Algebraic Number Theory — problem sheet #3

1. (i) Show that \( A_Q = \mathbb{R} \times (\hat{\mathbb{Z}} \otimes \mathbb{Q}) \).

(ii) Use Theorem 5.4 to show that for any extension \( L/K \) of number fields \( A_L \cong A_K \otimes_K L \).

(iii) Let \( L/K \) be a finite extension of number fields. Show that the trace maps \( \text{tr}_{L_w/K_v} : L_w \to K_v \) define a continuous additive homomorphism \( \text{tr}_{L/K} : A_L \to A_K \), whose restriction to \( L \) is the trace map \( L \to K \).

(iv) Repeat (iii) for the norm map and the idele group.

2. (i) Let \( F \) be a local field. If \( f \in S(F) \) and \( g(x) = f(ax + b) \), compute \( \hat{g} \) in terms of \( \hat{f} \).

(ii) Let \( F/\mathbb{Q}_p \) be finite. Compute the Fourier transform of the characteristic function of \( b + \pi_n^p \sigma_F \). Deduce that the Fourier transform maps \( S(F) \) to itself, and that the Fourier inversion formula holds for \( S(F) \).

(iii) Verify that if \( F = \mathbb{R} \) or \( \mathbb{C} \) then \( \zeta(f, s) = \Gamma_F(s) \), where

\[
\zeta(f, s) = \begin{cases} 
 e^{-\pi x^2} & \text{if } F = \mathbb{R} \\
 e^{-2\pi x^2} & \text{if } F = \mathbb{C}
\end{cases}
\]

3. Let \( F/\mathbb{Q}_p \) be finite, and \( f \in S(F) \).

(i) Show that if \( f(0) = 0 \), then \( \zeta(f, s) \in \mathbb{C}[q^s, q^{-s}] \), hence is analytic for all \( s \in \mathbb{C} \).

(ii) Show that in general \( \zeta(f, s) \) is analytic except for a simple pole at \( s = 0 \) with residue \( f(0)/\log q \). If you like analysis, investigate the analyticity of \( \zeta(f, s) \) for an arbitrary \( f \in S(F) \) when \( F \) is archimedean.

4. Let \( L/K \) be an extension of number fields of degree \( n \). Suppose that almost all places \( v \) of \( K \) split completely in \( L \) (i.e. \( \# \{ w|v \} = n \) ). By comparing \( \zeta_L(s) \) and \( \zeta_K(s)^n \) show that \( L = K \).

5. Let \( K \) be a number field, and \( S \subset \Sigma_K \) a finite set of places. Let \( m = \sum_{v \in S} m_v \cdot (v) \) be a finite formal linear combination of places of \( K \) with positive coefficients \( m_v \). Define

\[
U_{v,m} = \begin{cases} 
 F_v^* & v \text{ complex, or } v \notin S \text{ real} \\
 F_v^{*,+} = \mathbb{R}^*_+ & v \in S \text{ real} \\
 1 + \pi_v^{m_v}\mathcal{O}_v^* & v \in S \text{ finite} \\
 \mathcal{O}_v^* & \text{otherwise}
\end{cases}
\]

and set \( U_{K,m} = \prod_{v} U_{v,m} \)

(i) Show that \( U_{K,m} \) is an open subgroup of \( J_K \), and that every open subgroup of \( J_K \) contains some \( U_{K,m} \).

(ii) Show that \( Cl_m(K) = J_K/\mathcal{O}_K^m \) is finite. (It is called the ray class group modulo \( m \).)

(iii) Show that if \( K = \mathbb{Q} \) and \( \infty \in S \) then \( Cl_m(\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^* \), where \( N = \prod_{p \in S} p^{m_p} \).

(iv)* Let \( I_m(K) \) be the free abelian group on the finite places of \( K \) not in \( S \), viewed as a subgroup of the group of fractional ideals of \( K \). Let \( P_m(K) \) be the group of principal ideals \( x \mathcal{O}_K \), where \( x \in \mathcal{O}_K^* \) satisfies:

(a) for all finite \( v \in S \), \( v(x-1) \geq m_v \); and

(b) for all real \( v \in S \), \( x \) is positive under the embedding \( K \hookrightarrow \mathbb{R} \) determined by \( v \).

Show that there is a natural isomorphism \( \theta : Cl_m(K) \xrightarrow{\sim} I_m(K)/P_m(K) \) — more precisely, that there is a unique isomorphism \( \theta \) with the property that for every finite \( w \notin S \), if \( x = (x_v) \) is the idele with \( x_w = \pi_w \) and \( x_v = 1 \) for all \( v \neq w \), then \( \theta(x) \) is the class of the fractional ideal \( P_w^{-1} \).

[The \(-1\) is a convenient normalisation. It is not the case that for every idele \( x \), \( \theta(x) \) is the class of the fractional ideal \( c_S(x)^{-1} \). Fröhlich used to call this the “fundamental mistake of class field theory”.]
6. (Characters) (i) Let $F/\mathbb{Q}_p$ be finite. A character\footnote{Many people call these quasi-characters, and reserve the term “character” for a continuous homomorphism $\chi: F^* \to \mathbb{C}^*$.} of $F^*$ is a continuous homomorphism $\chi: F^* \to \mathbb{C}^*$. Show that the kernel of any character is an open subgroup of $F^*$. (Hint: it’s enough to show that $\ker \chi(\sigma_F^*)$ is open. First show that $\chi(\sigma_F^*) \subset U(1)$; then observe that the open subset $V = \{z \mid |\arg z| < \pi/2\}$ does not contain any non-trivial subgroup of $U(1)$.)

(ii) We say $\chi: F^* \to \mathbb{C}$ is unramified if $\chi(\sigma_F^*) = 1$. Show that a character is unramified if and only if it is of the form $\chi(x) = |x|^s$ for some $s \in \mathbb{C}$.

(iii) [* from here on, increasing in number.] An idele class character or Hecke character of a number field $K$ is a continuous homomorphism $\chi: J_K/K^* \to \mathbb{C}^*$. Show that for any Hecke character $\chi$, there exist continuous $\chi_v: K_v^* \to \mathbb{C}^*$ ($v \in \Sigma_K$) such that

(a) for almost all finite $v$, $\chi_v$ is unramified;
(b) if $x \in K_v^*$, then $\prod_v \chi_v(x) = 1$;
(c) if $x = (x_v) \in J_K$ then $\chi(x) = \prod_v \chi_v(x_v)$.

Conversely, show that for any family of characters $(\chi_v)_v$ satisfying (a-b), there is a unique Hecke character $\chi$ such that (c) holds.

(iii) Show that if for each infinite place $v$, $\chi_v(x) \in \{\pm 1\}$, then $\chi$ factors through some ray class group $Cl_m(K)$, and in particular has finite order.

(iv) A Hecke character is said to be algebraic if there are integers $m_v$, $n_v$, $n'_v$ such that:

(a) if $v$ is real, $\chi_v(x) = x^{m_v}$ for all positive $x \in K_v^* = \mathbb{R}^*$;
(b) if $v$ is complex, $\chi_v(z) = z^{n_v} \bar{z}^{n'_v}$ for every $z \in K_v^* \simeq \mathbb{C}^*$.

Show that if $K$ is totally real (i.e. has no complex places) and $\chi$ is an algebraic Hecke character, then $m = m_v$ is independent of $v$, and that $\chi(x) = \phi(x) |x|^{m_v}_{\mathfrak{m}}$ for some Hecke character $\phi$ of finite order. (Hint: show that there exists $d \geq 1$ such that if $e \in \sigma_F^*$ is a unit, $\prod_v \chi_v(e^d) = 1$. Then use the unit theorem.)

(v) Let $K = \mathbb{Q}(\sqrt{-7}) \subset \mathbb{C}$. Write $w$ for the unique place over 7, and if $x \in \mathcal{O}_w$, let $(\frac{x}{7})$ denote the Legendre symbol of $(x \mod \pi_w) \in k_w = \mathbb{F}_7$.

Show that if $v \in \Sigma_K \setminus \{w\}$ then there is a unique $\pi_v \in \mathcal{O}_v$ with $P_v = (\pi_v)$ and $(\frac{\pi_v}{v}) = 1$.

Show that there is a unique Hecke character $\chi$ of $K$ such that

- for all finite $v \neq w$, $\chi_v$ is unramified and $\chi_v(\pi_v) = \pi_v$;
- for all $x \in \mathcal{O}_w^*$, $\chi_w(x) = (\frac{x}{7})$.

Show that $\chi$ is algebraic, and that $(n_w, n'_w) = (-1, 0)$ for the complex place $v$ of $K$.\label{6}