}(k) \nmid n$$

Before leaving curves of genus 1 let's just explain what happens when $k = \mathbb{C}$. Consider a pair $\tau_1, \tau_2 \in \mathbb{C}$ of complex numbers, linearly independent over $\mathbb{R}$. Let $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2 \subset \mathbb{C}$. Theory of elliptic functions (see Riemann surfaces course) tells us that there is a meromorphic function $\wp(z)$, holomorphic on $\mathbb{C}$ apart from double poles at every $z \in \Lambda$, such that $\wp(z + \lambda) = \wp(z)$ for all $\lambda \in \Lambda$. Moreover $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 = g_2\wp(z) - g_3, \quad \text{certain } g_2, g_3 \in \mathbb{C}.$$

The functions $\wp$, $\wp'$ are therefore meromorphic functions on the Riemann surface $T = \mathbb{C}/\Lambda$, and one shows that the map

$$z \mapsto \begin{cases} (1 : \wp(z) : \wp'(z)/2) & \text{if } z \in \mathbb{C} \setminus \Lambda \\ (0 : 0 : 1) & \text{if } z \in \Lambda \end{cases}$$

is then a bijection between T and a smooth plane cubic curve in $\mathbb{P}^2_{\mathbb{C}}$. Now T has an obvious group structure (as a quatient group of $\mathbb{C}$) and this map is an isomorphism of groups (for the group law on the cubic we have defined earlier).

Finally notice that there is an isomorphism

$$\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} T, \ (x_1, x_2) \mapsto x_1\tau_1 + x_2\tau_2 \mod \Lambda$$

and so the subgroup of elements of order dividing n in T is isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}.n\mathbb{Z}$.

Finally consider a curve V of genus $g \geq 2$. If V is hyperelliptic, then we have already seen a fairly precise description of V.

If not, we have in any case $\ell(K) = g \geq 2$. Consider the morphism $\phi_K \colon V \to \mathbb{P}^{g-1}$ given by a canonical divisor K.

Theorem 10.4. *Suppose V is not hyperelliptic. Then $\phi_K \colon V \to \mathbb{P}^{g-1}$ is an embedding.*

Proof. Suppose ϕ_K is not an embedding. Then by the theorem, there exist P and Q with $\ell(K - P - Q) \geq g - 1$. Apply Riemann-Roch to $D = P + Q$. We get $\ell(D) = \ell(K - D) + 1 - g + \deg(D) \geq 2$. So as $g \neq 0$, $\ell(D) = 2$, say $L(D) = k \oplus k.x$ with $(x) = -P - Q + D'$. Then $\phi_D \colon V \to \mathbb{P}^1$ satisfies $\phi_D^*(\infty) = D$, so ϕ_D has degree 2, i.e. V is hyperelliptic. □