Algebraic Number Theory - Ex Sheet 3

1. (i) $\mathbb{A}_{\mathbb{Q}} = \{ (x_v) \in \prod_{\text{all } v} \mathbb{Q}_v \mid \text{for a.all } p,\ x_p \in \mathbb{Z}_p \}$

$= \mathbb{R} \times V$ say, $V = \{ (x_p) \in \prod_p \mathbb{Q}_p \mid \cdots\cdots \}$.

Now $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \hookrightarrow V$ and as V is a $\mathbb{Q}$-vector space this extends to an embedding $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow V$.

Finally, if $x = (x_p) \in V$, let $S = \{ p \mid x_p \notin \mathbb{Z}_p \}$

$$m_p = \begin{cases} -v_p(x_p) & \text{if } p \in S \\ 0 & p \notin S \end{cases}$$

$\Rightarrow p^{m_p} x_p \in \mathbb{Z}_p \quad \forall p$

$\Rightarrow Nx \in \prod \mathbb{Z}_p \subset V$ if $N = \prod_{p \in S} p^{m_p}$

ie $\hat{\mathbb{Z}} \otimes \mathbb{Q} \xrightarrow{\sim} V$.

(ii) $L = K(a)$ say, $a \in \mathcal{O}_L$.

$$\mathbb{A}_L \subset \prod_w L_w = \prod_v \Big(\prod_{w \mid v} L_w \Big)$$

$$= \prod_v (K_v \otimes L) \qquad \text{by Th. 5.4}$$

$$\cong \Big(\prod_v K_v \Big) \otimes_K L \qquad \text{since}^* \dim_K L < \infty.$$

Let $S = \{ v \mid \infty \} \cup \{ v \text{ finite s.t. } v(N_{L/K}\, g'(a)) \neq 0 \}$.

Then $v \notin S,\ w \mid v \Rightarrow \mathcal{O}_{L_w} = \mathcal{O}_{K_v}[a] = \mathcal{O}_{K_v} \cdot \mathcal{O}_L$
(by 3.6)

$*$ $L \cong K^n$ so $\prod_v (K_v \otimes L) \cong \prod_v (K_v^n) \cong \Big(\prod_v K_v\Big)^n \cong \Big(\prod_v K_v\Big) \otimes L$.

So if $z = (z_w) \in \prod K_w$, then

$z \in \mathbb{A}_L \iff$ for a. all w, $z_w \in \mathcal{O}_{K_v}\mathcal{O}_L$

$\iff$ for a. all v, $(z_w)_{w|v} \in \prod_{w|v} L_w = K_v \otimes L$ lies in $\mathcal{O}_{K_v} \otimes \mathcal{O}_L$.

(iii) Let $t_v = (\mathrm{tr}_{L_w/K_v})_{w|v} : \prod_{w|v} L_w \longrightarrow K_v$.

Then as $\prod_{w|v} L_w = L \otimes_K K_v$, t_v is just the K_v-linear extension of $\mathrm{tr}_{L/K} : L \to K$ to $L \otimes K_v \longrightarrow K_v$.

Clearly $t_v\left(\prod_{w|v} \mathcal{O}_w\right) \subset \mathcal{O}_v$. So

$$(\mathrm{tr}_{L_w/K_v}) : \prod_w L_w \longrightarrow \prod_v K_v$$

maps $\mathbb{A}_L$ to $\mathbb{A}_K$.

(iv) is the same.

2. (i) $\mathfrak{m} = \sum m_v (v)$

$$J_K \supset U_{K,\mathfrak{m}} = \prod_{v \in Z_K} U_v^{m_v}$$

Under inclusion $\prod_{v \in S} \mathcal{O}_v^* \hookrightarrow J_K$

clearly $\prod_{v \in S} 1 + \pi_v^{m_v} \mathcal{O}_v \hookrightarrow U_{K,\mathfrak{m}}$

$$\therefore \prod_{v \in S} \frac{\mathcal{O}_v^*}{1 + \pi_v^{m_v} \mathcal{O}_v} \longrightarrow \frac{J_K}{U_{K,\mathfrak{m}}} \twoheadrightarrow \frac{J_K}{K^* U_{K,\mathfrak{m}}} = Cl_{\mathfrak{m}}(K).$$

(ii) $Cl(K) = J_K / K^* . U_{K,1}$ ($U_{K,1} = U_{K,\mathfrak{m}}$ for $\mathfrak{m} = 0$)

$\therefore \ker \left(Cl_{\mathfrak{m}}(K) \twoheadrightarrow Cl(K) \right)$

$$= \frac{K^* U_{K,1}}{K^* U_{K,\mathfrak{m}}} = \frac{U_{K,1}}{(K^* \cap U_{K,1}) . U_{K,\mathfrak{m}}}$$

$$= \frac{U_{K,1}}{\mathcal{O}_K^* . U_{K,\mathfrak{m}}} = \frac{\prod_v \mathcal{O}_v^* / U_v^{m_v}}{(\text{image of } \mathcal{O}_K^*)} . \quad (*)$$

If $m_v = 0$ $\forall$ real places v, then $U_K^{m_v} = K_v^*$ for all $v | \infty$.

$$\therefore (*) = \prod_{v \in S} \left(\mathcal{O}_v^* / U_v^{m_v} \right) \Big/ (\text{image of } \mathcal{O}_K^*).$$

(iii) $Cl^+(K) = J_K / K^* U_{K,\infty}$ $\left(\begin{array}{l} U_{K,\infty} = U_{K,m} \\ \text{for } m = \sum_{v|\infty} (v) \end{array} \right)$

Hypothesis $\Rightarrow$ $U_{K,m} \subset U_{K,\infty}$, hence

$Cl_m(K) \twoheadrightarrow Cl^+(K)$ with kernel

$$\frac{K^* U_{K,\infty}}{K^* U_{K,m}} = \frac{U_{K,\infty}}{\underbrace{(K^* \cap U_{K,\infty})}_{= \mathcal{O}_{K,+}^*} U_{K,m}}$$

and rest as in (ii)

3 (i) $K = \mathbb{Q}(i)$, $Cl(K) = 0$. So by 2(ii)

$$\underset{\{\pm 1, \pm i\}}{\overset{\|}{\mathcal{O}_K^*}} \longrightarrow \frac{\mathcal{O}_{K,3}^*}{1 + 3\mathcal{O}_{K,3}} \longrightarrow Cl_{(3)}(K) \longrightarrow 0$$

$$\frac{\mathcal{O}_{K,3}^*}{1 + 3\mathcal{O}_{K,3}} = \mathbb{F}_3[i]^* = \mathbb{F}_9^*$$

i.e. $Cl_{(3)}(\mathbb{Q}(i)) = \mathbb{F}_3[i]^* / \{\pm 1, \pm i\} \simeq \mathbb{Z}/2\mathbb{Z}$.

(ii) $K = \mathbb{Q}(\sqrt{5})$. $Cl(K) = 0$. $\varepsilon = \frac{1+\sqrt{5}}{2}$ is a unit of norm -1, hence $Cl^+(K) = 0$ as well (since if $x \in K^*$, one of $\pm x, \pm \varepsilon x$ is totally positive)

$\therefore$ by 2(iii), with $v = (\sqrt{5})$

$$\begin{array}{ccccccc} \mathcal{O}_{K,+}^{*} & \longrightarrow & \dfrac{\mathcal{O}_v^{*}}{1+\sqrt{5}\,\mathcal{O}_v} & \longrightarrow & Cl_{m}(K) & \longrightarrow & 0 \\ \parallel & & \parallel & & & & \\ \langle \varepsilon^2 \rangle & & \mathbb{F}_5^{*} & & & & \\ \parallel & & & & & & \\ \left\langle \frac{3+\sqrt{5}}{2} \right\rangle & \longmapsto & \{\pm 1\} & & & & \end{array}$$

as $\frac{3}{2} \equiv -1 \bmod 5$

$$\therefore \ Cl_m(K) \cong \mathbb{F}_5^{*}/\{\pm 1\} \simeq \mathbb{Z}/2\mathbb{Z}.$$

(iii) $K = \mathbb{Q}(\sqrt{-5})$, $m = 3(v)$, $v \leftrightarrow (2, 1+\sqrt{-5}) =: \mathfrak{p}_2$

$\mathfrak{p}_2$ generates $Cl(K) \simeq \mathbb{Z}/2\mathbb{Z}$. Have:

$$\begin{array}{ccccccccc} \mathcal{O}_K^{*} & \longrightarrow & \mathcal{O}_v^{*}/1+\pi_v^3\mathcal{O}_v & \longrightarrow & Cl_m(K) & \longrightarrow & Cl(K) & \longrightarrow & 0 \\ \parallel & & \parallel & & & & & & \\ \{\pm 1\} & & (\mathcal{O}_v/\pi_v^3\mathcal{O}_v)^{*} & & & & & & \end{array}$$

$\mathcal{O}_v/\pi_v^3\mathcal{O}_v = \mathbb{Z}_2[\pi_v]/(\pi_v^3)$ as $K_v/\mathbb{Q}_2$ totally ramified

$\pi_v = 1+\sqrt{-5}$, satisfying Eisenstein poly. $T^2 - 2T + 6$

$\Rightarrow \pi_v^2 \equiv -6 \equiv 2 \bmod (\pi_v^3)$

$\therefore \left(\mathcal{O}_v/(\pi_v^3)\right)^{*} = \{\pm 1, \pm 1 + \pi_v \bmod \pi_v^3\}$ order 4

so quotient by image of $\mathcal{O}_K^{*}$ is $\cong \mathbb{Z}/2$,

generated by image of $-1+\pi_v = \sqrt{-5}$.

So $Cl_m(K)$ has order 4. Is it cyclic?

We need to lift the non-trivial element of $Cl(K)$ to an element $x \in Cl_m(K)$ and see whether or not $x^2 = 1$.

Take $x = (x_w) \in J_K$, $x_w = 1$ if $w \neq v$
$x_v = 1 + \sqrt{-5} = \pi_v$.

The image of x in $Cl(K)$ represents the class of $\mathfrak{f}_2$, which is nontrivial, and

$$x^2 = y = (y_w) \text{ with } y_w = \begin{cases} 1 & w \neq v \\ 2(-2+\sqrt{-5}) = \pi_v^2, & w = v \end{cases}$$

Is $y \in K^* U_{K,m}$?

$$U_{K,m} = \mathbb{C}^* \times \prod_{\substack{w \neq v \\ \text{finite}}} \mathcal{O}_w^* \times (1 + \pi_v^3 \mathcal{O}_v)$$

Let $z \in K^*$. Then $yz \in U_{K,m}$

$\iff$ (a) $\forall$ finite $w \neq v$, $w(z) = 1$

(b) $z y_v \in 1 + \pi_3 \mathcal{O}_v$

But (a) $\Rightarrow$ ideal (z) = power of $\mathfrak{f}_2$

$\Rightarrow z = \pm 2^n$, $n \in \mathbb{Z}$ as $\mathfrak{f}_2^2 = (2)$

$\therefore$ $z y_v = \pm 2^n \pi_v^2$. Then (b) $\Rightarrow n = -1$.

$$z y_v = \pm \frac{\pi_v^2}{2} = \pm(-2+\sqrt{-5}) \equiv \pm\sqrt{-5} \mod (2)$$
$$\not\equiv \pm 1 \mod (2)$$

So $x^2 \neq 0$ in $Cl_m(K)$ i.e. $Cl_m(K) \cong \mathbb{Z}/4\mathbb{Z}$

4 (i) $$\Phi_w(x) = \sum_{v \in \Sigma = \Sigma_{K,\infty}} e_v . (u_v |x_v|_v)^{2/e_v}$$

$u = (u_v) \in \mathbb{R}^{\Sigma}$ with $\prod_v u_v = 1$

– quadratic form on

$$K \otimes \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} = \{((a_i)_i, (b_j + c_j\sqrt{-1}))_j\}$$

$$\Phi_w(x) = \sum_{i=1}^{r_1} u_i^2 a_i^2 + 2\sum_{j=1}^{r_2} u_j (b_j^2 + c_j^2)$$

ON basis : $\left.\begin{matrix} (\frac{1}{u_1}, 0 \, \cdots\cdots\cdots \, 0) \\ \vdots \\ (0, \cdots \frac{1}{u_{r_1}}, \cdots\cdots 0) \end{matrix}\right\}$ real v

(*) $\left.\begin{matrix} (0, \cdots\cdots, \frac{1}{\sqrt{2u_{r_1+1}}}, \cdots 0) \\ (0, \cdots \frac{i}{\sqrt{2u_{r_1+1}}}, \cdots 0) \end{matrix}\right\}$ complex v

$\cdots\cdots$

So multiplying u_i by α_i $(\prod \alpha_i = 1)$ scales basis by

$$\begin{pmatrix} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_{r_1} & & & \\ & & & \sqrt{\alpha_{r_1+1}} & & \\ & & & & \sqrt{\alpha_{r_1+1}} & \\ & & & & & \ddots \end{pmatrix}$$

which has det = 1

(ii) $w=0$: ie. $u_i=1$ $\forall i$, so measure induced by Q_w is $\prod_{i=1}^{r_1} da_i \times \prod_{j=1}^{r_2} 2\, db_i\, dc_i$ by $(*)$

for $(\underline{a}, \underline{b}+i\underline{c}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$

Now if $\mathcal{O}_K = \sum_{i=1}^{n} \mathbb{Z} x_i$, $|d|^{1/2}$ is the absolute value of

$$\begin{vmatrix} \sigma_1(x_1) & \cdots & \sigma_n(x_1) \\ \vdots & & \vdots \\ \sigma_{r_1}(x_1) & & \vdots \\ \sigma_{r_1+1}(x_1) & & \vdots \\ \overline{\sigma_{r_1+1}}(x_1) & & \vdots \\ \vdots & & \vdots \\ \overline{\sigma_{r_1+r_2}}(x_1) & \cdots & \overline{\sigma_{r_1+r_2}}(x_n) \end{vmatrix}$$

and if $z = b+ic$, $|dz|.|d\bar{z}| = 2\, db\, dc$.

$$\Rightarrow \operatorname{vol}(K \otimes \mathbb{R} / \mathcal{O}_K) = |d_K|^{1/2}$$

(iii) integral representation: if $I_0 \in \mathcal{C}^{-1}$,

$$\Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_K(\mathcal{C}, s) =$$

$$\frac{n 2^{r-1} R_K}{w_K} (N I_0)^s \int_{(\mathbb{R}/\mathbb{Z})^{r-1}} \mathcal{E}\left(I_0, Q_w, \frac{ns}{2}\right) dw.$$

Recall $\operatorname{Res}_{s=n/2} \mathcal{E}(\Lambda, s) = \mu(V/\Lambda)^{-1}$.

$\therefore \ \operatorname{Res}_{s=1} \mathcal{E}(I_0, Q_w, \tfrac{ns}{2}) =$

$$\frac{2}{n} \operatorname{Res}_{s=n/2} \mathcal{E}(I_0, Q_w, s) = \frac{2}{n}\, \mu(K \otimes \mathbb{R} / I_0)^{-1}$$

$$= \frac{2}{n} (N I_0)^{-1} \mu(K \otimes \mathbb{R} / O_K)^{-1}$$

$$= \frac{2}{n} (N I_0)^{-1} . \, |d_K|^{-1/2}$$

$\Gamma_{\mathbb{R}}(1) = \pi^{-1/2} \Gamma(1/2) = 1$ $\qquad \displaystyle\int_{(\mathbb{R}/\mathbb{Z})^{r-1}} dw = 1.$

$\Gamma_{\mathbb{C}}(1) = 2(2\pi)^{-1} . \Gamma(1) = \pi^{-1}$

$$\therefore \ \operatorname{Res}_{s=1} \zeta_K(C, s) = \frac{\pi^{r_2} . 2^{r} \, R_K}{w_K \, |d_K|^{1/2}}$$

& summing over ideal classes gives $\operatorname{Res} \zeta_K(s)$

5. Let S be the set of finite places of K which don't split completely in L/K. By hypothesis, S is finite. So if $v \notin S$ then there are $n = [L:K]$ places w_i of L over v, and so $f(w_i/v) = 1$, $q_{w_i} = q_v$.

$$\therefore \zeta_L(s) = \prod_{\substack{v \in S \\ w|v}} (1 - q_w^{-s})^{-1} \prod_{v \notin S} \underbrace{\prod_{i=1}^{n} (1 - q_{w_i}^{-s})^{-1}}_{= (1-q_v^{-s})^{-n}}$$

i.e. $\zeta_L(s) = \zeta_K(s)^n \prod_{v \in S} \left[(1 - q_v^{-s}) . \prod_{w|v} (1 - q_w^{-s})^{-1} \right]$

The product over S is clearly holomorphic and $\neq 0$ in a neighbourhood of $s = 1$. So as both ζ_L and ζ_K have a simple pole at $s = 1$, we have $n = 1$ i.e. $L = K$.