5 Examples

Here explore some examples of projective varieties in low dimensions. First start with plane curves of degree d: $V = V(F) \subset \mathbb{P}^2$, F irreducible of degree d.

Proposition 5.1. *$V \subset \mathbb{P}^2$ an irreducible plane curve of degree $d > 1$, $L \subset \mathbb{P}^2$ a line. Then $\#(V \cap L) \leq d$. More precisely, there exist integers $m_P(V, L) \geq 1$ for $P \in V \cap L$ such that*

$$\sum_{P \in V \cap L} m_P(V, L) = d$$

and $m_P(V, L) = 1 \iff L \not\subset T^{proj}_{V,P}$.

Proof. Choose coordinates in $\mathbb{P}^2$ such that $(0:1:0) \notin V$ and $L = \{X_2 = 0\}$. Then $V \cap L \subset \{X_0 \neq 0\} \simeq \mathbb{A}^2$.

Let $V \cap \mathbb{A}^2$ have affine equation $f(x, y) = F(1, x, y)$. Since $(0:1:0) \notin V$, f has degree d in the variable x. Consider $P = (a, 0) \in L \cap \mathbb{A}^2$. Let $m_P(V, L) =$ multiplicity of $x = a$ as root of $f(x, 0)$. Then

$$m_P(V, L) = 1 \iff \frac{\partial f}{\partial x}(a, 0) \neq 0 \iff L \not\subset T_{V,P}$$

since $T^{aff}_{V,P}$ is the line $(\partial f/\partial x)(a, 0)(x - a) + (\partial f/\partial y)(a, 0)y = 0$. □

Proposition 5.2. *Every irreducible conic V is nonsingular and is isomorphic to $\mathbb{P}^1$. Moreover there exist quadratic forms Q_0, Q_1, $Q_2 \in k[X_0, X_1]$) such that $(Q_0 : Q_1 : Q_2) \colon \mathbb{P}^1 \xrightarrow{\sim} V$.*

Proof. Let $P, Q \in V$, $P \neq Q$, $L =$ line PQ. Then 5.1 $\implies L \cap V = \{P, Q\}$ and $L \not\subset T_{V,P}$, so P is a smooth point of V.

Choose coordinates so that $P = (0:0:1)$, $T_{V,P} = \{X_0 = 0\}$. Then $F = X_0(aX_0 + bX_1 + cX_2) - X_1^2$. If $c = 0$ then F is reducible. So $c = 0$ and after further change of coordinates, $F = X_0X_2 - X_1^2$, and so V is the image of $\phi = (Y_0^2 : Y_0Y_1 : Y - 1^2) \colon \mathbb{P}^1 \to \mathbb{P}^2$. In any other system of coordinates ϕ will be given by $(Q_0 : Q_1 : Q_2)$ for quadratics $Q_i \in k[Y_0, Y_1]$. The morphism ϕ is an isomorphism with projection from P as inverse. □

Proposition 5.3. *Let F, $G \in k[X_0, X_1, X_2]$ be coprime homogeneous non-zero polynomials,* $\deg(F) \leq 2$. *Then $\#V(F) \cap V(G) \leq \deg F \cdot \deg G$.*

Proof. If $\deg(F) = 1$ or $F = F_1F_2$ $\deg(F_i) = 1$ then 5.1 gives the result. Otherwise, $V(F)$ is a conic. As $F \nmid G$ there exists $P \in V(F) \setminus V(G)$. Choose coordinates so that $P = (0:0:1)$ and $F = X_0X_2 - X_1^2$. Then $V(F) \cap V(G) \subset \mathbb{A}^2$ so is given by $y - x^2 = 0 = g(x, y)$ i.e. $g(x, x^2) = 0$ where $g(x, y) = G(1, x, y)$. As $\deg(g) \leq d$, $\#V(F) \cap V(G) \leq 2d$. □

(See Thm.7.2 for the result without the hypothesis $\deg(F) \leq 2$.)

Theorem 5.4. *Let $P_1, \ldots, P_5$ be distinct points in $\mathbb{P}^2$, no 3 collinear. Then there exists a unique conic containing $\{P_i\}$ and it is irreducible.*

Proof. If $V = \{F = 0\}$ is a conic through $\{P_i\}$ then F is irreducible (otherwise at least 3 of $\{P_i\}$ would lie on a line). Next, if $V' = \{F' = 0\}$ is another then by 5.3 since $\#V \cap V' > 4$ we have $V' = V$ i.e. F' is a multiple of F.

Finally, the equation $F(P_i) = 0$ is a linear equation in the 6 coefficients of F, so the 5 equations $F(P_i) = 0$ have a nonzero solution, hence F exists. $\square$

Remark. Useful formula:

$$\dim\{F \in k[X_0, \ldots, X_n] \mid F \text{ homogeneous of degree } d\} = \binom{n+d}{n}.$$

An easy way to see this: we have to count the monomials $X_0^{d_0} \ldots X_n^{d_n}$ with $\sum d_i = d$, or equivalent the n-tuples $(d_1, \ldots, d_n) \in \mathbb{N}^n$ with $\sum d_i \leq n$. Associate to this the increasing sequence

$$1 \leq d_1 + 1 < d_1 + d_2 + 2 < \cdots < d_1 + d_2 + \cdots + d_n + n \leq d + n \quad \square$$

The rational normal curve

Let $d \geq 1$. Then

$$\phi_d = (X_0^d : X_0^{d-1} X_2 : \ldots : X_1^d) \colon \mathbb{P}^1 \to \mathbb{P}^d$$

is a morphism. Let $I_d \subset k[Y_0, \ldots, Y_d]$ be the ideal generated by the 2×2 minors of

$$\begin{pmatrix} Y_0 & Y_1 & \cdots & Y_{d-1} \\ Y_1 & Y_2 & \cdots & Y_d \end{pmatrix}$$

Theorem 5.5. *The map ϕ_d is an isomorphism between $\mathbb{P}^1$ and $C_d = V(I_d) \subset \mathbb{P}^d$.*

Proof. Obvious that for all $P \in \mathbb{P}^1$, $\phi_d(P) \in C_d$. Suppose $P = (y_i) \in C_d$. If $y_0 \neq$ then WLOG $P = (1 : y_1 : \ldots : y_d)$ and the relation $Y_0 Y_r - Y_1 Y_{r-1}$ for $1 < r \leq d$ implies that $y_r = t^r$ with $t = y_1$, so $P = \pi((1:t))$. If however $y_0 = 0$ then the relation $Y_2^2 - Y_{r-1} Y_{r+1}$, $1 \leq r \leq d-1$ shows that $y_r = 0$ for $r < d$, hence $P = (0 : \ldots : 0 : 1) = \phi((0:1))$. So $C_d = \phi_d(\mathbb{P}^1)$. Consider the projection $\psi \colon C \longrightarrow \mathbb{P}^1$ given by $(Y_0 : Y_1)$. It is clearly regular except possibly at $(0 : \ldots : 0 : 1) = P_0$. But the relation $Y_0 Y_d - Y_1 Y_{d-1}$ shows that ψ can also be represented by $(Y_{d-1} : Y_d)$ hence is regular at P_0. It is trivial that ϕ_d, ψ are mutual inverses. $\square$

Remarks. (i) Notices that C_d does not lie in any hyperplane, since the monomials $\{X_0^i X_1^{d-i}$ are linearly independent.

(ii) One can show (somewhat harder) that $I_d = I(C_d)$ (equivalently, that I_d is its own radical).

(iii) Any curve projectively equivalent to C_d is called a **rational normal curve of degree** d. For $d = 1$ this is $\mathbb{P}^1$, for $d = 2$ a conic. The next case, $d = 3$ is called a **twisted cubic** — it is the simplest example of a curve in $\mathbb{P}^3$ which does not lie in a plane.

Plane cubics

Theorem 5.6. *Let $V = V(F) \subset \mathbb{P}^2$ where F is an irreducible homogeneous cubic. Then V has at most one singular point. If it does, then there exists a birational morphism $\phi \colon \mathbb{P}^1 \to V$.*

Proof. Let $P_0 \in V$ be a singular point, $P \in V \setminus \{P_0\}$. Consider L_P, the line PP_0. As $L_P \subset \mathbb{P}^2 = T_{V,P_0}^{\text{proj}}$, $m_{P_0}(L_P, V) \geq 2$. As $\deg(F) = 3$, by 5.1 this implies that

(i) $L_P \cap V = \{P_0, P\}$

(ii) $m_{P_0}(L_P, V) = 2$ and

(iii) $m_P(L_P, V) = 1$, so P is a smooth point.

Consider projection from P_0, $\phi\colon \mathbb{P}^2 \to \mathbb{P}^1$. This is regular on $V \setminus \{P_0\}$ and by (i) is injective on $V \setminus \{P_0\}$. Fix coordinates $P_0 = (0:0:1)$, $\phi = (X_0 : X_1)$. Then

$$F = F_3(X_0, X_1) + X_2F_2(X_0, X_1), \quad \deg(F_j) = j,\ \{F_j\} \text{ coprime.}$$

Let L be line $\{X_1 = tX_0\}$ through P_0. Then $L \cap V(F)$ is given by the polynomials $X_1 - tX_0$ and

$$F_3(X_0, tX_0) + X_2F_2(X_0, tX_0) = X_0^2(F_3(1,t)X_0 + F_2(1,t)X_2)$$

so $\phi(P) = (1:t)$ iff $P = (a_i)$, $a_1 = ta_0$, $a_2F_2(1,t) + a_0F_3(1,t) = 0$, and so

$$\psi = (F_2(Y_0, Y_1) : F_2(Y_0, Y_1)Y_1 : -F_3(Y_0, Y_1))$$

is an inverse to ϕ, and is a morphism since F_2 and F_3 are coprime. □

Remark. Later (see after 7.3) we'll show that if $V \subset \mathbb{P}^2$ is a nonsingular cubic then V is not birational to $\mathbb{P}^1$.

Higher-dimension analogue of 5.1 for hypersurfaces:

Proposition 5.7. *Let $V = V(F) \subset \mathbb{P}^n$ with $\deg F = d$ and $L \subset \mathbb{P}^n$ a line not contained in V. Then:*

(i) $\#L \cap V \leq d$

(ii) If there exists $P \in L \cap V$ with $L \subset T_{V,P}$ then $\#L \cap V \leq d - 1$

Remark. If $P \in L \subset V$ then by definition $L \subset T^{\text{proj}}_{V,P}$.

Proof. L is the image of some $\phi = (G_0 : \ldots : G_n)\colon \mathbb{P}^1 \to \mathbb{P}^n$, $G_i \in k[X_0, X_1]$ linear, coprime. Then $H(X_0, X_1) = F(G_0, \ldots, G_n)$ is homogeneous of degree d and $\phi(P) \in V \iff H(P) = 0$. If $H = 0$ then $L \subset V(F)$. Otherwise H is a product of at most d linear factors. If $L \subset T^{\text{proj}}_{V,\phi(P)}$ then H has a repeated factor (by definition of tangent space) so $\#L \cap V \leq d - 1$. □

We now consider **quadric surfaces** $V = V(F)$ where F is homogeneous of degree 2. If F is reducible then V = union of 2 planes (possibly equal).

Proposition 5.8. *If V is an irreducible quadric surface which is singular, then V has exactly one singular point P_0, If $H \subset \mathbb{P}^3$ is any plane through P_0 then $C = H \cap V$ is a nonsingular conic in H and*

$$V = \{P_0\} \cup \{P \in \mathbb{P}^3 \setminus \{P_0\} \mid \textit{the line } PP_0 \textit{ meets } H \textit{ in a point of } C\}$$

Say that V is the **cone** on C with **vertex** P_0.

Proof. Let $P_0 \in V = V(F)$ be a singular point. By 5.7 since $\deg(F) = 2$, for every line $L \subset \mathbb{P}^3$ through P_0, either $L \cap V = \{P_0\}$ or $L \subset V$. Let $H \subset \mathbb{P}^2$ be a plane not containing P_0, $C = V \cap H$. Then V is the union of the lines PP_0, $P \in C$. Now C is nonsingular — if not, then C is a line pair and so V is a union of 2 planes. So taking suitable coordinates we can assume $P = (1:0:0:0)$, $H = \{X_0 = 0\}$ and $C = \{X_0 = 0 = X_1X_2 - X_3^2\}$. Then $V = V(X_1X_2 - X_3^2$ from which it is easy to check that V has no other singular point. □

Theorem 5.9. *Let $V \subset \mathbb{P}^3$ be a smooth quadric. Then for each $P \in V$ there exists exactly two lines in V containing P. The set of all lines on V is the union $\mathcal{L} \cup \mathcal{L}'$ of two disjoint sets of lines satisfying:*

(i) $L, L' \in \mathcal{L} \implies L = L'$ *or* $L \cap L' = \emptyset$ *(and similarly for $\mathcal{L}'$)*

(ii) $L \in \mathcal{L}$, $L' \in \mathcal{L}' \implies \# L \cap L' = 1$.

The families $\mathcal{L}$, $\mathcal{L}'$ are called the **generators** or the quadric.

Proof. Consider $T_P = T^{\text{proj}}_{V,P} \subset \mathbb{P}^3$. By 5.7, if $L \subset T_P$ with $P \in L$, then eight $L \cap V = \{P\}$ or $L \subset V$. So $T_P \cap V$ is a union of lines through P. If $T_P = V(X_3)$ then $T_P = V(X_0, F(X_0, X_1, X_2, 0)))$ so is a conic in the plane T_P, hence is either a line or a line pair. If a line, then $F = GX_3 - H^2$ for some linear forms $G, H \in k[\underline{X}]$ and then V is singular at the point $G = H = X_3 = 0$. So $\forall P \in V$, $T_P \cap V = L_P \cup L'_P$ where L_P, L'_P are distinct lines through P. As any line through P lies in T_P, the first part is proved.

Let $\mathcal{C} = \{\text{all lines on } V\}$. Define a relation $\sim$ on $\mathcal{C}$ by: $L_1 \sim L_2$ iff L_1, L_2 are either equal or disjoint. Then $\sim$ is an equivalence relation: the only nontrivial thing to check is that $L_1 \sim L_3 \sim L_2 \implies L_1 \sim L_2$. If not then $L_1 \cap L_2 = \{P\} \neq \emptyset$ and $L_1 \cap L_3 = L_2$
$calL_3 = \emptyset$, so $T_P \cap V = L_1 \cup L_2$ and therefore $T_P \cap L_3 = \emptyset$, impossible (a line and a plane in $\mathbb{P}^3$ always meet).

As $L_P \not\sim L'_P$ there are at least 2 equivalence classes. If there are more there would be 3 distinct lines L_1, L_2, L_3 on V with $L_i \cap L_j \neq \emptyset$. Then $\{L_j\}$ would be coplanar, which is impossible (the plane would intersect V is a curve of degree ≥ 3). □

Compute equations: choose $P \in V$, $Q \in V \setminus T_P$. Then P, Q, $R = L_p \cap L'_Q$ and $R = L'_P \cap L_Q$ are not on a plan, so can choose coordinates such that

$$P = (1\ 0\ 0\ 0),\ Q = (0\ 0\ 0\ 1), R = (0:1:0:0),\ S = (0:0:1:0$$

in which case

$$\begin{aligned} T_P &= \{X_3 = 0\} \ni P, Q, R \qquad & T_P \cap V &= \{X_1 = X_3 = 0\} \cup \{X_2 = X_3 = 0\} \\ T_Q &= \{X_0 = 0\} \ni Q, R, S \qquad & T_Q \cap V &= \{X_0 = X_1 = 0\} \cup \{X_0 = X_2 = 0\} \end{aligned}$$

so

$$\left.\begin{aligned} F(X_0, X_1, X_2, 0) &= cX_1X_2 \\ F(0, X_1, X_2, X_3) &= c'X_1X_2 \end{aligned}\right\} \implies F = cX_1X_2 + dX_0X_3,\ cd \neq 0$$

or after scaling $F = X_0X_3 - X_1X_2$. In particular, any two nonsingular quadrics are isomorphic by a linear automorphism of $\mathbb{P}^3$.

Segre embedding [I only mentioned this briefly in 2013 lectures.]

The product of affine varieties is an affine variety, since $\mathbb{A}^m \times \mathbb{A}^n \simeq \mathbb{A}^{m+n}$ and if $V \subset \mathbb{A}^m$, $W \subset \mathbb{A}^n$ are varieties the $V \times W = V(I) \subset \mathbb{A}^{m+n}$, where I is the ideal generated by polynomials $f(X_1, \dots, X_m)$ for $f \in I(V)$ and $g(X_{m+1}, \dots, X_{m+n})$ for $g \in I(W)$.

But this does not extend to an isomorphism between $\mathbb{P}^m \times \mathbb{P}^n$ and $\mathbb{P}^{m+n}$. For example, if $m = n = 1$ then $\mathbb{P}^2 = \mathbb{A}^2 \cup (\text{line})$ but $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{A}^2 \cup (2 \text{ lines})$.

Definition: the **Segre embedding** is the map

$$\begin{aligned} \sigma_{mn} \colon \mathbb{P}^m \times \mathbb{P}^n &\to \mathbb{P}^{mn+m+n} \\ ((x_i), (y_j)) &\mapsto (x_i y_j) \end{aligned}$$

where the $(m+1)(n+1)$ variables in $\mathbb{P}^{mn+m+n}$ are labelled Z_{ij}, $0 \le i \le m$, $0 \le j \le n$.

Note this is (just) a map of sets. However, for Q fixed, $P \mapsto \sigma_{mn}(P, Q)$ is a linear morphism $\mathbb{P}^m \hookrightarrow \mathbb{P}^{mn+m+n}$, and likewise for P fixed.

Theorem 5.10. *σ_{mn} is a bijection between $\mathbb{P}^m \times \mathbb{P}^n$ and the projective variety $V = V(I) \subset \mathbb{P}^{mn+m+n}$, where I is the homogeneous ideal generated by polynomials*

$$Z_{ij} Z_{pq} - Z_{iq} Z_{pj}, \quad i, p \in \{0, \dots, m\},\ j, q \in \{0, \dots, n\},\ i \neq p,\ j \neq q.$$

V is irreducible and smooth.

Proof. Clearly $\sigma_{mn}(\mathbb{P}^m \times \mathbb{P}^n) \subset V$. Consider the affine piece $V_{00} = V \cap \{Z_{00} \neq 0\} \subset \mathbb{A}^{mn+m+n}$. The inhomogeneous ideal I_{00} defining V_{00} is (setting $Y_{ij} = Z_{ij}/Z_{00}$) generated by the polynomials

$$Y_{ij} - Y_{i0} Y_{j0}, \quad 1 \le i \le m,\ 1 \le j \le n$$

which contains automatically all the other elements $Y_{ij} Y_{pq} - Y_{iq} Y_{pj}$. So σ_{mn} defines an isomorphism $\mathbb{A}^m \times \mathbb{A}^n \xrightarrow{\sim} V(I_{00})$ with inverse

$$(Y_{ij}) \mapsto ((Y_{10}, \dots, Y_{m0}), (Y_{01}, \dots, Y_{0n})).$$

So $V(I_{00})$ is smooth and irreducible. Repeating this for the other affine pieces $\{Z_{ij} \neq 0\}$ gives the result. □

Consider the case $m = n = 1$. Then $\sigma_{11} \colon \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} V \subset \mathbb{P}^3$ where $V = V(Z_{00} Z_{11} - Z_{01} Z_{10})$ is a smooth quadric, and $\sigma_{11}(\{P\} \times \mathbb{P}^1)$, $\sigma_{11}(\mathbb{P}^1 \times \{q\})$ are the lines in V through the point $\sigma_{11}(P, Q)$.

Veronese surface [I didn't lecture on this in 2013]

This is a higher-dimensional analogue of the rational normal curve. Consider the morphism

$$\begin{aligned} \phi \colon \mathbb{P}^2 &\to \mathbb{P}^5 \\ (X_0 : X_1 : X_2) &\mapsto (X_0^2 : X_0 X_1 : X_0 X_2 : X_1^2 : X_1 X_2 : X_2^2) = (Y_{ij})_{0 \le i \le j \le 2} \end{aligned}$$

Then ϕ is an isomophism between $\mathbb{P}^2$ and $V = V(I) \subset \mathbb{P}^5$, where I is the ideal generated by the 2×2 minors of

$$\begin{pmatrix} Y_{00} & Y_{01} & Y_{02} \\ Y_{01} & Y_{11} & Y_{12} \\ Y_{02} & Y_{12} & Y_{22} \end{pmatrix}$$

(proof similar to that of 5.5 or 5.10). The surface $V \subset \mathbb{P}^5$ is called the **Veronese surface**. It is related to conics in $\mathbb{P}^2$ is two different ways:

(1) Let $P = (x_i) \in \mathbb{P}^2$. Consider

$$\{ \text{ all homogeneous quadratics } F \in k[\underline{X}] \text{ vanishing at } P\} = \{(a_{ij}) \in k^6 | \sum a_{ij}x_ix_j = 0\}$$

(identifying $F = \sum a_{ij}X_iX_j$ with its coefficient vector). This is a codimension 1 subspace of k^6, so corresponds to a dimension 1 subspace of the dual k^6, i.e. to a point in $\mathbb{P}^5$, which is none other than $\phi(P)$.

Let $H \subset \mathbb{P}^5$ be the hyperplane $\sum a_{ij}Y_{ij} = 0$. Then

$$\begin{aligned} H \cap V &= \{\phi(P) \mid P = (x_i), \ \sum a_{ij}x_ix_j = 0\} \\ &= \phi(V(\sum a_{ij}X_iX_j)) \end{aligned}$$

i.e hyperplane sections of V are the images of conics in $\mathbb{P}^2$ under ϕ.

(2) Here we assume $\mathrm{char}(k) \neq 2$. Identify $\underline{y} = (y_{ij}) \in k^6$ with the quadratic polynomial

$$F_{\underline{y}}(T_0, T_1, T_2) = \sum_{0\leq i\leq 2} a_{ii}T_i^2 + 2 \sum_{0\leq i<j\leq 2} a_{ij}T_iT_j.$$

Under this correspondence, points of $\mathbb{P}^5$ correspond to (possibly reducible) conics. If $Q = (y_{ij}) \in \mathbb{P}^5$, then

$$\begin{aligned} Q = \phi(P), \ P = (x_i) &\iff (y_{ij}) = (x_ix_j) \\ &\iff F_{\underline{y}}(T_0, T_1, T_2) = c(x_iT_i)^2 \end{aligned}$$

So the Veronese surface can be identified is the set of those conics which are doubled lines.

Similarly (and more easily), if to $P = (a:b:c) \in \mathbb{P}^2$ we associate the quadratic form $F_P = aU_0^2 + 2bU_0U_1 + cU_1^2$, then since the discrimnant of F_P is $4(B^2 - ac)$, F_P is a square iff P lies on the conic $V(X_0X_2 - X_1^2) \subset \mathbb{P}^2$, which is the image of the 2-tuple embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$.

The Veronese surface has many other fine properties. Let's just note that if $L \subset \mathbb{P}^5$ is a linear subspace of dimension 2, then $L = H_1 \cap H_2$ for distinct hyperplanes H_i, and $H_i \cap V = \phi(C_i)$ for conics $C_i \subset \mathbb{P}^2$. So $\#L \cap V = \#C_1 \cap C_2$ is in general 4 distinct points.

6 Curves

Rest of course will study curves — varieties of dimension 1.

If V is an irreducible curve then its proper subvarieties are finite. For plane curves with was Sheet 1, Ex.5. For the general case, it is enough to prove: let $V \subset \mathbb{A}^n$ be an irreducible affine curve, $W \subset V$ a proper irreducible subvariety. Then W is a point.

Proof: we have $I(V) \subset\neq I(W)$ (by Nullstellensatz) and the homomorphism

$$\phi^* \colon k[V] = k[\underline{X}]/I(V) \to k[W] = k[\underline{X}]/I(W)$$

induced by the inclusion $\phi\colon W \to V$. If W is not a point, then $k[W] \neq k$. If $t \in k[W] \setminus k$ then t is transcendental over k (since k is algebraically closed). Let $0 \neq x \in k[V]$ with $\phi^*(x) = 0$ and $y \in k[V]$ with $\phi^*(y) = y$. Then easy to see that x, y are algebraically independent. But this contradicts $\dim V = \text{tr.deg.}(k(V)/k) = 1$.

So now let $V \subset \mathbb{P}^n$ be an irreducible curve. Associated to V we have:

function field $k(V)$ of V; we know there exists $t \in k(V)$ such that $k(V)/k(t)$ is a finite (and even separable) extension.

Local ring $\mathcal{O}_{V,P} = \mathcal{O}_P = \{f/g \mid g(P) \neq 0\} \subset k(V)$ at a point $P \in V$; $\mathfrak{m}_P$ unique maximal ideal.

Theorem 6.1. *P is a smooth point of V iff $\mathfrak{m}_P \subset \mathcal{O}_P$ is a principal ideal.*

Any π_P such that $\mathfrak{m}_P = (\pi_P)$ is called a **local parameter** at P.

Proof. We'll only prove $\implies$ which is all we need. Assume P lies in an affine piece $V_0 \subset \mathbb{A}^n$ of V and WLOG $P = (0, \dots, 0)$. Then

$$\begin{aligned} k[V_0] &= k[X_1, \dots, X_n]/I(V_0) = k[x_1, \dots, x_n] \quad \text{where } x_i = \text{image of } X_i \\ \mathcal{O}_P &= \left\{ \frac{f}{g} \mid f, g \in k[V_0],\ g \notin (x_1, \dots, x_n) \right\} \\ \mathfrak{m}_P &= \left\{ \frac{f}{g} \mid f \in (x_1, \dots, x_n),\ g \notin (x_1, \dots, x_n) \right\} \\ &= x_1\mathcal{O}_P + \dots + x_n\mathcal{O}_P \end{aligned}$$

More generally, if $J \subset \mathcal{O}_P$ is **any** ideal then

$$J = \left\{ \frac{f}{g} \mid f \in J \cap k[V_0],\ g \in k[V_0],\ g(P) \neq 0 \right\}$$

so in particular is finitely generated (since the ideal $J \cap k[V_0] \subset k[V_0]$ is, by the Hilbert basis theorem).

As P is smooth, after change of coordinates we may assume $T_P^{aff} = \{X_2 = \dots = X_n = 0\}$. Will show $\mathfrak{m}_P = (x_1)$. There exists $f_2, \dots, f_n \in I(V_0)$ such that

$$f_j = X_j - h_j \quad (2 \leq j \leq n)$$

where h_j has no terms of degree < 2. So in $\mathcal{O}_P$ we have

$$x_j = h_j(x_1, \dots, x_n) \in (x_1^2, x_1x_2, \dots, x_n^2) = \mathfrak{m}_P^2, \quad (2 \leq j \leq n)$$

Thus

$$\mathfrak{m}_P = \sum_{j=1}^n x_i\mathcal{O}_P = x_1\mathcal{O}_P + \mathfrak{m}_P^2.$$

Lemma: this implies $\mathfrak{m}_P = (x_1)$.

Lemma 6.2 (Nakayama's Lemma)**.** *R a ring, M a finitely generated R-module, $J \subset R$ an ideal. Then:*

(i) $JM = M \implies \exists r \in R$ *such that* $(1 + r)M = 0$.

(ii) Let $N \subset M$ be a submodule such that $JM + N = M$. Then $\exists r \in J$ such that $(1+r)M \subset N$.

Proof. (i) Let $M = y_1 R + \cdots + y_n R$, $y_i \in M = JM$. Then $y_i = \sum_{j=1}^n x_{ij} y_j$ with $x_{ij} \in J$. Let $X = (x_{ij})$, then have matrix equation $(I_n - X)\underline{y} = 0$. Multiply by adjugate of $(I_n - X)$ to get $\det(I_n - X) f_i = 0\ \forall i$. And $\det(I_n - X) = 1 + z$ for some $z \in J$.

(ii) Apply (i) to the R-module M/N. ☐

Then apply (ii) with $R = \mathcal{O}_P \supset J = \mathfrak{m}_P = M \supset N = (x_1)$. ☐

The local parameter at a smooth point is not unique, but if π_P is one every other is of the form $u\pi_P$, $u \in \mathcal{O}_P^*$ a unit.

Exs: $\mathcal{O}_{\mathbb{P}^1,0}$. Also $\mathcal{O}_{C,0}$ for curves $y^2 = x^3 - x$, $y^2 = x^3 + x^2$. For an affine plane curve $V(f) \subset \mathbb{A}^2$, $f \in k[x,y]$, $x - x(P)$ is a local parameter at a smooth point P iff $(\partial f/\partial y)(P) \neq 0$ (by the proof of 6.1).

Corollary 6.3. *$P \in V$ a smooth point. Then there exists a surjective homomorphism $v_P \colon k(V)^* \to \mathbb{Z}$ (called the **valuation** at P) such that*

$$\mathcal{O}_P = \{0\} \cup \{f \in k(V)^* \mid v_P(f) \geq 0\}$$
$$\mathfrak{m}_P = \{0\} \cup \{f \in k(V)^* \mid v_P(f) > 0\}.$$

and if $f \in k(V)^*$ then for any local parameter π_P at P, $f = \pi_P^{v_P(f)} u$ for some $u \in \mathcal{O}_P^*$.

Proof. We know $\mathfrak{m}_P = (\pi_P)$ so $\mathfrak{m}_P^n = (\pi_P^n)$. Consider $J = \cap_n \mathfrak{m}_P^n$. As $J \subset \mathcal{O}_P$ is an ideal it is finitely generated, and obviously $\mathfrak{m}_P J = \pi_P J = J$. So by Nakayama again, $J = 0$. So for every $f \in \mathcal{O}_P \setminus \{0\}$ there exists unique $n \geq 0$ such that $f \in \mathfrak{m}_P^n \setminus \mathfrak{m}_P^{n+1}$. Set $v_P(f) = n$. If $f \in k(V) \setminus \mathcal{O}_P$ then $f^{-1} \in \mathcal{O}_P$ and we set $v_P(f) = -v_P(f^{-1})$. Thus since $\mathcal{O}_P \setminus \mathfrak{m}_P = \mathcal{O}_P^*$ (local ring) every $0 \neq f \in k(V)$ has $f = u\pi_P^n$, $n = v_P(f)$, $u \in \mathcal{O}_P^*$. Obviously v_P is then a surjective homomorphism. ☐

By convention we write $v_P(0) = \infty$, so $v_P(f)$ is now defined for all $f \in k(V)$.

A **discrete valuation ring** or **DVR** is an integral domain with an element $t \neq 0$ such that every $0 \neq x \in R$ has a unique expression ut^n. Equivalently, it is a local PID.

Consequences for geometry:

Corollary 6.4. *V an irreducible curve, $f \in k(V)$, $P \in V$ a smooth point. Then one of f, f^{-1} is regular at P.*

Proof. f regular at P iff $v_P(f) \geq 0$. ☐

Corollary 6.5. *V a projective nonsingular curve. Then any rational map $\phi \colon V \dashrightarrow \mathbb{P}^m$ is a morphism.*

Proof. Assume the image of ϕ isn't contained in $\{X_0 = 0\}$. Then $\phi = (G_0 : \ldots : G_m) = (1 : g_1 : \ldots : g_m)$ say, $g_i = G_i/G_0 \in k(V)$. If all $g_i \in \mathcal{O}_P$ then ϕ is regular at P. Otherwise $t = \min\{v_P(g_i) \mid 1 \leq i \leq m\}$ is < 0, so $\min\{v_P(\pi_P^{-t} g_i)\} = 0$, hence $\phi = (\pi_P^{-t} : \pi_P^{-t} g_1 : \ldots)$ is regular at P. ☐

Examples: $P = (a) \in \mathbb{A}^1$; here $x - a$ is a local parameter; at $\infty = (0:1) \in \mathbb{P}^1$ local parameter is $1/x = X_0/X_1$ (or $1/(x-a)$ for any a)

$V = V(f) \subset \mathbb{A}^2$, $f \in k[x,y]$ irreducible. Then if $P = (a,b) \in V$, $x - a$ is a local parameter provided $\partial f/\partial y(P) \neq 0$ (tangent not vertical).

Now study morphisms between curves in more detail. Let $\phi\colon V \to W$ be a non-constant morphism of irreducible curves.

Proposition 6.6. *(i) For all $Q \in W$ the set $\phi^{-1}(Q)$ is finite;*

(ii) ϕ induces an inclusion of function fields $\phi^*\colon k(W) \hookrightarrow k(V)$ which makes $k(V)$ a finite extension of $k(W)$.

Proof. (i) $\phi^{-1}(Q)$ is a closed subvariety of V, so it either V (in which case ϕ is constant) or is a finite set.

(ii) V is infinite (since $\dim V > 0$ for example) so by (i) $\phi(V)$ is infinite, hence dense in W. Therefore ϕ is dominant and so $\phi^*\colon k(W) \to k(V)$ is defined (and is injective as $k(W)$ is a field). Let $t \in k(W) \setminus k$, $x = \phi^* t \in k(V)$. Then since $k(V)/k$ is finitely generated and has transcendence degree 1, $k(V)$ is a finite extension of $k(x)$ hence *a fortiori* also of $\phi^* k(W)$. □

The degree $[k(V) : \phi^* k(W)]$ is called the **degree** $\deg(\phi)$ of the morphism ϕ.

Suppose $P \in V$ and $Q = \phi(P) \in W$ are smooth points. We may then define the **ramification degree** of ϕ at P to be

$$e_P = e(\phi, P) = v_P(\phi^* \pi_Q)$$

for any local parameter π_Q on W at Q — note that this doesn't depend on the choice of local parameter.

The next theorem is key to the study of curves.

Theorem 6.7. *(i) Let $\phi\colon V \to W$ be a morphism of* ***projective*** *curves. Then ϕ is surjective.*

(ii) If in addition V and W are smooth, then for any $Q \in W$,

$$\sum_{P \in V,\ \phi(P) = Q} e_P = \deg(\phi).$$

(iii) If $\mathrm{char}(k) = 0$ or more generally if $k(V)/k(W)$ is separable, then for all but finitely many $P \in V$, $e_P = 1$ (equivalently: for all but finitely many $Q \in W$, $\#\phi^{-1}(Q) = \deg(\phi)$).

Statement (i) is a special case of the important theorem: if $\phi\colon V \to W$ is a morphism of projective varieties, then $\phi(V)$ is a closed subvariety of W. Morally, "projective curves have no missing points" (for varieties of dimension > 1 this is an over-simplification).

Statement (ii) is sometimes called the **finiteness theorem** for curves. In short-hand, the fibres of ϕ (this means the sets $\phi^{-1}(\{Q\})$) have the same size, when multiplicities are included in the count.

Remark: this is very similar to the theorem in Number Fields which says that if K is a number field and $p\mathfrak{o}_K = \prod \mathfrak{p}_i^{e_i}$ where $\mathfrak{p}_i$ is a prime ideal of norm p^{f_i}, then $\sum e_i f_i = [K : \mathbb{Q}]$.

We'll prove (iii) a bit later on (time permitting) or you can take it on trust. See the 3rd example sheet for an example of what can go wrong in characteristic p.

Proof. **(Non-examinable and not lectured)** $\square$

Important consequence:

Corollary 6.8. *V smooth projective irreducible curve, $f \in k(V)^*$. Then:*
(i) f regular for all $P \in V \implies f \in k$.
(ii) The set of P such that $v_P(f) \neq 0$ is finite, and $\sum_{P \in V} v_P(f) = 0$.

Proof. Consider the morphism $\phi = (1 : f) \colon V \to \mathbb{P}^1$. Then $\phi(P) = \infty = (0:1) \iff$ f not regular at P, so if f is regular everywhere then $\infty \notin \phi(V)$, and ϕ (and so f) is constant, so (i).

For (ii), the statement is trivial for f constant, so assume not. Then $t = X_1/X_0$ is a local parameter at $0 = (1:0) \in \mathbb{P}^1$ and $\phi^* t = f$ (since as a rational map $\mathbb{P}^1 \to \mathbb{P}^1$, t is the identity). So $f(P) = 0 \implies e_P = v_P(\phi^* t) = v_P(f)$. Likewise, $1/t$ is a local parameter at ∞ so $f(P) = \infty \implies e_P = v_P(\phi^*(1/t)) = -v_P(f)$. Finally, if $\phi(P) \notin \{0, \infty\}$ then $v_P(f) = 0$ so the number of P with $v_P(f) \neq 0$ is finite, and by finiteness theorem,

$$\deg \phi = \sum_{\phi(P)=0} v_P(f) = \sum_{\phi(P)=\infty} -v_P(f).$$

Hence $\sum_P v_P(f) = 0$. $\square$

Morally this says: number of zeros of f = number of poles of f, where a pole is any point P at which $v_P(f) < 0$.