9 Idele class group

Already seen that the content map induces surjections from $C_K = J_K/K^*$ and J_K^1/K^* to the ideal class group $Cl(K) = I(K)/P(K)$. It's important for Class Field Theory to understand *all* the finite quotients of C_K.

Proposition 9.1. *Let G be a discrete group.*
(i) Any continuous homomophism $\alpha\colon C_K \to G$ has finite image.
(ii) There is a bijection:

$$\left\{\begin{matrix}\text{continuous homomor-}\\ \text{phisms } \alpha\colon J_K \to G\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{families } (\alpha_v\colon K_v^* \to G)_v \text{ of continu-}\\ \text{ous homomorphisms with } \alpha_v(\mathfrak{O}_v^*) =\\ \{1\} \text{ for almost all finite } v\end{matrix}\right\}$$

Proof. (i) As $J_K/J_K^1 \simeq \mathbb{R}$, we must have $\alpha(C_K) = \alpha(J_K^1/K^*)$ and result follows from compactness of J_K^1/K^*.

(ii) The subgroup $\bigoplus_v K_v^* \subset J_K$ is dense (since $\oplus\mathfrak{O}_v^* \subset \prod \mathfrak{O}_v^*$ is dense). So if $\alpha\colon J_K \to G$ is continuous, it is determined by the family $(\alpha_v = \alpha|_{K_v^*}\colon K_v^* \to G)_v$. As $\ker\alpha$ is open, $\alpha_v(\mathfrak{O}_v^*) = 1$.

Conversely, if $(\alpha_v)_v$ is such a family, then the formula $\alpha(x) = \prod_v \alpha_v(x_v)$ is a finite product if $x \in J_K$ and defines a continuous homomorphism α. □

Definition. A *modulus* is a finite formal sum $\mathfrak{m} = \sum_{v\in\Sigma_K} m_v(v)$ with $m_v \in \mathbb{N}$. The *support* and *finite support* of $\mathfrak{m}$ are the sets

$$\mathrm{supp}(\mathfrak{m}) = \{v \in \Sigma_K \mid m_v > 0\}, \quad \mathrm{supp}_f(\mathfrak{m}) = \mathrm{supp}(\mathfrak{m}) \cap \Sigma_{K,f}.$$

Define $U_{K,\mathfrak{m}} = \prod_{v\in\Sigma_K} U_v^{m_v} \subset J_K$ where

$$U_v^m \begin{cases} \mathfrak{O}_v^* & v \text{ finite, } m = 0 \\ 1 + \pi_v^m \mathfrak{O}_v & v \text{ finite, } m > 0 \\ K_v^* & v \text{ real and } m = 0, \text{ or } v \text{ complex} \\ K_v^{*,+} = \mathbb{R}_{>0}^* & v \text{ real, } m > 0. \end{cases}$$

Then $U_{K,\mathfrak{m}}$ is an open subgroup of J_K, since $U_v^{m_v} \subset K_v^*$ is open for all v and equals $\mathfrak{O}_v^*$ for almost all v.

Proposition 9.2. *Any open subgroup of J_K contains some $U_{K,\mathfrak{m}}$, and $J_K/K^*U_{K,\mathfrak{m}}$ is finite.*

Proof. The first statement follows from the definition of the topology on J_K, since the open subgroups of $\mathbb{R}^*$ are $\mathbb{R}^*$ and $\mathbb{R}^{*,+}$ and $\mathbb{C}^*$ has no proper open subgroup. Since $U_{K,\mathfrak{m}}$ is open, $J_K/K^*U_{K,\mathfrak{m}}$ is a discrete quotient of C_K hence is finite by Prop.9.1(i). □

Definition. The group $Cl_{\mathfrak{m}}(K) = J_K/K^*U_{K,\mathfrak{m}}$ is called the *ray class group* of K modulo $\mathfrak{m}$.

Clearly every (continuous) finite quotient of C_K factors through some $Cl_{\mathfrak{m}}(K)$. If $\mathfrak{m} = 0$ then $U_{K,\mathfrak{m}} = U_K = \ker\big(c\colon J_K \to \mathcal{I}(K)\big)$ and $Cl_{\mathfrak{m}}(K)$ is the class group $Cl(K)$.
More notations: let $\mathfrak{m} = \sum m_v(v)$ be a modulus.

- For $x \in K^*$, write $x \equiv 1 \bmod^* \mathfrak{m}$ if:

– for $v \in \mathrm{supp}_f(\mathfrak{m})$, $v(x-1) \geq m_v$; and
– for every real $v \in \mathrm{supp}(\mathfrak{m})$, $x \in K_v^{*,+}$.

- $K^*_{\mathfrak{m}} = \{x \in K^* \mid x \equiv 1 \bmod^* \mathfrak{m}\}$.
- $\mathcal{I}_{\mathfrak{m}}(K) =$ free abelian group on $\{v \in \Sigma_{K,f} \mid m_v = 0\}$,
- $\mathcal{P}_{\mathfrak{m}}(K) = \{x\mathfrak{o}_K \mid x \in K^*_{\mathfrak{m}}\} \subset \mathcal{I}_{\mathfrak{m}}(K)$.

Theorem. $C_{\mathfrak{m}}(K) \simeq \mathcal{I}_{\mathfrak{m}}(K)/\mathcal{P}_{\mathfrak{m}}(K)$.

Example. Suppose $\mathfrak{m} = \sum_{v\mid\infty}(v)$. Then $\mathcal{I}_{\mathfrak{m}}(K) = \mathcal{I}(K)$ and $\mathcal{P}_{\mathfrak{m}}(K)$ is the group of principal fractional ideals $x\mathfrak{o}_K$ where for every $\sigma\colon K \hookrightarrow \mathbb{R}$, $\sigma(x) > 0$. (We say x is *totally positive.*) The group $Cl(\mathfrak{m}(K)$ is called the *narrow ideal class group*, written $Cl^+(K)$. The kernel of the obvious surjection $Cl^+(K) \to Cl(K$ is killed by multiplication by 2.

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More precisely:

Theorem 9.3. *Let $S \subset \Sigma_{K,f}$ be a finite subset containing* $\mathrm{supp}_f(\mathfrak{m})$.. *There is a unique continuous homomorphism $\alpha = (\alpha_v)\colon J_K \to \mathcal{I}_{\mathfrak{m}}(K)/\mathcal{P}_{\mathfrak{m}}(K)$ such that $\alpha(K^*) = 0$ and, for every finite $v \notin S$, $\alpha_v(\pi_v) = P_v^{-1}$. It is surjective with kernel $K^*U_{K,\mathfrak{m}}$.*

Remark. We use P_v^{-1} rather than P_v to make subsequent things nicer.

Proof. More notation! Let

$$J_{K,\mathfrak{m}} = \{(x_v) \in J_K \mid \forall v \in \mathrm{supp}_f(\mathfrak{m}),\ x_v \in U_v^{m_v}\}$$

which is the subgroup of J_K generated by $U_{K,\mathfrak{m}}$ and all the K_v^*'s. The strong approximation theorem then implies that $J_K = K^*J_{K,\mathfrak{m}}$. If $x \in K^* \subset J_K$ then $x \in K^*_{\mathfrak{m}}$ iff $x \in J_{K,\mathfrak{m}}$, i.e. $K^*_{\mathfrak{m}} = K^* \cap J_{K,\mathfrak{m}}$.

First show α is unique: if α, α' are two such homomorphisms set $\phi = \alpha^{-1}\alpha'$. Then (enlarging S is necessary) we may assume that $\phi(U_{K,\mathfrak{m}'}) = 1$ for some modulus $\mathfrak{m}'$ with finite support contained in S. By hypothesis $\phi_v(\pi_v) = 1$ for all finite $v \notin S$. Therefore $\phi(J_{K,\mathfrak{m}'}) = 1$. As $\phi(K^*) = 1$, this means $\phi = 1$.

So it suffices to construct α. But as $K^*J_{K,\mathfrak{m}} = J_K$ and $K^* \cap J_{K,\mathfrak{m}} = K^*_{\mathfrak{m}}$,

$$\frac{J_K}{K^*U_{K,\mathfrak{m}}} \xleftarrow{\sim} \frac{J_{K,\mathfrak{m}}}{(K^*U_{K,\mathfrak{m}}) \cap J_{K,\mathfrak{m}}} = \frac{J_{K,\mathfrak{m}}}{K^*_{\mathfrak{m}}U_{K,\mathfrak{m}}} \tag{1}$$

We also have an isomorphism $-c\colon J_{K/\mathfrak{m}}/U_{K,\mathfrak{m}} \xrightarrow{\sim} \mathcal{I}_{\mathfrak{m}}(K)$, which maps an idele x to the inverse of its content: $-c(x) = \prod_{v\nmid\infty} P_v^{-v(x_v)}$. As $c(K^*_{\mathfrak{m}}) = \mathcal{P}_{\mathfrak{m}}(K)$, this induces an isomorphism

$$\overline{-c}\colon \frac{J_{K,\mathfrak{m}}}{K^*_{\mathfrak{m}}U_{K,\mathfrak{m}}} \xrightarrow{\sim} \frac{\mathcal{I}_{\mathfrak{m}}(K)}{\mathcal{P}_{\mathfrak{m}}(K)}.$$

We define α to be the composite of (1) and $\overline{-c}$. □

Remark. It is important to note that the homomorphism $J_K/K^* \to \mathcal{I}_{\mathfrak{m}}/\mathcal{P}_{\mathfrak{m}}$ just constructed is *not* induced by the content map on all of J_K (Neukirch VI.1.9 is at best ambiguous in this regard). It only agrees with the content map on $J_{K,\mathfrak{m}}$. (Fröhlich called this the “fundamental mistake of class field theory”).

Example. Suppose $K = \mathbb{Q}$, $m \geq 1$, and let $\mathfrak{m} = m\infty$ denote the modulus $(\infty) + \sum_{p|m} v_p(m) \cdot (p)$. If $I \in \mathcal{I}_\mathfrak{m}(\mathbb{Q})$ then $I = (a/b)\mathbb{Z}$ for unique positive coprime integers a, b with $(ab, m) = 1$, set $\theta(I) = (a/b) \pmod{m}$. This clearly defines an isomorphism $\theta\colon \mathcal{I}_\mathfrak{m}(\mathbb{Q})/\mathcal{P}_\mathfrak{m}(\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^*$.

On the idelic side, the inclusion $\prod_{p|m} \mathbb{Z}_p^* \hookrightarrow J_\mathbb{Q}$ induces an inclusion

$$\beta\colon (\mathbb{Z}/m\mathbb{Z})^* = \prod_{p|m} (\mathbb{Z}_p/m\mathbb{Z}_p)^* \hookrightarrow J_{\mathbb{Q},\mathfrak{m}}/U_{\mathbb{Q},\mathfrak{m}}$$

I claim that the composite map:

$$(\mathbb{Z}/m\mathbb{Z})^* \xrightarrow[\beta]{\sim} Cl_\mathfrak{m}(\mathbb{Q}) \xrightarrow[\alpha]{\sim} \mathcal{I}_\mathfrak{m}(\mathbb{Q})/\mathcal{P}_\mathfrak{m}(\mathbb{Q}) \xrightarrow[\theta]{\sim} (\mathbb{Z}/m\mathbb{Z})^*$$

is the identity map! To see this, let $a > 0$ be an integer prime to m. Then $\beta(a)$ is represented by the idele with components a at every $p|m$ and 1 elsewhere. Multiplying by the principal idele a^{-1} shows that $\beta(a)$ is also represented by the idele x with components 1 at every $p|m$ and a^{-1} at all others. So (as $a > 0$) $x \in J_{\mathbb{Q},\mathfrak{m}}$, and $\alpha(x)$ is the class of the ideal I with $v_p(I) = v_p(a)^{-1}$ for all $p|m$, $v_p(I) = 0$ for all other p — in other words, $I = a\mathbb{Z}$.

10 Dedekind zeta function

Theta functions of lattices

Let V be a real vector space of dimension $n \geq 1$, $\langle , \rangle\colon V \times V \to \mathbb{R}$ an inner product. Let $\{e_i\}$ be an ON basis for V, and μ (or dv) the associated measure (for which $\mu(V/\sum \mathbb{Z}e_i) = 1$) — it doesn't depend on the choice of ON basis.

Fourier transform: Let $f\colon V \to \mathbb{C}$ be a nice[2] function. Define

$$\hat{f}\colon V \to \mathbb{C}, \qquad \hat{f}(u) = \int_V e^{-2\pi i \langle u,v \rangle} f(v)\, dv$$

The **Fourier inversion formula** for $\mathbb{R}^n$ says: $\hat{\hat{f}}(v) = f(-v)$.

Theorem 10.1 (Poisson summation formula)**.** *Let $\Lambda \subset V$ be a lattice, $\Lambda' = \{y \in V | \langle x, y \rangle \in \mathbb{Z}\ \forall x \in \Lambda\}$ the dual lattice. Then*

$$\sum_{x \in \Lambda} f(x) = \mu(V/\Lambda)^{-1} \sum_{y \in \Lambda'} \hat{f}(y).$$

Proof. Let $g(v) = \sum_{x \in \Lambda} f(v + x)\colon V/\Lambda \to \mathbb{C}$. Then g can be written as a Fourier series:

$$g(v) = \sum_{y \in \Lambda'} c_y\, e^{2\pi i \langle y,v \rangle}$$

with coefficients

$$\begin{aligned} c_y &= \mu(V/\Lambda)^{-1} \int_{V/\Lambda} g(v) e^{-2\pi i \langle y,v \rangle}\, dv \\ &= \mu(V/\Lambda)^{-1} \int_V f(v) e^{-2\pi i \langle y,v \rangle}\, dv = \mu(V/\Lambda)^{-1} \hat{f}(y) \end{aligned}$$

Then $\sum_{x \in \Lambda} f(x) = g(0) = \sum_{y \in \Lambda'} c_y = \mu(V/\Lambda)^{-1} \sum_{y \in \Lambda'} \hat{f}(y)$. □

[2] "Nice" here means that the derivatives $f^{(m)}$ ($m \in \mathbb{N}^n$) satisfy: for every polynomial function P on V, $P(v) f^{(m)}(v)$ is bounded.

Lecture 20

Now we define the **theta function** of Λ to be

$$\Theta_\Lambda(t) = \sum_{x\in\Lambda} e^{-\pi t\langle x,x\rangle} \qquad (0 < t \in \mathbb{R}).$$

Note that since $\#\{x \in \Lambda \mid \langle x,x\rangle < N\} = O(N^n)$, the series is bounded by a constant times $\sum_{N=1}^{\infty} N^n e^{-\pi t N}$ so converges. (The same argument shows that the series g in the proof of Poisson summation converges.)

Theorem 10.2.

$$\Theta_\Lambda(t) = t^{-n/2}\mu(V/\Lambda)^{-1}\Theta_{\Lambda'}(1/t).$$

Proof. We let $f(v) = e^{-\pi\langle v,v\rangle} = \prod_i e^{-\pi v_i^2}$, if $v = \sum v_i e_i$ in the ON basis $\{e_i\}$. Then $\hat{f} = f$ (standard result[3] for $n = 1$).

Moreover as $(c\Lambda)' = c^{-1}\Lambda'$, by Poisson summation

$$\Theta_\Lambda(t) = \sum_{x\in t^{1/2}\Lambda} f(x) = \mu(V/t^{1/2}\Lambda)^{-1} \sum_{y\in t^{-1/2}\Lambda'} f(y) = t^{-n/2}\mu(V/\Lambda)^{-1}\Theta_{\Lambda'}(1/t).$$

□

As $\Theta_{\Lambda'}(t) \to 1$ as $t \to \infty$, deduce:

Corollary 10.3. *$\Theta_\Lambda(t) \sim \mu(V/\Lambda)^{-1}t^{-n/2}$ as $t \to 0$.*

The **Epstein zeta function** of the quadratic lattice Λ is

$$E(\Lambda, s) = E(\Lambda, \langle -,-\rangle; s) = \sum_{0\neq x\in\Lambda} \frac{1}{\langle x,x\rangle^s}.$$

It converges absolutely for $\mathrm{Re}(s) > n/2$. In fact, if $\{f_i\}$ is a basis for Λ and $x = \sum x_i f_i$ then $\langle x,x\rangle \geq c\max(x_i^2)$ for some $c > 0$, so

$$\begin{aligned}
\sum_{0\neq x\in\Lambda} \langle x,x\rangle^{-s} &\leq c^{-s} \sum_{0\leq\in\mathbb{Z}^n} (\max|x_i|)^{-2s} \\
&= c^{-s}\sum_{N\geq 1} N^{-2s} \times \#\{\underline{x} \in \mathbb{Z}^n \mid \max|x_i| = N\} \\
&\ll c^{-s}\sum_{N\geq 1} N^{n-1-2s}
\end{aligned}$$

which converges for $\mathrm{Re}(2s) > n$.

Theorem 10.4. *$\mathcal{E}(\Lambda, s) = \pi^{-s}\Gamma(s)\, E(\Lambda, s)$ has a meromorphic continuation to $\mathbb{C}$, analytic apart from simple poles at $s = 0$, $n/2$ with residues -1, $\mu(V/\Lambda)$ respectively. It satisfies the **functional equation***

$$\mathcal{E}(\Lambda, s) = \mu(V/\Lambda)^{-1}\mathcal{E}(\Lambda', \frac{n}{2} - s).$$

In particular, $E(\Lambda, 0) = -1$.

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[3]Proof: if $f(x) = \exp(-\pi x^2)$ then

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x y}\, dx = e^{-\pi y^2}\int_{-\infty}^{\infty} e^{-\pi(x+iy)^2}\, dx = f(y)\int_{-\infty+iy}^{\infty+iy} e^{-\pi z^2}\, dz\,.$$

Now shift the path of integration to $[-\infty,\infty]$ and use $\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$.

Recall $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$ is analytic apart for simple poles at $s = 0, -1, -2, \ldots$, and $\Gamma(n+1) = n!$, $\Gamma(s+1) = s\Gamma(s)$. In particular, $\mathrm{Res}_{s=0}\,\Gamma(s) = 1$.

Remark. $\Lambda = \mathbb{Z} \subset \mathbb{R}$ with Euclidean inner product. Then $E(\mathbb{Z}, s) = 2\zeta(2s)$. We get $\zeta(s) \sim 1/(s-1)$, $\zeta(0) = -1/2$ and the usual FE for $\zeta(s)$. (See Analytic Number Theory next term.)

Proof.

$$\begin{aligned}\pi^{-s}\Gamma(s)\,E(\Lambda, s) &= \int_0^\infty e^{-t}\pi^{-s}t^s \sum_{0\neq x\in\Lambda} \langle x, x\rangle^{-s}\,\frac{dt}{t} \\ &= \int_0^\infty \sum_{x\neq 0} e^{-\pi t\langle x,x\rangle} t^s\,\frac{dt}{t} = \int_0^\infty (\Theta_\Lambda(t) - 1)\,t^s\,\frac{dt}{t}\end{aligned}$$

Break up as $\int_0^1 + \int_1^\infty$. Then if $\mathrm{Re}(s) > n/2$, we can by Cor.10.2 compute:

$$\begin{aligned}\int_0^1 &= \int_0^1 \Theta_\Lambda(t)t^s\,\frac{dt}{t} - \frac{1}{s} = -\frac{1}{s} + \int_0^1 \mu(V/\Lambda)^{-1}\Theta_{\Lambda'}(t^{-1})t^{s-n/2}\,\frac{dt}{t} \\ &= -\frac{1}{s} + \mu(V/\Lambda)^{-1}\int_1^\infty \Theta_{\Lambda'}(t)t^{n/2-s}\,\frac{dt}{t} \\ &= -\left(\frac{1}{s} + \frac{\mu(V/\Lambda)^{-1}}{n/2-s}\right) + \mu(V/\Lambda)^{-1}\int_1^\infty (\Theta_{\Lambda'}(t) - 1)\,t^{n/2-s}\,\frac{dt}{t}\end{aligned}$$

So $\mathcal{E}(\Lambda, s)$ equals

$$-\left(\frac{1}{s} + \frac{\mu(V/\Lambda)^{-1}}{n/2-s}\right) + \int_1^\infty (\Theta_\Lambda(t) - 1)\,t^s + \mu(V/\Lambda)^{-1}\,(\Theta_{\Lambda'}(t) - 1)\,t^{n/2-s}\cdot\frac{dt}{t}$$

Now $\Theta_\Lambda - 1$ tends rapidly to 0 as $t \to \infty$, so the integral is analytic for all $s \in \mathbb{C}$. This gives the residues, and using the FE for Θ and $\mu(V/\Lambda)^{-1} = \mu(V/\Lambda')$ we get the FE. □

Lecture 21

Definition. The *Dedekind zeta function* is the function

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}$$

the sum taken over non-0 ideals $I \subset \mathfrak{o}_K$.

Proposition 10.5. $\zeta_K(s) = \prod_{v\nmid\infty}(1 - q_v^{-s})^{-1}$, *and the product converges absolutely for* $\mathrm{Re}(s) > 1$.

Proof. As formal series, the product follows from unique factorisation of ideals: writing $I = \prod P_v^{n_v}$ gives $NI = \prod q_v^{n_v}$, hence

$$\zeta_K(s) = \prod_v (1 + q_v^{-s} + q_v^{-2s} + \cdots)$$

Now $\#\{v|p\} \leq n$ and $q_v \geq p$ if $v|p$, so product converges by comparison with

$$\prod_p (1 - p^{-s})^{-n} = (\sum_{N\geq 1} N^{-s})^n = \zeta(s)^n.$$

□

We are now going to prove:

Theorem 10.6. *$\zeta_K(s)$ has a meromorphic continuation to $\mathbb{C}$ whose only singularity is a simple pole at $s = 1$. Moreover*

$$\lim_{s\to 0} \frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K R_K}{w_K}.$$

This is the famous **analytic class number formula**. Here:

- $h_K = \#Cl(K)$, the *class number* of K
- $w_K = \#\mu(K)$ the order of the group of roots of unity of K
- R_K is the *regulator* of K (defined below).

Let $\Sigma = \Sigma_{K,\infty}$ be the infinite places of K. Write the group of units of K as $\mathfrak{o}_K^* = \mu_K \times \langle \varepsilon_1, \ldots, \varepsilon_{r-1}\rangle$, $r = r_1 + r_2 = \#\Sigma$. Let $e_v = e(v/\infty) = 1$ if v is real, 2 if v is complex.

Consider the $(r-1) \times r$ real matrix $(\log |\varepsilon_j|_v)$ ($v \in \Sigma$, $1 \le j \le r-1$). Since $\prod_v |\varepsilon_j|_v = 1$, the columns of this matrix sum to zero. So all of its $(r-1)\times(r-1)$ minors have the same absolute value, which is by definition R_K. The proof of the unit theorem shows that $R_K \neq 0$ (it is, for suitably chosen measure, the volume of $\mathbb{R}^{r-1}/\lambda(\mathfrak{o}_K^*)$.)

We begin by breaking the sum up into ideal classes:

$$\zeta_K(s) = \sum_{\mathcal{C}\in Cl(K)} \zeta_K(\mathcal{C}, s), \quad \text{where } \zeta_K(\mathcal{C}, s) = \sum_{I\subset \mathfrak{o}_K,\ I\in\mathcal{C}} NI^{-s}.$$

Fix $I_0 \in \mathcal{C}^{-1}$. Then $\mathcal{C} = \{xI_0^{-1} \mid x \in I_0\}$ and

$$\zeta_K(\mathcal{C}, s) = (NI_0)^s \sum_{x\in I_0}^{*} \left|N_{K/\mathbb{Q}}(x)\right|^{-s}$$

where * means to take the sum over nonzero elements of I_0 modulo $\mathfrak{o}_K^*$.

If $K = \mathbb{Q}$ which is just $(1/2)Z(\mathbb{Z}, s/2)$. If $\mathbb{Q}(\sqrt{-D})$ is imaginary quadratic, then as $\mathfrak{o}_K$ is finite, the sum is just $1/w_K$ times the sum over all nonzero $x \in I_)$, and $N_{K/Q}(x) = x\bar{x}$ is a positive definite quadratic form. So $\zeta_K(\mathcal{C}, s)$ is itself an Epstein zeta function. For other K, this isn't the case, but we can write each "partial zeta" $\zeta_K(\mathcal{C}, s)$ as an *integral* of Epstein zeta functions.

Define

$$\phi\colon \mathbb{R}^{r-1} \xrightarrow{\sim} \{u = (u_v) \in \mathbb{R}^{\Sigma}_{>0} \mid \prod u_v = 1\}, \quad \phi(w) = \Big(\prod_j |\varepsilon_j|_v^{w_j}\Big)_v$$

For $x = (x_v) \in K \otimes \mathbb{R} = \prod_{v\in\Sigma} K_v \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and $u = \phi(w)$ set

$$Q_w(x) = \sum_{v\in\Sigma} e_v (u_v |x_v|_v)^{2/e_v} = \sum_{v \text{ real}} u_v^2 |x_v|_v^2 + 2 \sum_{v \text{ complex}} u_v |x_v|_v$$

which is a positive definite quadratic form (remember that for v complex, $|-|_v$ is *square* of complex modulus).

Lemma 10.7. *For all $m \in \mathbb{Z}^{r-1}$ and $\zeta \in \mu(K)$, $Q_w(\zeta \prod_j \varepsilon_j^{m_j} \cdot x) = Q_{w+m}(x)$.*

(Proof trivial.)

Lecture 22

Recall

$$Q_w(x) = \sum_{v\in\Sigma} e_v (u_v \left|x_v\right|_v)^{2/e_v} \quad \text{where} \quad u = \phi(w) = \Big(\prod_j \left|\varepsilon_j\right|_v^{w_j}\Big)_v .$$

The next result is the key to rewriting $\zeta_K(s)$ in terms of Epstein zeta functions.

Proposition 10.8. *Let $x \in K^*$. Then*

$$\Gamma(s) \int_{\mathbb{R}^{r-1}} Q_w(x)^{-s}\, dw = R_K^{-1} \left(\frac{\Gamma(s/n)^{r_1}\, \Gamma(2s/n)^{r_2}}{n 2^{r_1+2r_2 s/n-1}} \right) \left|N_{K/\mathbb{Q}}(x)\right|^{-2s/n} .$$

Proof.

$$\text{LHS} = \int_{\mathbb{R}_{>0}\times\mathbb{R}^{r-1}} e^{-t} t^s Q_w(x)^{-s} \frac{dt}{t}\, dw = \int_{\mathbb{R}_{>0}\times\mathbb{R}^{r-1}} e^{-tQ_w(x)} t^s \frac{dt}{t}\, dw. \qquad (*)$$

Change variables to $(y_v) \in \mathbb{R}^{\Sigma}_{>0}$ where $y_v = t^{e_v} u_v^2$ and $(u_v) = \phi(w)$ as above. Then

$$t^n = \prod_v t^{e_v} = \prod_v y_v \quad \text{as} \prod u_v = 1, \qquad tQ_w(x) = \sum_v e_v y_v^{1/e_v} \left|x\right|_v^{2/e_v} .$$

Compute the Jacobian:

$$\frac{dy_v}{y_v} = e_v \frac{dt}{t} + 2\sum_{j=1}^{r-1} \log\left|\varepsilon_j\right|_v \, dw_j \,, \qquad \prod \frac{dy_v}{y_v} = |J|\, \frac{dt}{t} \prod_j dw_j$$

where

$$\begin{aligned} |J| &= \left\| \begin{matrix} \cdots & \cdots & \cdots & \cdots \\ e_v & 2\log\left|\varepsilon_1\right|_v & \cdots & 2\log\left|\varepsilon_{r-1}\right|_v \\ \cdots & \cdots & \cdots & \cdots \end{matrix} \right\| \\ &= 2^{r-1} \left\| \begin{matrix} n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ e_v & \log\left|\varepsilon_1\right|_v & \cdots & \log\left|\varepsilon_{r-1}\right|_v \\ \cdots & \cdots & \cdots & \cdots \end{matrix} \right\| = 2^{r-1} n R_K \end{aligned}$$

adding all the remaining rows to the first and using $\sum_v \log\left|\varepsilon\right|_v = 0$. Then

$$\begin{aligned} (*) \times |J| &= \int_{\mathbb{R}^{\Sigma}_{>0}} \exp\Big(-\sum_v \underbrace{e_v y_v^{1/e_v} \left|x\right|_v^{2/e_v}}_{=z}\Big) \prod_v y_v^{s/n} \prod_v \frac{dy_v}{y_v} \\ &= \prod_v \int_0^\infty \exp(-z) \Big(e_v^{-2} \left|x\right|_v^{-2} z^{e_v}\Big)^{s/n} e_v \frac{dz}{z} \\ &= 2^{r_2(1-2s/n)} \left|N_{K/\mathbb{Q}}(x)\right|^{-2s/n} \Gamma\Big(\frac{s}{n}\Big)^{r_1} \Gamma\Big(\frac{2s}{n}\Big)^{r_2} . \end{aligned}$$

where we have made the further change of variables $y_v = e_v^{-2} \left|x_v\right|_v^{-2} z^{e_v}$ in the integral (note that $e_v^{e_v} = e_v^2$). ☐

Proof of Theorem. Apply to $\zeta_K(\mathcal{C}, s)$ (replacing s in the Proposition with $ns/2$) to get:

$$\Gamma\Big(\frac{s}{2}\Big)^{r_1}\,\Gamma(s)^{r_2}\,\zeta_K(\mathcal{C},s) = R_K\,n2^{r_1+r_2s-1}\Gamma\Big(\frac{ns}{2}\Big)\,(NI_0)^s \sum_{x\in I_0}^{*} \int_{\mathbb{R}^{r-1}} Q_w(x)^{-ns/2}\,dw$$

Now break up the domain of integration into boxes $m+[0,1]^{r-1}$, $m \in \mathbb{Z}^{r-1}$ and apply Lemma 10.7 with $\varepsilon = \zeta\prod_j \varepsilon_j^{m_j}$, $\zeta \in \mu(K)$, to write the sum as:

$$\begin{aligned}\sum_{x\in I_0}^{*} \int_{[0,1]^{r-1}} \sum_{m\in\mathbb{Z}^{r-1}} Q_{w+m}(x)^{-ns/2}\,dw &= \sum_{x\in I_0}^{*} \int_{[0,1]^{r-1}} \frac{1}{\#\mu(K)} \sum_{\varepsilon\in\mathfrak{o}_K^*} Q_w(\varepsilon x)^{-ns/2}\,dw \\ &= \int_{[0,1]^{r-1}} \frac{1}{\#\mu(K)} \sum_{0\neq x\in I_0} Q_w(x)^{-ns/2}\,dw \\ &= \frac{1}{\#\mu(K)} \int_{[0,1]^{r-1}} E(I_0, Q_w, ns/2)\,dw.\end{aligned}$$

Therefore we get:

$$\Gamma\Big(\frac{s}{2}\Big)^{r_1}\,\Gamma(s)^{r_2}\,\zeta_K(\mathcal{C},s) = n2^{r_1+r_2s-1}\pi^{\frac{ns}{2}}\,(NI_0)^s\frac{R_K}{\#\mu(K)} \int_{(\mathbb{R}/\mathbb{Z})^{r-1}} \mathcal{E}(I_0, Q_w, \frac{ns}{2})\,dw$$

Now (analytic fact!) $\Gamma(s)$ has no zeroes, so by the analytic continuation of $\mathcal{E}$ we obtain that of $\zeta_K(\mathcal{C}, s)$, with only possible poles at $s = 1$ and 0. The leading term of the Taylor series at $s = 0$ of the various terms are:

$$\Gamma\Big(\frac{s}{2}\Big)^{r_1}\,\Gamma(s)^{r_2} \sim 2^{-r_1}s^{r_1+r_2}, \qquad \mathcal{E}(I_0, Q_w, \frac{ns}{2}) \sim -\frac{2}{ns}$$

so $\zeta_K(\mathcal{C}, s) \sim -w_K^{-1}R_K s^{r-1}$. Summing over ideal classes we get the Theorem! □

Remark. It is standard to write $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The integral representation then becomes

$$\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(\mathcal{C},s) = \frac{n2^{r-1}R_K}{w_K}(NI_0)^s \int_{(\mathbb{R}/\mathbb{Z})^{r-1}} \mathcal{E}(I_0, Q_w, \frac{ns}{2})\,dw.$$

from which it is an exercise to derive the functional equation for $\zeta_K(s)$ (see example sheet 3).