Algebraic Number Theory — problem sheet #3

1. (i) Show that $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times (\widehat{\mathbb{Z}} \otimes \mathbb{Q})$.

   (ii) Use Theorem 5.4 to show that for any extension L/K of number fields $\mathbb{A}_L \simeq \mathbb{A}_K \otimes_K L$.

   (iii) Let L/K be a finite extension of number fields, Show that the trace maps $\mathrm{tr}_{L_w/K_v} \colon L_w \to K_v$ define a continuous additive homomorphism $\mathrm{tr}_{L/K} \colon \mathbb{A}_L \to \mathbb{A}_K$, whose restriction to L is the trace map $L \to K$.

   (iv) Repeat (iii) for the norm map and the idele group.

2. (i) Let F be a local field. If $f \in \mathcal{S}(F)$ and $g(x) = f(ax+b)$, compute $\widehat{g}$ in terms of $\widehat{f}$.

   (ii) Let $F/\mathbb{Q}_p$ be finite. Compute the Fourier transform of the characteristic function of $b + \pi_F^n \mathfrak{o}_F$. Deduce that the Fourier transform maps $\mathcal{S}(F)$ to itself, and that the Fourier inversion formula holds for $\mathcal{S}(\mathcal{F})$.

   (iii) Verify that if $F = \mathbb{R}$ or $\mathbb{C}$ then $\zeta(f,s) = \Gamma_F(s)$, where

$$f(x) = \begin{cases} e^{-\pi x^2} & \text{if } F = \mathbb{R} \\ e^{-2\pi x \bar{x}} & \text{if } F = \mathbb{C} \end{cases}$$

3. Let $F/\mathbb{Q}_p$ be finite, and $f \in \mathcal{S}(F)$.

   (i) Show that if $f(0) = 0$, then $\zeta(f,s) \in \mathbb{C}[q^s, q^{-s}]$, hence is analytic for all $s \in \mathbb{C}$.

   (ii) Show that in general $\zeta(f,s)$ is analytic except for a simple pole at $s = 0$ with residue $f(0)/\log q$.

   If you like analysis, investigate the analyticity of $\zeta(f,s)$ for an arbitrary $f \in \mathcal{S}(F)$ when F is archimedean.

4. Let L/K be an extension of number fields of degree n. Suppose that almost all places v of K split completely in L (i.e. $\#\{w|v\} = n$). By comparing $\zeta_L(s)$ and $\zeta_K(s)^n$ show that $L = K$.

5. Let K be a number field, and $S \subset \Sigma_K$ a finite set of places. Let $\mathfrak{m} = \sum_{v \in S} m_v \cdot (v)$ be a finite formal linear combination of places of K with positive coefficients m_v. Define

$$U_{v,\mathfrak{m}} = \begin{cases} F_v^* & v \text{ complex, or } v \notin S \text{ real} \\ F_v^{*,+} = \mathbb{R}^*_{>0} & v \in S \text{ real} \\ 1 + \pi_v^{m_v} \mathcal{O}_v^* & v \in S \text{ finite} \\ \mathcal{O}_v^* & \text{otherwise} \end{cases}$$

   and set $U_{K,\mathfrak{m}} = \prod_v U_{v,\mathfrak{m}}$

   (i) Show that $U_{K,\mathfrak{m}}$ is an open subgroup of J_K, and that every open subgroup of J_K contains some $U_{K,\mathfrak{m}}$.

   (ii) Show that $Cl_\mathfrak{m}(K) = J_K/K^* U_{K,\mathfrak{m}}$ is finite. (It is called the *ray class group modulo* $\mathfrak{m}$.)

   (iii) Show that if $K = \mathbb{Q}$ and $\infty \in S$ then $Cl_\mathfrak{m}(\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^*$, where $N = \prod_{p \in S} p^{m_p}$.

   (iv)$*$ Let $I_\mathfrak{m}(K)$ be the free abelian group on the finite places of K not in S, viewed as a subgroup of the group of fractional ideals of K. Let $P_\mathfrak{m}(K)$ be the group of principal ideals $x\mathfrak{o}_K$, where $x \in K^*$ satisfies:

   (a) for all finite $v \in S$, $v(x-1) \geq m_v$; and

   (b) for all real $v \in S$, x is positive under the embedding $K \hookrightarrow \mathbb{R}$ determined by v.

   Show that there is a natural isomorphism $\theta \colon Cl_\mathfrak{m}(K) \xrightarrow{\sim} I_\mathfrak{m}(K)/P_\mathfrak{m}(K)$ — more precisely, that there is a unique isomorphism θ with the property that for every finite $w \notin S$, if $x = (x_v)$ is the idele with $x_w = \pi_w$ and $x_v = 1$ for all $v \neq w$, then $\theta(x)$ is the class of the fractional ideal P_w^{-1}.

   [The -1 is a convenient normalisation. It is *not* the case that for every idele x, $\theta(x)$ is the class of the fractional ideal $c_S(x)^{-1}$. Fröhlich used to call this the "fundamental mistake of class field theory".]

6. (Characters) (i) Let $F/\mathbb{Q}_p$ be finite. A *character*[1] of F^* is a continuous homomorphism $\chi\colon F^* \to \mathbb{C}^*$. Show that the kernel of any character is an open subgroup of F^*. (Hint: it's enough to show that $\ker \chi \cap \mathfrak{o}_F^*$ is open. First show that $\chi(\mathfrak{o}_F^*) \subset U(1)$; then observe that the open subset $V = \{z \mid |\arg z| < \pi/2\}$ does not contain any non-trivial subgroup of $U(1)$.)

(ii) We say $\chi\colon F^* \to \mathbb{C}$ is *unramified* if $\chi(\mathfrak{o}_F^*) = 1$. Show that a character is unramified if and only if it is of the form $\chi(x) = |x|^s$ for some $s \in \mathbb{C}$.

(iii) [$*$ from here on, increasing in number.] An *idele class character* or *Hecke character* of a number field K is a continuous homomorphism $\chi\colon J_K/K^* \to \mathbb{C}^*$. Show that for any Hecke character χ, there exist continuous $\chi_v\colon K_v^* \to \mathbb{C}^*$ ($v \in \Sigma_K$) such that

(a) for almost all finite v, χ_v is unramified;
(b) if $x \in K^*$, then $\prod_v \chi_v(x) = 1$;
(c) if $x = (x_v) \in J_K$ then $\chi(x) = \prod_v \chi_v(x_v)$.

Conversely, show that for any family of characters $(\chi_v)_v$ satisfying (a-b), there is a unique Hecke character χ such that (c) holds.

(iii) Show that if for each infinite place v, $\chi_v(x) \in \{\pm 1\}$, then χ factors through some ray class group $Cl_{\mathfrak{m}}(K)$, and in particular has finite order.

(iv) A Hecke character is said to be *algebraic* if there are integers m_v, n_v, n'_v such that:

(a) if v is real, $\chi_v(x) = x^{m_v}$ for all positive $x \in K_v^* = \mathbb{R}^*$;
(b) if v is complex, $\chi_v(z) = z^{n_v}\bar{z}^{n'_v}$ for every $z \in K_v^* \simeq \mathbb{C}^*$.

Show that if K is totally real (i.e. has no complex places) and χ is an algebraic Hecke character, then $m = m_v$ is independent of v, and that $\chi(x) = \phi(x)\,|x|_{\mathbb{A}}^m$ for some Hecke character ϕ of finite order. (Hint: show that there exists $d \geq 1$ such that if $\varepsilon \in \mathfrak{o}_K^*$ is a unit, $\prod_{v|\infty} \chi_v(\varepsilon^d) = 1$. Then use the unit theorem.)

(v) Let $K = \mathbb{Q}(\sqrt{-7}) \subset \mathbb{C}$. Write w for the unique place over 7, and if $x \in \mathcal{O}_w$, let $\left(\frac{x}{7}\right)$ denote the Legendre symbol of $(x \bmod \pi_w) \in k_w = \mathbb{F}_7$.

Show that if $v \in \Sigma_{K,f} - \{w\}$ then there is a unique $\pi_v \in \mathfrak{o}_K$ with $P_v = (\pi_v)$ and $\left(\frac{\pi_v}{7}\right) = 1$.

Show that there is a unique Hecke character χ of K such that

- for all finite $v \neq w$, χ_v is unramified and $\chi_v(\pi_v) = \pi_v$;
- for all $x \in \mathcal{O}_w^*$, $\chi_w(x) = \left(\frac{x}{7}\right)$.

Show that χ is algebraic, and that $(n_v, n'_v) = (-1, 0)$ for the complex place v of K.

[1] Many people call these *quasi-characters*, and reserve the term "character" for a continuous homomorphism $\chi\colon F^* \to U(1)$