Algebraic Geometry IID 2013

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These are rough lecture notes, in which the details of the examples are mostly left out (as are all the pictures). On the other hand there are some proofs here which I won't give in the lectures (for reasons of time or public decency). These were written when I gave the course in 2009, and although I have changed some things to reflect differences in the way I gave the course this time, the text doesn't follow the lectures word-for-word.

Brief introduction

What the subject is about. Examples of plane curves. Rational parameterisations. Plane cubics (singular and nonsingular).

1 Affine varieties

k here any field

Affine n**-space** $\mathbb{A}^n = k^n$ (as a set); elements are **points** $P = (a_i) = (a_1, \dots, a_n)$

Affine subspace of $\mathbb{A}^n$: any subset of the form $v + U$, $v \in k^n$, $U \subset k^n$ a vector subspace.

$k[\underline{X}] = k[X_1, \dots, X_n]$ (polynomial ring in n variables)

Recall basic facts: (1) $k[\underline{X}]$ is a UFD (Gauss's Lemma). (2) Hilbert basis theorem: Every ideal of $k[\underline{X}]$ is finitely generated (i.e. $k[\underline{X}]$ is Noetherian).

$f \in k[\underline{X}] \implies$ function

$$f\colon \mathbb{A}^n \to k$$
$$P = (a_i) \mapsto f(P) = f(a_1, \dots, a_n)$$

NB: if k is finite funny things can happen (two polynomials can represent the same function). Doesn't arise with infinite fields. Whatever the field, by a **constant** polynomial we always mean an element of $k \subset k[\underline{X}]$.

Linear algebra $\implies$ affine subspaces of $\mathbb{A}^n$ are just those subsets which can be defined by linear equations (not necessarily homogeneous).

Affine closed algebraic set or **affine variety** $V(S)$ determined by $S \subset k[\underline{X}]$ is

$$V(S) = \{P \in \mathbb{A}^n \mid \forall f \in S,\ f(P) = 0\}$$

Although this notion makes sense in general, it is really only useful if k is algebraically closed, which will assume later on.

Ex: $n = 1$ and $0 \neq f \in k[X]$ then $V(f) = \{\text{zeros of } f\}$, a finite subset of $\mathbb{A}^1$, and if $V \subset \mathbb{A}^1$ is finite then $V = V(f)$ with $f = \prod_{a \in V}(x - a)$.

Define **hypersurface** $V(f) = V(\{f\})$, f any non-constant polynomial (i.e. $f \notin k$).

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Theorem 1.1. *(i) $S \subset k[\underline{X}]$, $I =$ ideal generated by S. Then $V(I) = V(S)$.*

(ii) $V(S)$ an affine variety $\implies \exists$ finite set $\{f_j\} \subset S$ of polynomials with $V(S) = V(\{f_j\})$.

Proof. (i) $P \in \mathbb{A}^n$; then $f(P) = 0$ for all $f \in S$ iff $f(P) = 0$ for all $f \in I$ (obviously).

(ii) $V(S) = V(I)$ as in (i); take $\{h_1, \dots, h_r\}$ generators for I. Then can find finite subset $\{f_1, \dots, f_m\} \subset S$ and $g_{ij} \in k[\underline{X}]$ such that $h_i = \sum_{j=1}^m g_{ij} f_j$. Then $\{f_j\}$ also is a set of generators for I, so $V(S) = V(\{f_j\})$. □

Review defn. of $V(S) \subset \mathbb{A}^n$.

Proposition 1.2. *(i) $S \subset T \implies V(T) \subset V(S)$.*

(ii) $V(0) = \mathbb{A}^n$; and $V(k[\underline{X}]) = \emptyset = V(c)$ for any $0 \neq c \in k$.

(iii) $\bigcap_j V(I_j) = V(\sum_j I_j)$ for any family of ideals I_j (finite or not)

(iv) $V(I) \cup V(J) = V(I \cap J)$.

Proof. (i), (ii) trivial

(iii) $\bigcap V(I_j) = V(\bigcup I_j)$ by definition; then apply Thm 1.1(i).

(iv) By (i) $V(I) \cup V(J) \subset V(I \cap J)$. Let $P \in V(I \cap J)$, and suppose $P \notin V(I)$. Then $\exists g \in I$ with $g(P) \neq 0$; and $\forall f \in J$, $fg \in I \cap J$ so $(fg)(P) = 0$, hence $f(P) = 0$ i.e. $P \in V(J)$. □

V is **irreducible** if $V \neq V_1 \cup V_2$ for varieties $V_i \neq V$ $(i = 1, 2)$. **Reducible** = not irreducible.

NB: some people use "variety" to mean "irreducible variety". Later will come across more general types of varieties (projective, quasi-projective).

Ex: $V(X_1(X_2 - X_1^2)) = V(X_1) \cup V(X_2 - X_1^2) \subset \mathbb{A}^2$.

Proposition 1.3. *Every (affine) variety V is a finite union of irreducible varieties.*

Proof. (usual bisection argument) Suppose not. Then V reducible so $= V_1 \cup V_1'$ say. If both of V_1, V_i' are finite unions of irreducibles then done. Otherwise V_1 say isn't. So we get a chain $V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \dots$ of varieties $V_j = V(I_j)$. Let $W = \bigcap_j V_j = V(\sum I_j)$. As $I = \sum I_j$ is finitely generated, must have $I = \sum_{j \le N} I_j$ for some N. Then $W = \bigcap_{j \le N} V_j$ so chain terminates; contradiction. □

Can also show that a minimal decomposition $V = \cup V_i$ into distinct irreducibles is unique up to ordering (exercise). The irreducibles V_i that occur are called the **irreducible components** of V.

Zariski topology on $\mathbb{A}^n$ — say $U \subset \mathbb{A}^n$ is open if U is complement of a variety. Prop.1.2(ii-iv) shows this is a topology.

Ex: $\mathbb{A}^1$. Closed sets are either $\mathbb{A}^1$ itself of finite subsets of k. In particular $\mathbb{A}^1$ is not Hausdorff if k is infinite (since any two nonempty open sets intersect).

Proof of 1.3 shows that Zariski topology on $\mathbb{A}^n$ is **Noetherian** — every descending chain of closed subsets is ultimately stationary.

The topology doesn't say much about a variety. (See example sheet.) More a convenience of language.

When if $V(I) = \emptyset$? E.g. $k = \mathbb{R}$, $V(X_1^2 + X_2^2 + 1) = \emptyset$. So we should look at $k = \bar{k}$.

For $V \subset \mathbb{A}^1$ the answer is easy: every ideal of $k[X]$ is principal, and f nonconstant $\implies V(f) = \{\text{roots of } f \text{ in } k\}$. So if k is algebraically closed, $I \neq k[X] \implies V(I) \neq \emptyset$. General result is harder:

Theorem 1.4 (Hilbert's Nullstellensatz I)**.** *If k is algebraically closed and $I \subsetneq k[\underline{X}]$ is a proper ideal then $V(I) \neq \emptyset$.*

(proof later)

When is $V(I) = V(J)$?

Even for algebraically closed k, can have $V(I) = V(J)$ with $I \neq J$. E.g. let $I = (f)$, $J = (f^d)$ for any $d > 1$.

Given some affine variety $V \subset \mathbb{A}^n$, there is a largest possible ideal for which $V = V(I)$. Namely: define

$$I(V) = \{f \in k[\underline{X}] \mid \forall P \in V \ f(P) = 0\}.$$

Proposition 1.5. *(i) $V = V(S) \implies S \subset I(V)$*

(ii) $V = V(I(V))$.

(iii) $V = W \iff I(V) = I(W)$.

(proof obvious)

So get an injective map: (affine varieties in n-space) $\to$ (ideals $\subset k[\underline{X}]$) given by $V \mapsto I(V)$, which has $V(-)$ as left inverse.

Proposition 1.6. *(i) $V \subset W$ iff $I(V) \supset I(W)$.*

(ii) V irreducible iff $I(V)$ prime.

Proof. (i) $\implies$ obvious. If $V \not\subset W$ let $P \in V \setminus W$. Then $P \notin W = V(I(W)) \implies \exists f \in I(W)$ with $f(P) \neq 0$, i.e. $f \notin I(V)$.

(ii) Obviously $I(V_1 \cup V_2) = I(V_1) \cap I(V_2)$. Suppose $V = V_1 \cup V_2$ is reducible. Let $I_j = I(V_j)$. Then $I(V) = I_1 \cap I_2$ and by (i), $I_1 \not\subset I_2 \not\subset I_1$. Let $f_1 \in I_1 \setminus I_2$, $f_2 \in I_2 \setminus I_1$. Then $f_i \notin I(V)$ but $f_1 f_2 \in I_1 \cap I_2 = I(V)$, so $I(V)$ not prime.

Conversely, suppose $f_1 f_2 \in I(V)$ with $f_i \notin I(V)$. Let $V_i = V \cap V(f_i) = \{P \in V \mid f_i(P) = 0\}$. As $f_i \notin I(V)$, $V_i \neq V$. Then $P \in V \implies f_1(P) f_2(P) = 0 \implies P \in V_1 \cup V_2$ hence $V = V_1 \cup V_2$. ∎

$V \subset \mathbb{A}^n$ affine variety. Then $f \in k[\underline{X}]$ determines a function $V \to k$. And f, g determine the same function iff $f(P) = g(P) \forall P \in V$ i.e. iff $f - g \in I(V)$. The set of all such functions is therefore the quotient ring $k[V] := k[\underline{X}]/I(V)$, the **ring of regular functions** or **coordinate ring** of V — also written $\mathcal{O}(V)$.

Corollary 1.7. *V irreducible iff $k[V]$ is an integral domain.*

Theorem 1.8 (Hilbert's Nullstellensatz II)**.** *Let $k = \bar{k}$, $V = V(I)$. Then $f \in I(V)$ iff for some $m > 0$, $f^m \in I$.*

Ex: let $k = \bar{k}$ and $V = V(f)$ a hypersurface with f irreducible. Then (f) is a prime ideal, so Nullstellensatz implies that $I(V) = (f)$ rather easily. Therefore $I(V)$ is prime and $V(f)$ is irreducible.

For an ideal I (in any commutative ring R) define the **radical** of I to be

$$\sqrt{I} = \{f \in R \mid \exists m > 0 \; st \; f^m \in I\}.$$

Can check that $\sqrt{I}$ is an ideal (exercise), and it's obvious that $\sqrt{\sqrt{I}} = \sqrt{I}$. It follows that $V(I) = V(J)$ iff $\sqrt{I} = \sqrt{J}$.

Summing up: if $k = \bar{k}$, have bijection (inclusion-reversing)

$$\{\text{ideals } I \subset k[\underline{X}] \text{ with } I = \sqrt{I}\} \leftrightarrow \{\text{affine varieties in } \mathbb{A}^n\}$$
$$I \mapsto V(I), \quad V \mapsto I(V)$$

Irreducible varieties correspond to prime ideals.

From now on, assume $k = \bar{k}$ unless explicitly stated to the contrary.

Let $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$. A mapping $\phi\colon V \to W$ is a **regular mapping** or **morphism** if $\exists f_1, \dots f_m \in k[V]$ such that $\phi(P) = (f_1(P), \dots, f_m(P))$. Denote them $\mathrm{Mor}(V, W)$.

For example, $\mathrm{Mor}(V, \mathbb{A}^1) = k[V]$.

Examples: projection $\mathbb{A}^n \to \mathbb{A}^m$, linear and affine transformations, inclusion morphism, d-tuple embedding $\mathbb{A}^1 \to \mathbb{A}^d$.

Composition of polynomials is a polynomial, so $\phi\colon V \to W$, $\psi\colon W \to Z \implies \psi \circ \phi\colon V \to Z$. An **isomorphism** is a morphism with a 2-sided inverse.

Ex: $\mathbb{A}^1 \xrightarrow{\sim}$ conic in $\mathbb{A}^2$.

In particular if $g \in k[W] = \mathrm{Mor}(W, \mathbb{A}^1)$, and $\phi\colon V \to W$ is any morphism, define $\phi^* g = g \circ \phi \in k[V] = \mathrm{Mor}(V, \mathbb{A}^1)$, the **pullback** of g to V. The map $\phi^*\colon k[W] \to k[V]$ is a ring homomorphism (easy) which is identity on k (so a k-algebra homomorphism).

Theorem 1.9. *$V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$. Then $\phi \mapsto \phi^*$ is a bijection*

$$\mathrm{Mor}(V, W) \xrightarrow{\sim} \{k\text{-algebra homomorphisms } k[W] \to k[V]\ \}$$

Rational functions: $V \subset \mathbb{A}^n$ irreducible. **Function field** $k(V) = \operatorname{Frac} k[V]$ (fraction field of integral domain). If $f, g \in k[\underline{X}]$, $g \notin I(V)$ then f/g represents an element of $k(V)$ and determines a map

$$\phi\colon V \setminus V(g) \to k$$

Say ϕ is a **rational function** on V and that $P \in V$ is a **regular point** for ϕ if can find f/g with $\phi = f/g$ and $g(P) \neq 0$.

Ex: $X_1/X_2\colon \mathbb{A}^2 \setminus \{Y_2 = 0\} \to k$

The **local ring** at point $P \in V$ (irreducible) is $\mathcal{O}_{V,P} = \{f \in k(V) \mid f \text{ regular at } P\}$. If $f \in \mathcal{O}_{V,P}$, $f(P) \neq 0$ then $f \in \mathcal{O}^*_{V,P}$. Maximal ideal $\mathfrak{m}_{V,P} = \{f \in \mathcal{O}_{V,P} \mid f(P) = 0\} = \ker(f \mapsto f(P))$.

Define a **local ring** to be a ring with a **unique** maximal ideal. Simple fact:

Proposition 1.10. *R is a local ring iff $R \setminus R^*$ is an ideal. If so then $R \setminus R^*$ is the maximal ideal of R.*

Proof. In any ring, if $A \subset R$ is an ideal, then $A \subsetneq R$ iff $A \cap R^* = \emptyset$ (obvious).

Suppose $\mathfrak{m} = R \setminus R^*$ is an ideal. Then by the previous sentence it is a maximal ideal and contains every proper ideal of R. So it is the unique maximal ideal of R.

Conversely, let $(R, \mathfrak{m})$ be a local ring. Then $\mathfrak{m} \subset R \setminus R^*$, and if $x \in R \setminus R^*$ then $(x) \neq R$ so $(x) \subset \mathfrak{m}$ by uniqueness. Therefore $\mathfrak{m} = R \setminus R^*$. □

2 Projective varieties

Brief introduction — point at infinity on Riemann sphere, parallel lines in $\mathbb{A}^2$.

U f-d vector space over k. Define $\mathbb{P}(U) = \{\text{lines in } U \text{ through } 0\}$. In particular, define $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ **projective n-space**

Usually index the coordinates on k^{n+1} by $0, \ldots, n$. If line is $\{(a_0 t, a_1 t, \ldots, a_n t) \mid t \in k\}$, write $(a_0 : a_1 : \ldots : a_n)$ or simply (a_i) for corresponding element of $\mathbb{P}^n$. Thus

$$\mathbb{P}^n = \{(a_0 : \ldots : a_n) \mid a_i \in k, \text{ not all } 0\}/\sim$$

where $(a_i) \sim (b_i)$ iff $\exists t \in k^*$ with $a_i = t b_i$.

$\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. $\mathbb{P}^2 = \mathbb{A}^2$ plus line at infinity. In general can write $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$.

Affine patches: let $H_j = \{(a_i) \in \mathbb{P}^n \mid a_j = 0\}$ and $U_j = \mathbb{P}^n \setminus H_j = \{(a_i) \in \mathbb{P}^n \mid a_j \neq 0\}$. Then $U_j \xrightarrow{\sim} \mathbb{A}^n$ (set bijection) by

$$(a_0 : \ldots : a_n) \mapsto (a_0/a_j, \ldots, a_{j-1}/a_j, a_{j+1}/a_j, \ldots a_n/a_j) = (a_0/a_j, \ldots, \widehat{a_j/a_j}, \ldots a_n/a_j)$$

(hat means omit the term underneath). Other way:

$$(b_1, \ldots, b_n) \mapsto (b_1, \ldots, b_i, 1, b_{i+1}, \ldots, b_n)$$

Eg usual covering of $\mathbb{P}^1$. Picture for $\mathbb{P}^2$.

Polynomials $k[\underline{X}] = k[X_0, \ldots, X_n]$ aren't functions on $\mathbb{P}^n$.

Terminology:

monomial: $X_0^{d_0} X_1^{d_1} \cdots X_n^{d_n}$, $d_i \geq 0$.
term: $c \times$(monomial), $c \in k^*$.
total degree of term $= \sum d_i$.
homogeneous polynomial of degree d: (possibly empty) sum of terms of total degree d. (So the zero polynomial is homogeneous of every degree).

Every poly of total degree $\leq d$ has a decomposition (unique) as sum of homogeneous parts $f = \sum_{i=0}^{d} f_{[i]}$

f homogeneous of degree d iff $f(TX_0, \ldots, TX_n) = T^d f(X_0, \ldots, X_n) \in k[X_0, \ldots, X_n, T]$.

Partial derivatives defined formally by

$$\partial(X_i^m)/\partial X_j = \begin{cases} m X_i^{m-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Euler's formula: if f is homogeneous of degree $m \geq 0$ then

$$\sum_i X_i \frac{\partial f}{\partial X_i} = m \times f$$

$f \in k[\underline{X}]$ homogeneous of degree d, $(a_i) \in \mathbb{P}^n$. Suppose $b_i = ta_i$ some $t \in k^*$. Then $f((b_i)) = t^d f((a_i))$, so $f((b_i)) = 0 \iff f((a_i)) = 0$. So although f isn't a function on $\mathbb{P}^n$ its zeroes form a well-defined subset of $\mathbb{P}^n$.

For ideals, need a definition:

An ideal $I \subset k[\underline{X}]$ is **homogeneous** if it is generated by a set of homogeneous polys (not necessarily of the same degree)

Lemma 2.1. *$I \subset k[\underline{X}]$. TFAE:*

(i) I is homogeneous;

(ii) If $f \in I$ then its homogeneous parts $f_{[r]}$ are in I.

Proof. (i) $\implies$ (ii): Let g_j be generators of I, homogeneous of degrees d_j. If $f = \sum h_j g_j \in I$ then split each h_j into homogeneous pieces $h_{j[r]}$, then $h_{j[r]} g_j \in I$ so $f = \sum f_{[r]}$ with $f_{[r]} = \sum_j h_{j[r-d_j]} g_j \in I$ homogeneous of degree r.

(ii) $\implies$ (i) trivial (decompose generators of I). □

Definition: let I be a homogeneous ideal. Define

$$V(I) = \{P = (a_i) \in \mathbb{P}^n \mid f((a_i) = 0 \;\forall f \in I\}$$

$V(I)$ is a **projective variety**. (By the lemma, $V(I)$ is the same if we add the condition "f homogeneous".)

Note: if $f_1, \ldots, f_m$ is a set of homogeneous generators for I then $V(I)$ is the set of simultaneous zeros of the f_i.

Examples. Linear subspaces: let $U \subset k^{n+1}$ be a vector subspace, then $\mathbb{P}^(U) \subset \mathbb{P}^n$. If $U = \{\underline{v} \in k^{n+1} \mid \sum_{i=0}^n a_i^{(j)} v_i = 0 \,\forall j\}$ for a subset $\{\underline{a}^{(j)}\} \subset k^{n+1}$ then $\mathbb{P}(U) = V(I)$ where I is the (homogeneous) ideal generated by the linear forms $F_j = \sum_i a_i^{(j)} X_i$. Conversely, any projective variety defined by linear homogeneous polynomials is of this form. Have $\mathbb{P}(U \cap V) = \mathbb{P}(U) \cap \mathbb{P}(V)$. Hypersurfaces.

Affine pieces of projective variety $V = V(I) \subset \mathbb{P}^n$. Let

$$I_0 = \{f = F(1, Y_1, \ldots, Y_n) \mid F \in I \text{ homogeneous}\} \subset k[Y_1, \ldots, Y_n]$$

which is an ideal. Let $V_0 \subset \mathbb{A}^n$ be the affine variety defined by I_0. Then $V_0 = V \cap \mathbb{A}^n$ thinking of $\mathbb{A}^n$ as $U_0 \subset \mathbb{P}^n$.

Likewise, setting $X_j = 1$ defines an ideal I_j whose associated affine variety is $V \cap U_j$.

Projective closure V^* of affine variety V: start with $f \in k[Y_1, \ldots, Y_n]$ of total degree d. Then

$$F(X_0, \ldots, X_n) = X_0^d f(X_1/X_0, \ldots, X_n/X_0) \in k[\underline{X}]$$

is a homogeneous polynomial of degree d, not divisible by X_0, and $F(1, Y_1, \ldots, Y_n) = f$. Consider the (homogeneous) ideal I^* generated by all such F as f runs over $I(V)$. It is the ideal of a projective variety $V^* \subset \mathbb{P}^n$ with $V^* \cap \mathbb{A}^n = V$, called the **projective closure** of V.

Example: projective closure of a plane curve.

Proposition 1.2 holds (same proofs) for projective varieties.

$I^h(V)$ = ideal generated by all homogeneous polys vanishing on V. Assuming k algebraically closed then have:

Theorem 2.2 (Projective Nullstellensatz). *(i) If $V(I) = \emptyset$ then $I \supset (X_0^m, \ldots, X_n^m)$ for some $m > 0$.*

(ii) If $V = V(I) \neq \emptyset$ then $I^h(V) = \sqrt{I}$.

The proof is an easy consequence of the affine result and we omit it.

Let $V \subset \mathbb{P}^n$ be a projective variety. If $W \subset \mathbb{P}^n$ is a projective variety with $W \subset V$ we say that W is a **closed subvariety** of V, and that the complement $V \setminus W$ is an **open subvariety** of V. These satisfy same properties as open and closed sets in topology (by 1.2).

We say V is **irreducible** if $V \neq V_1 \cup V_2$ for proper closed subvarieties V_i.

Proposition 2.3. *(i) Every projective variety is a finite union of irreducibles.*

(ii) V irreducible iff $I^h(V)$ is prime.

The proofs are the same as for affines, once you notice that if I is a homogeneous ideal which is not prime, can find homogeneous F, $G \notin I$ with $FG \in I$.

We say a subset $S \subset V$ is **(Zariski) dense** in V if, for $f \in k[\underline{X}]$ homogeneous, f vanishes on $S \implies f$ vanishes on V.

Proposition 2.4. *Let $V \subset \mathbb{P}^n$ be irreducible and $W \subset V$ a proper closed subvariety. Then $V \setminus W$ is dense in V.*

Proof. Let $f \in k[\underline{X}]$ be homogeneous, vanishing on $V \setminus W$. As $W \neq V$ there exists $g \in I^h(W) \setminus I^h(V)$ (by Nullstellensatz). Then fg vanishes on all of V. As $g \in I^h(V)$ a prime ideal, $f \in I^h(V)$. □

Moral: proper closed subvarieties of an irreducible variety are "smaller" than V. (Later: they have smaller **dimension**.)

Rational functions: $V \subset \mathbb{P}^n$ irreducible variety. Define

$$k(V) = \{F/G \mid F, G \in k[\underline{X}] \text{ homogeneous of same degree, } G \notin I^h(V)\}/\sim$$

where $F_1/G_1 \sim F_2/G_2$ iff $F_1G_2 = F_2G_1$. Easy to check an equivalence relation (using fact that $I^h(V)$ is prime) and that $k(V)$ is a field, the **function field** of V. It is a finitely-generated extension of k (if $X_0 \notin I^h(V)$ it is generated by the rational functions represented by X_i/X_0, $1 \leq i \leq n$).

$\phi \in k(V)$ **regular** at $P \in V$ iff $\phi = F/G$ for some F, G with $G(P) \neq 0$. In this case $\phi(P) := F(P)/G(P)$ is independent of representation, and $\phi\colon V \setminus \{\text{points where } \phi \text{ isn't regular}\} \to k$.

Suppose $V \not\subset \{X_0 = 0\}$. Let $V_0 = V \cap \{X_0 \neq 0\} \subset \mathbb{A}^n$ be the complement of the hyperplane, an affine variety. Then the coordinate functions on V_0 are just the rational functions X_i/X_0 on V, and in particular $k(V) = k(V_0)$.

Local ring of V at P defined the same way as for affine varieties.

Rational maps:

$\boxed{\mathbb{P}^m \dashrightarrow \mathbb{P}^m}$

Let $F_0, \ldots F_m \in k[\underline{X}]$ be homogeneous of same degree d. If $P = (a_i) \in \mathbb{P}^n$ and not all $f_j(\underline{a})$ are zero, can consider $(F_0(\underline{a}) : \ldots : F_m(\underline{a})) \in \mathbb{P}^m$, which is well-defined, so get a map

$$\mathbb{P}^n \setminus \bigcap_j V(f_j) \to \mathbb{P}^m$$

called a **rational map**. Notation: $\phi = (F_i)\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ (the broken arrow to indicate that the map is only partially-defined).

Multiplying F_i by a common G gives essentially the same map (except the set where it isn't defined will be possibly larger). As $k[\underline{X}]$ is a UFD there is a best choice of $\{F_i\}$ got by cancelling common factors. The points where ϕ is defined are the **regular points** of ϕ. If every point is regular, ϕ is a **morphism** written with $\to$. It is an **isomorphism** if there is a morphism $\psi\colon W \to V$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity morphisms on W and V respectively.

Examples:

Linear maps $\phi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, given by any $(m+1)\times(n+1)$ matrix $(a_{ij}) \neq 0$ so $\phi = (F_j)$ with $F_j = \sum_i a_{ij}X_i$. If (a_{ij}) has rank $n+1 \leq m+1$ then ϕ is a morphism, whose image a linear projective subspace (and an isomorphism if $m = n$, with inverse given by the inverse matrix).

Remark: all automorphisms of $\mathbb{P}^n$ are linear. Any morphism $\mathbb{P}^n \to \mathbb{P}^m$ with $n > m$ is necessarily constant.

Example where not a morphism: projection from a point $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$.

Now let $V \subset \mathbb{P}^n$ be an irreducible variety. If $F_0, \dots, F_m \in k[\underline{X}]$ are homogeneous of the same degree, not all in $I^h(V)$, they are said to determine a rational map $\phi\colon V \dashrightarrow \mathbb{P}^m$. It is a mapping of the nonempty open subvariety

$$V \setminus \bigcap_j (V \cap V(F_j)) \to \mathbb{P}^m$$

Two sets of polyns (F_j), (G_j) are said to determine the same rational map if $F_iG_j - F_jG_i \in I^h(V)$ for all i, j. The point $P \in V$ is a **regular point** of ϕ if ϕ has a representation (F_i) with not all $F_i(P) = 0$. If all P are regular, ϕ is a morphism (may need to use different representations of ϕ at different points!). The **domain** $\operatorname{dom}(\phi)$ of ϕ is its set of regular points. It is a nonempty open subvariety of V (easy to see). If $W \subset \mathbb{P}^m$ and $\phi(\operatorname{dom}(\phi)) \subset W$ then ϕ is a rational map from V to W, written $\phi\colon V \dashrightarrow W$.

Example: rational function are rational maps $V \dashrightarrow \mathbb{P}^1$ (all of them apart from the morphism which maps every point of V to $\infty \in \mathbb{P}^1$).

Example: conic to point by projection from a point.

Conic $X_1^2 = X_0X_2$. First project from $(0:1:0)$ (not on conic) by $(X_0 : X_2)$. Regular at all points.

Next project from $(0:0:1)$ by $(X_0 : X_1) = (X_1 : X_2)$. Note both $(0:0:1)$ and $(1:0:0)$ are on V so we need both forms to get the morphism.

This is an **isomorphism**, inverse $(Y_0 : Y_1) \mapsto (Y_0^2 : Y_0Y_1 : Y_2^2)$.

Let $\phi\colon V \to W$ be an morphism of (projective or affine) varieties. Let $Z \subset W$ be a closed subvariety. The set $\phi^{-1}(Z) = \{P \in V \mid (\phi(P) \in Z\}$ is the **inverse image** of Z under ϕ. It is easily seen to be a closed subvariety of V (since the condition "$\phi(P) \in Z$" is equivalent to the vanishing of certain polynomials in the coordinates of P).

Suppose $\phi\colon V \dashrightarrow W$, $\psi\colon W \dashrightarrow Z$ are rational maps. The composite $\psi \circ \phi$ isn't always defined (since the image of ϕ could consist entirely of points at which ψ is not regular).

Suppose $\phi(\operatorname{dom}\phi) \subset W$ is **dense** in W. Then we say ϕ is **dominant** and in this case $\psi \circ \phi$ is defined for any ψ.(This is the analogue of surjectivity for rational maps.)

If $\psi\colon W \dashrightarrow V$ is such that $\psi \circ \phi$, $\phi \circ \psi$ are defined and equal the identity maps of V, W respectively, then we say ϕ is **birational** (or a **birational equivalence** or **birational isomorphism**).

Exs: Obviously any isomorphism is birational, but there are lots of other important examples.

Cremona transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by $(X_1X_2 : X_0X_2 : X_0X_1)$ (think of this as $(1/X_0 : 1/X_1 : 1/X_2)$, so obviously $\phi \circ \phi$ is the identity).

See ex. sheet for others.

Last lecture defined rational map, said what it meant for a rational map $\phi\colon V \dashrightarrow W$ to be **dominant**, **birational**.

If ϕ is dominant then can compose rational functions on W with ϕ to give a map $\phi^*\colon k(W) \to k(V)$ which is a homomorphism (easy to check). So if ϕ is birational, it induces an isomorphism $k(W) \xrightarrow{\sim} k(V)$. Very important fact:

Theorem 2.5. *Let U, V be irreducible varieties. Then U, V are birationally isomorphic iff $k(U) \simeq k(V)$.*

So study of varieties up to birational equivalence is equivalent to the study of their function fields.

Proof. (Sketch as the details are tedious) Let $V \subset \mathbb{P}^n$ not contained in $\{X_0 = 0\}$, $W \subset \mathbb{P}^m$ not contained in $\{Y_0 = 0\}$. Then $k(V) = k(x_1, \ldots, x_n)$, $x_i = X_i/X_0$ and $k(W) = k(y_1, \ldots, y_m)$, $y_j = Y_j/Y_0$. An isomorphism $k(V) \simeq k(W)$ identifies y_j with $f_j(\underline{x})$, for some rational functions f_j in n variables. Clearing denominators and homogenising, get $m+1$ homogeneous $F_j \in k[\underline{X}]$ with

$$f_j(X_1/X_0, \ldots, X_n/X_0) = \frac{F_j(X_0, \ldots, X_n)}{F_0(X_0, \ldots, X_n)}$$

and $(F_0 : \ldots : F_m)$ determines a rational map $V \dashrightarrow W$. Writing the x_is in terms of $\{y_j\}$ defines a map in the other direction. It is tedious but straightforward to check these are mutually inverse rational maps. □

Remark: can also regard a rational map $V \dashrightarrow W \subset \mathbb{P}^m$, $W \not\subset \{X_0 = 0\}$ as an m-tuple of rational functions $(f_1, \ldots, f_m)$ (get an $(m+1)$-tuple (F_j) by clearing denominators). Likewise can define rational maps between affine varieties.

Finish this section with:

Defn: a **variety** is an open subvariety of a projective variety.

This includes affine and projective varieties as special cases. Note that there are varieties which are neither: an example is $\mathbb{A}^2 \setminus \{(0,0)\}$.

3 Some commutative algebra

Hilbert Nullstellensatz

Recall k algebraically closed.

Theorem 3.1 (= Theorem 1.4, Hilbert Nullstellensatz I)**.** *(i) Every maximal ideal of $k[\underline{X}]$ is of the form $(X_1 - a_1, \dots, X_n - a_n)$ for $a_i \in k$.*

(ii) If $I \subsetneq k[\underline{X}]$ then $V(I) \neq \emptyset$.

Proof. I'll prove this only in the case k uncountable (eg $k = \mathbb{C}$). For the general case see Reid or Hulek.

(i) We know every ideal of this form is maximal (example sheet). So let $\mathfrak{m} \subset k[\underline{X}]$ be a maximal ideal, $K = k[\underline{X}]/\mathfrak{m}$ and $a_i = X_i + \mathfrak{m} \in K$. Then K is a field and $K = k[a_1, \dots, a_n]$. If $K = k$ then $a_i \in k$ and $X_i - a_i \in \mathfrak{m}$, and are done.

Otherwise, let $t \in K \setminus k$. As $k = \bar{k}$ have $k \subset k(t) \subset K$ and t is transcendental over k, i.e. $k(t)$ is the field of rational functions in t. Let $U_m \subset K$ be the k-vector subspace spanned by the products $\{a_1^{r_1} \cdots a_n^{r_n}\}$ with $\sum r_i \leq m$. Clearly $\dim U_m < \infty$ and and $K = \bigcup U_m$. Now $\{1/(t-c) \mid c \in k\}$ are linearly independent over k, so only finitely many of them can lie in each V_m. Therefore the number belonging to $K = \bigcup U_m$ is countable. As K is uncountable, we have a contradiction.

(ii) By Zorn's lemma (or in the this case using ACC for ideals) there exists a maximal ideal $\mathfrak{m} \subset k[\underline{X}]$ containing I. By (i), $\mathfrak{m} = V(P)$ for some $P = (a_1, \dots, a_n) \in \mathbb{A}^n$, so $P \in V(I)$. □

Theorem 3.2 (= Theorem 1.8, Nullstellensatz II)**.** $V = V(I) \implies I(V) = \sqrt{I}$.

Proof. **(Not given in lectures and not examinable)** Let $f \in I(V)$. Consider the ideal $J \subset k[X_1, \dots, X_n, T]$ generated by the elements of I and the polynomial $1 - fT$. If $P = (a_1, \dots, a_{n+1}) \in V(J)$ then $f(a_1, \dots, a_n) = 0$ (as $f \in I$) but $1 - a_{n+1} f(\underline{a}) = 0$. So $V(J) = \emptyset$, hence by Nullstellensatz I, $J = k[\underline{X}, T]$. So $1 \in J$ which can be written as

$$1 = \sum_{r=0}^{m} T^r h_r + (1 - fT) g$$

for some $h_r \in I$ and $g \in k[\underline{X}, T]$. WLOG may assume that $m \geq$ the T-degree of g. Multiplying by f^m we then get

$$f^m = \sum_{r=0}^{m} f^m T^r h_r + (1 - fT) f^m g(\underline{X}, T) = \sum_{r=0}^{m} f^{m-r} h_r (fT)^r + (1 - fT) g_1(\underline{X}, fT)$$

for some polynomial g_1. Set $T = 1/f$ in this identity[2] to get $f^m = \sum_{r=0}^{m} f^{m-r} h_r$, i.e. $f^m \in I$. □

Transcendence basis

Terminology: Let K/k be a finitely generated field extension. K/k is a **pure transcendental** extension if $K = k(x_1, \dots, x_n)$ for $x_1, \dots, x_n \in K$ algebraically independent over k.

Proposition 3.3. *Let K/k be a finitely generated field extension. Then there exists a pure transcendental subextension $K_0 = k(x_1, \dots, x_n) \subset K$ such that K/K_0 is finite and separable. Moreover $K = K_0(y)$ for some $y \in K$.*

Remark. Later will see (remark (ii) following Thm.4.5) that the integer n is unique. It's called the **transcendence degree** of K/k.

[2] This means compute the image under the homomorphism $k[\underline{X}, T] \to k(\underline{X})$ given by $X_i \mapsto X_i$, $T \mapsto 1/f$.

Proof (char$(k) = 0$). The last part is just the primitive element theorem.

For the rest, suppose $k = K(x_1, \ldots, x_m)$. There is a maximal subset of $\{x_i\}$ which is algebraically independent. After reordering let it be $\{x_1, \ldots, x_n\}$. Then each of $x_{n+1}, \ldots, x_m$ is algebraic over $k(x_1, \ldots, x_n)$ so $K/k(x_1, \ldots, x_n)$ is finite. When char$(k) = 0$ it is automatically separable. □

Proof of Propn. 3.3 when char$(k) = p$. **This was not given in the lectures and is not examinable.** (See also Reid who gives a more general result. However there is a subtle error in the proof he gives of his (3.16), also reproduced in Hulek, Prop. 1.33 — see if you can spot it. The result as Reid states it is true but needs a different proof — see for example Zariski–Samuel *Commutative Algebra* vol.I, Ch.5, sec.4 Thm.8.)

There certainly exist subfields $K_0 \subset K_1 \subset K$ with K_0 pure transcendental, K_1/K_0 finite and separable and K/K_1 finite (e.g. $K_1 = K_0 = k(x_1, \ldots, x_n)$ as in the char = 0 proof). So there exists $K_0 \subset K_1 \subset K$ with K_1 maximal. Let $K_0 = k(x_1, \ldots, x_n)$ and $K_1 = K_0(y)$, with y algebraic and separable over K_0. By maximality of K_1, K/K_1 is purely inseparable. If $K = K_1$ we're done; otherwise there exists $z \in K \setminus K_1$ with $z^p = t \in K_1$. By the proposition there is an irreducible polynomial $g(X_1, \ldots, X_n, T)$ such that $g(x_1, \ldots, x_n, z^p) = 0$. As K_1/K_0 is separable, g is separable in the variable T. Suppose g is not of the form $h(X_1^p, X_2, \ldots, T)$. Then x_1 is separable over $K_0' = k(x_2, \ldots, x_n, z)$ and y is separable over $K_0 \subset K_0'(x_1)$. Therefore $K_1(t)$ is separable over K_0' which is a pure transcendental extension of k, by the proposition. This contradicts the maximality of K_1.

Therefore we must have $g(X_1, \ldots, X_n, T) = h(X_1^p, \ldots, X_n^p, T)$. Let h^* be the polynomial whose coefficients are the p-th roots of those of h. Then $h^*(x_1, \ldots, x_n, z) = 0$ which shows that z is separable over K_0, contradiction. □

Proposition 3.4. *Let $K = k(x_1, \ldots, x_n)$ with $(x_1, \ldots, x_n)$ algebraically independent, and let x_{n+1} be algebraic over K. Then*

$$I = \{g \in k[X_1, \ldots, X_{n+1}] \mid g(\underline{x}) = 0\}$$

is a principal ideal (f) generated by some irreducible $f \in k[\underline{X}]$. Moreover if f contains the variable X_i then $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$ is algebraically independent.

In other words, $k[x_1, \ldots, x_{n+1}] = k[\underline{X}]/I = k[\underline{X}]/(f)$.

Proof. As $x_1, \ldots, x_n$ are algebraically independent, the ring $R = k[x_1, \ldots x_n]$ is isomorphic to the polynomial ring $k[X_1, \ldots, X_n]$ so is a UFD. Let $h \in K[T]$ be the minimal polynomial of x_{n+1} over K, and let $b \in R$ be the least common denominator of its coefficients. Then bh is irreducible in $R[T]$ by Gauss's Lemma. Therefore $bh = f[x_1, \ldots, x_n, T]$ for some irreducible $f \in k[X_1, \ldots, X_{n+1}]$.

Let $g \in k[\underline{X}]$. Then in the ring $K[T]$, $g(x_1, \ldots, x_n, T)$ is a multiple of h, so again applying Gauss's Lemma g is a multiple of f.

For the last part, can assume $1 \le i \le n$. Suppose $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$ is not algebraically independent. Then there exists $0 \neq g \in I$ which does not involve X_i. But as g is a multiple of f have a contradiction. □

Corollary 3.5. *Let $k = \bar{k}$. Then any irreducible variety V is birational to a hypersurface.*

Proof. Let $K = k(V)$. By Propositions 3.3, we can write $K = k(x_1, \ldots, x_{n+1})$ where $x_1, \ldots, x_{n+1}$ are algebraically independent and x_{n+1} is algebraic over $k(x_1, \ldots, x_n)$, and by proposition 3.4 $k[x_1, \ldots, x_{n+1}] = k[\underline{X}]/(f)$ for some irreducible polynomial $f(X_1, \ldots, X_{n+1})$. Therefore K equals the function field of the hypersurface $V(f)$. Result follows by Thm. 2.5. □

4 Singularities, smoothness and dimension

Motivation: $V = V(f) \subset \mathbb{A}^n$ affine hypersurface, f irreducible, $P = (a_i) \in V$. Consider affine line through P

$$L = \{(a_1 + tb_1, \ldots, a_n + tb_n) \mid t \in k\}, \quad 0 \neq \underline{b} \in k^n$$

Compute $V \cap L$ by

$$0 = f(a_1 + tb_1, \ldots) = g(t) = \sum_r c_r t^r$$

with $c_0 = f(\underline{a}) = 0$, $c_1 = \sum_i b_i (\partial f/\partial X_i)(\underline{a})$. Then g vanishes at $t = 0$ because $P \in V \cap L$. Also, g has a zero of order > 1 at $t = 0$ (i.e. L is tangent to V at P) iff L is contained in the affine subspace

$$T^{\text{aff}}_{V,P} = V(g) \subset \mathbb{A}^n, \quad g = \sum_{i=1}^{n} (\partial f/\partial X_i)(P)(X_i - a_i).$$

Definition: $T^{\text{aff}}_{V,P}$ is the **(affine) tangent space** of V at P.

So $T^{\text{aff}}_{V,P}$ is either an affine space of dimension $n - 1$ or the whole of $\mathbb{A}^n$. The point P is **smooth** (or **nonsingular** or **regular**) in the first case, and is **singular** otherwise.

Example: $f = X_2^2 - X_1^2(X_1 + 1)$.

Need to be able to compute also for projective $V = V(F) \subset \mathbb{P}^n$, $F \in k[X_0, \ldots, X_n]$ homogeneous, irreducible.

Defn: **(projective) tangent space** of $V = V(F)$ at $P = (a_0 : \ldots : a_n)$ is

$$T^{\text{proj}}_{V,P} = V(G) \subset \mathbb{P}^n, \quad G = \sum_{i=0}^{n} X_i (\partial F/\partial X_i)(\underline{a})$$

Remarks. (i) $T^{\text{proj}}_{V,P}$ is a linear projective subspace containing P, since $\deg(F)G(P) = F(P) = 0$ (Euler's formula).

(ii) Assume $V \not\subset \{X_0 = 0\}$ and let $V_0 = V \cap \mathbb{A}^n \subset \mathbb{A}^n$ given by $f(X_1, \ldots, X_n)$ where

$$F(X_0, \ldots, X_n) = X_0^{\deg F} f(X_1/X_0, \ldots, X_n/X_0)$$

then computing $\partial F/\partial X_i$ shows that if $P \in V_0$ then $T^{\text{proj}}_{V,P} \cap \mathbb{A}^n = T^{\text{aff}}_{V_0,P}$.

In either case, it is a linear subvariety of dimension $n - 1$ or n.

As in the affine case, if the tangent space has dimension $n - 1$ we say V is **smooth/nonsingular/regular** at P; otherwise P is a **singular** point.

So P is a singular point iff all the partial derivatives $\partial f/\partial X_i$, $1 \leq i \leq n$ (in the affine case) or $\partial F/\partial X_i$, $0 \leq i \leq n$ (in the projective case) vanish at P.

Ex: plane curve $V(X_2^2 X_0 - X_1^2(X_1 + X_0))$ has one singular point $(1:0:0)$ (draw picture over $\mathbb{R}$).

Proposition 4.1. *The set of smooth points of an irreducible hypersurface is a nonempty open subvariety.*

Proof. (For V projective.) The set of singular points is $V \cap \bigcap_i V(\partial F/\partial X_i)$ which is a closed subvariety of V. If it were all of V then by Nullstellensatz, $\partial F/\partial X_i \in I^h(V) = (F)$ for all i. Since $\partial F/\partial X_i$ is homogeneous of degree $< \deg F$, would then have $\partial F/\partial X_i = 0$ for all i. Two cases:

- $\mathrm{char}(k) = 0$. Then F is constant, contradiction.
- $\mathrm{char}(k) = p > 0$. Then $F \in k[X_0^p, \dots, X_n^p]$ so $F = G^p$ for some polynomial G (remember k is algebraically closed), contradiction. □

Now consider a general variety V. It turns out best to consider the tangent space as a vector space, rather than affine or projective space.

Definition (i) Let $V \subset \mathbb{A}^n$ be an affine variety, $P \in V$. Define

$$T_{V,P} = \{\underline{v} \in k^n \mid \sum_{i=1}^{n} v_i \frac{\partial f}{\partial X_i}(P) = 0 \ \forall f \in I(V)\} \subset k^n$$

(ii) Let $V \subset \mathbb{P}^n$ a projective variety. Let $P \in V$ and let $V_j = V \cap \{X_j \neq 0\}$ be an affine piece of V containing P. Define $T_{V,P} = T_{V_j,P}$ as in (i).

If $V \subset \mathbb{A}^n$ is a hypersurface, then $T^{\text{aff}}_{V,P} = P + T_{V,P}$.

At a 'smooth' point of a variety V we expect the number of independent tangent directions to be a measure of the size/dimension of V. So we define:

Defn: V an affine or projective variety.

(i) V irreducible: define $\dim V = \min\{\dim T_{V,P} \mid P \in V\}$

(ii) $P \in V$ is **smooth/nonsingular** points if $\dim T_{V,P} = \dim V$, **singular** otherwise

(iii) In general, $\dim V =$ largest dimension of irreducible components of V.

The next result shows that this is a good notion.

Theorem 4.2. *The set of smooth points of V is a non-empty open subvariety.*

Proof. Obviously non-empty, by definition. We can assume that $V \subset \mathbb{A}^n$ is affine (if V is projective, just treat each affine pieces of V in turn) and that $I(V)$ is generated by polynomials f_j. Then if $P \in V$,

$$T_{V,P} = \{\underline{v} \in k^n \mid \sum_i v_i (\partial f_j/\partial X_i)(P) = 0\}$$

and so

$$\dim T_{V,P} = n - \operatorname{rank}\left(\frac{\partial f_j}{\partial X_i}(P)\right)$$

and for any $r \in \mathbb{N}$,

$$\{P \in V \mid \dim T_{V,P} \geq r\} = \{P \mid \operatorname{rank}((\partial f_j/\partial X_i)(P)) \leq n - r\}$$

is the closed subvariety of V given by the $(n-r) \times (n-r)$ minors of the matrix of polynomials $(\partial f_j/\partial X_i)$. □

Now suppose we have projective varieties $V \subset \mathbb{P}^n$, $W \subset \mathbb{P}^m$ and a rational map $\phi\colon V \dashrightarrow W$, and $P \in \operatorname{dom}(\phi)$. We will define a linear map $d\phi_P\colon T_{V,P} \to T_{W,\phi(P)}$. Assume that $P \in V \cap \mathbb{A}^n$, $\phi(P) = Q \in W \cap \mathbb{A}^m$, and that $\phi = (F_0 : \ldots : F_m)$ for homogeneous $F_j \in k[\underline{X}]$. Write $(F_j/F_o)(1, X_1, \ldots, X_n) = f_j \in k(X_1, \ldots, X_n)$, which represents a rational function on V, regular at P.

Definition: $d\phi_P$ is the map $T_{V,P} \to k^m$ given by

$$(d\phi_P)(\underline{v}) = \left(\sum_{i=1}^{n} v_i \frac{\partial f_j}{\partial X_i}(P) \right)_j \in k^m$$

Proposition 4.3. *(i)* $d\phi_P(T_{V,P}) \subset T_{W,\phi(P)}$.

(ii) $d\phi_P$ *depends only of* ϕ*, not on the polynomials* (F_i) *representing it.*

(iii) If $\psi\colon W \dashrightarrow Z$ *is a rational map with* $\phi(P) \in \operatorname{dom}(\psi)$ *then* $d(\psi \circ \phi)_P = d\psi_{\phi(P)} \circ d\phi_P$.

(iv) If ϕ *is birational and* ϕ^{-1} *is regular at* $\phi(P)$ *then* $d\phi_P$ *is an isomorphism.*

Proof. (i) We can replace V by the affine pieces $V \cap \mathbb{A}^n$, $W \cap \mathbb{A}^m$. Let $g \in I(W)$, so that $h = g(f_1, \ldots, f_m) \in k(\underline{X})$ is a rational function regular at P, vanishing on those points of V where it is regular. Then (chain rule)

$$\frac{\partial h}{\partial X_i}(P) = \sum_j \frac{\partial g}{\partial Y_j}(Q) \frac{\partial f_j}{\partial X_i}(P)$$

so if $\underline{v} \in T_{V,P}$, we see that $d\phi_P(\underline{v}) \in T_{W,Q}$.

(ii) If we take another representation (F'_j) for ϕ then the corresponding rational functions $f'_j \in k(\underline{X})$ will have the property that $f'_j - f_j$ vanishes on V wherever it is defined, so $f'_j - f_j = p_j/q_j$ where $p_j \in I(V)$ and $q_j \in k[\underline{X}]$, $q_j(P) \neq 0$. Then

$$\frac{\partial(f'_j - f_j)}{\partial X_i}(P) = \frac{1}{q_j(P)} \frac{\partial p'_j}{\partial X_i}(P)$$

Let $\underline{v} \in T_{V,P}$. Then the last equation shows that for every j

$$\sum_{i=1}^{n} v_i \frac{\partial(f'_j - f_j)}{\partial X_i}(P) = 0$$

so the map $d\phi_P$ is independent of the representation of ϕ.

(iii) This is just the chain rule.

(iv) follows from (iii), and implies by 4.2: □

Corollary 4.4. *Birational (irreducible) varieties have the same dimension.*

Proposition 4.3 shows that the tangent space $T_{V,P}$ is an **intrinsic** invariant of the variety at the point P. (In fact there is a way to define $T_{V,P}$ purely in terms of the local ring $\mathcal{O}_{V,P}$ and its maximal ideal $\mathfrak{m}_{V,P}$: it is isomorphic to the dual of the k-vector space $\mathfrak{m}_{V,P}/\mathfrak{m}^2_{V,P}$. See Reid or Hulek for details.)

Now give another characterisation of dimension: recall the definition of **transcendence degree** of a finitely-generated field extension K/k.

Theorem 4.5. *If* V *is an irreducible variety then* $\dim V = \text{tr.deg.}(k(V)/k)$.

Proof. We saw (§3) that V is birational to a hypersurface, and (§2) that birational varieties have isomorphic function fields. So by 4.4 we may assume that V is a hypersurface, say $V = V(f) \subset \mathbb{A}^n$, $f \in k[X_1, \ldots, X_n]$ irreducible, and WLOG $f \notin k[X_1, \ldots, X_{n-1}]$. So $k(V) = k(x_1, \ldots, x_n)$ where $x_1, \ldots, x_{n-1}$ are algebraically independent and $f(x_1, \ldots, x_n) = 0$. So tr.deg.$(k(V)/k) = n - 1 = \dim(V)$ □

Remarks. (i) Another characterisation of dimension: consider chains of closed *irreducible* subvarieties W_i

$$V \supset W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_d \neq \emptyset$$

Theorem. $\dim V =$ *maximum d for which such a chain exists.*

Cf. example sheet I, Q8 where you show that the only irreducible subvarieties of a curve are points. We won't prove this result in the course for varieties of higher dimension (for curves we'll do so at the beginning of §6).

(ii) Proof of Thm.4.5 also proves that transcendence degree is well-defined (since $\dim V$ only depends on $k(V)$ by 2.5 and 4.4)