1

Algebraic Number Theory – Ex Sheet 3

1. (i) $\mathbb{A}_{\mathbb{Q}} = \{ (x_v) \in \prod_{\text{all } v} \mathbb{Q}_v \mid \text{for a.all } p,\ x_p \in \mathbb{Z}_p \}$

$= \mathbb{R} \times V$ say, $V = \{ (x_p) \in \prod_p \mathbb{Q}_p \mid \cdots\cdots \}$.

Now $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \hookrightarrow V$ and as V is a $\mathbb{Q}$-vector space this extends to an embedding $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow V$.
(since $\hat{\mathbb{Z}}$ is torsion-free).

Finally, if $x = (x_p) \in V$, let $S = \{ p \mid x_p \notin \mathbb{Z}_p \}$

$$m_p = \begin{cases} -v_p(x_p) & \text{if } p \in S \\ 0 & p \notin S \end{cases}$$

$\Rightarrow p^{m_p} x_p \in \mathbb{Z}_p \quad \forall p$

$\Rightarrow Nx \in \prod \mathbb{Z}_p \subset V$ if $N = \prod_{p \in S} p^{m_p}$

ie $\hat{\mathbb{Z}} \otimes \mathbb{Q} \xrightarrow{\sim} V$.

(ii) $L = K(a)$ say, $a \in \mathcal{O}_L$.

$$\mathbb{A}_L \subset \prod_w L_w = \prod_v \Big(\prod_{w|v} L_w \Big)$$

$$= \prod_v (K_v \otimes L) \qquad \text{by Th. 5.4}$$

$$\simeq \Big(\prod_v K_v \Big) \otimes_K L \qquad \text{since* } \dim_K L < \infty.$$

Let $S = \{ v | \infty \} \cup \{ v \text{ finite s.t. } v(N_{L/K}\, g'(a)) \neq 0 \}$.

Then $v \notin S,\ w|v \Rightarrow \mathcal{O}_{L_w} = \mathcal{O}_{K_v}[a] = \mathcal{O}_{K_v} \cdot \mathcal{O}_L$
(by 3.6)

---

* $L \simeq K^n$ so $\prod_v (K_v \otimes L) \simeq \prod_v (K_v^n) \simeq \big(\prod_v K_v\big)^n \simeq \big(\prod_v K_v\big) \otimes L$.

$\boxed{2}$

So if $z = (z_w) \in \prod K_w$, then

$z \in \mathbb{A}_L \iff$ for a.all w, $z_w \in \mathcal{O}_{K_v}\mathcal{O}_L$

$\iff$ for a.all v, $(z_w)_{w|v} \in \prod_{w|v} L_w = K_v \otimes L$ lies in $\mathcal{O}_{K_v} \otimes \mathcal{O}_L$.

(iii) Let $t_v = (\mathrm{tr}_{L_w/K_v})_{w|v} : \prod_{w|v} L_w \longrightarrow K_v$.

Then as $\prod_{w|v} L_w = L \otimes_K K_v$, t_v is just the K_v-linear extension of $\mathrm{tr}_{L/K} : L \to K$ to $L \otimes K_v \longrightarrow K_v$.

Clearly $t_v\left(\prod_{w|v} \mathcal{O}_w\right) \subset \mathcal{O}_v$. So

$(\mathrm{tr}_{L_w/K_v}) : \prod_w L_w \longrightarrow \prod_v K_v$

maps $\mathbb{A}_L$ to $\mathbb{A}_K$.

(iv) is the same.

$\boxed{3}$

2. (i) $g(x) = f(ax+b)$, $a, b \in F$, $a \neq 0$

(otherwise $g(x) = f(b) \notin \mathcal{S}(F)$ if $\neq 0$.)

$$\hat{g}(y) = \int_F \psi(xy) f(ax+b)\, d_F x \qquad ax+b = w$$

$$\Rightarrow d_F w = |a|. \, d_F x$$

$$= \int_F \psi(a^{-1}(w-b)y) f(w) |a|^{-1} d_F w$$

$$= |a|^{-1} \psi\left(-\frac{by}{a}\right) \hat{f}\left(\frac{y}{a}\right).$$

(ii) $\hat{\mathbb{1}}_{\mathcal{O}_F} = q^{-\delta/2} . \mathbb{1}_{\mathcal{D}_F^{-1}}$ $\qquad \mathcal{D}_F = \mathcal{D}_{F/\mathbb{Q}_p}$.

If $f = \mathbb{1}_{\mathcal{O}_F}$, the $\mathbb{1}_{b+\pi_F^n \mathcal{O}_F}(x) = f(\pi^{-n}(x-b))$

$= g(x)$ say

So by (i),

$$\hat{g}(y) = |\pi^{-n}|^{-1} \psi(by). \hat{f}(\pi^n y)$$

$$= q^{-n-\delta/2} \psi(by) \mathbb{1}_{\pi^{-n}\mathcal{D}_F^{-1}}(y).$$

so $\hat{g} \in \mathcal{S}(F)$ (as ψ is locally constant)

and easily check $\hat{\hat{g}} = g$

(iii) $F = \mathbb{R}$ done in lectures

$$F = \mathbb{C} : \quad \zeta(f, s) = \frac{1}{\pi} \int_{\mathbb{C}^*} e^{-2\pi z\bar{z}} (z\bar{z})^s . \frac{|dz . d\bar{z}|}{z\bar{z}}$$

$$= 4. \int_0^\infty e^{-2\pi r^2} r^{2s} \frac{dr}{r} \qquad |dz\, d\bar{z}| = 2r\, dr\, d\theta$$

$$= 2 \int_0^\infty e^{-t} \left(\frac{t}{2\pi}\right)^s \frac{dt}{t} = \Gamma_{\mathbb{C}}(s) \qquad (t = 2\pi r^2).$$

3 (i). If $f \in \mathcal{S}(F)$ and $f(0) = 0$ then the support of f is a compact subset of F^*, hence contained in some $\{x \in F^* \mid |v(x)| \leq m\}$ $(m \geq 0)$.

So $$\zeta(f,s) = \sum_{n=-m}^{m} \int_{\pi^n \mathcal{O}_F^*} f(x).|x|^s \, d_F^* x$$

$$= \sum_{n=-m}^{m} q^{-ns} \int_{\pi^n \mathcal{O}_F^*} f(x) \, d_F^* x \in \mathbb{C}[q^s, q^{-s}].$$

(ii) In general, let $g = f - f(0).\mathbb{1}_{\mathcal{O}_F} \in \mathcal{S}(F)$.

Then $$\zeta(f,s) - \zeta(g,s) = f(0).\zeta(\mathbb{1}_{\mathcal{O}_F}, s)$$

$$= \frac{f(0)}{1-q^{-s}},$$ which has a simple pole at $s=0$, residue $f(0)/\log q$

By (i), $\zeta(g,s)$ is entire.

Hint for last part: write $\zeta(f,s) = \int_{|x|\geq 1} + \int_{|x|\leq 1} f(x)|x|^s d_F^x x$.

Show $\int_{|x|\geq 1}$ is entire (since $f \in \mathcal{S}(F)$); for $F = \mathbb{R}$ take a finite Taylor expansion $f(x) = \sum_{n=0}^{m} c_n x^n + x^{n+1} R(x)$ to compute $\int_0^1$, first for $\mathrm{Re}(s) > 1$.

5

4. Let S be the set of finite places of K which don't split completely in L/K. By hypothesis, S is finite. So if $v \notin S$ then there are $n = [L:K]$ places w_i of L over v, and so $f(w_i/v) = 1$, $q_{w_i} = q_v$.

$$\therefore \quad \zeta_L(s) = \prod_{\substack{v \in S \\ w | v}} (1 - q_w^{-s})^{-1} \prod_{v \notin S} \underbrace{\prod_{i=1}^{n} (1 - q_{w_i}^{-s})^{-1}}_{= (1 - q_v^{-s})^{-n}}$$

$$\underline{\text{i.e.}} \quad \zeta_L(s) = \zeta_K(s)^n \prod_{v \in S} \left[(1 - q_v^{-s}) \cdot \prod_{w | v} (1 - q_w^{-s})^{-1} \right]$$

The product over S is clearly holomorphic and $\neq 0$ in a neighbourhood of $s = 1$. So as both ζ_L and ζ_K have a <u>simple</u> pole at $s = 1$, we have $n = 1$ i.e. $L = K$.

$\boxed{6}$

5. (i) $U_{v,m} \subset K_v^*$ is open for every v, and

$U_{v,m} = \mathcal{O}_v^*$ for all finite $v \notin S$.

So $U_{K,m} \subset J_K$ is open.

Conversely, if $U \subset J_K$ is an open sgp, then as it is an open nbd. of 1, there exists a finite set $S \supset \Sigma_{K,\infty}$ and open neighborhoods W_v of $1 \in K_v^*$ ($v \in S$) such that $U \supset \prod_{v \in S} W_v \times \prod_{v \notin S} \mathcal{O}_v^*$.

If $v | \infty$ then as U is a subgroup, it contains the subgroup generated by W_v, which is also open, hence equals $\mathbb{R}^*$ or $\mathbb{R}^*_{>0}$ (v real) or $\mathbb{C}^*$ (v complex).

If $v \nmid \infty$ then W_v contains $1 + \pi_v^m \mathcal{O}_v$ for some m.

So $U \supset$ some $U_{K,m}$.

(ii) As $U_{K,m} \subset J_K$ is open, it is also closed, hence $J_K / U_{K,m}$ is discrete, hence so also is $J_K / K^* U_{K,m}$. But as $U_{K,m}$ contains $\mathbb{R}^*_{>0}$ at each infinite place, $|U_{K,m}|_{\mathbb{A}} = \mathbb{R}^*_{>0}$, hence $J_K^1 / K^* \longrightarrow J_K / K^* U_{K,m}$ is surjective. J_K^1 / K^* is compact, so $Cl_m(K)$ is finite.

7

(iii) $$U_{\mathbb{Q},m} = \mathbb{R}^*_{>0} \times \prod_{p\in S} (1+p^{m_p}\mathbb{Z}_p) \times \prod_{p\notin \mathbb{Z}} \mathbb{Z}_p^* .$$

As $Cl(\mathbb{Q}) = \{1\}$, $J_{\mathbb{Q}} = U_{\mathbb{Q}} . \mathbb{Q}^*$ where

$U_{\mathbb{Q}} = \mathbb{R}^* \times \prod_p \mathbb{Z}_p^*$. So

$$U_{\mathbb{Q}} \longrightarrow Cl_m(\mathbb{Q}) = J_{\mathbb{Q}} / \mathbb{Q}^*. U_{\mathbb{Q},m}$$

is surjective, with kernel

$$\begin{aligned} U_{\mathbb{Q}} \cap \mathbb{Q}^*. U_{\mathbb{Q},m} &= (U_{\mathbb{Q}} \cap \mathbb{Q}^*). U_{\mathbb{Q},m} \\ &= \{\pm 1\}. U_{\mathbb{Q},m} \end{aligned}$$

$$\therefore \; Cl_m(\mathbb{Q}) \cong U_{\mathbb{Q}} / \{\pm 1\}. U_{\mathbb{Q},m}$$

$$= \frac{\mathbb{R}^* \times \prod_p \mathbb{Z}_p^*}{\{\pm 1\}.\left(\mathbb{R}^*_{>0} \times \prod_{p\in S}(1+p^{m_p}\mathbb{Z}_p) \times \prod_{p\notin S} \mathbb{Z}_p^*\right)}$$

$$\cong \frac{\{\pm 1\} \times \prod_{p\in S} (\mathbb{Z}/p^{m_p}\mathbb{Z})^*}{\{\pm 1\} \text{ (embedded diagonally)}}$$

$$\xleftarrow{\sim} \prod_{p\in S} (\mathbb{Z}/p^{m_p}\mathbb{Z})^* \simeq (\mathbb{Z}/N\mathbb{Z})^*$$

(the $\xleftarrow{\sim}$ induced by inclusion $\prod_{p\in S} \hookrightarrow \{\pm 1\} \times \prod_{p\in S}$).

$\boxed{8}$

(iv) Let $J_{K,m} = \left\{ (x_v) \in J_K \;\middle|\; \forall v \in S, \begin{array}{ll} v(x_v - 1) \geq m_v & (v \text{ finite}) \\ x_v > 0 & (v \text{ real}) \end{array} \right\}$

So $U_{K,m} = J_{K,m} \cap U_K$

Define $\Theta_S : J_{K,m} \longrightarrow I_m(K)/P_m(K)$

by $(x_v) \longmapsto \sum_{\text{finite } v \notin S} -v(x_v) \cdot (v)$

(or $\prod_{v \notin S} P_v^{-v(x_v)}$ if you prefer)

- obviously surjective:

$\ker(J_{K,m} \to I_m(K)) = J_{K,m} \cap U_K = U_{K,m}$

and $P_m(K) = \{ x\mathcal{O}_K \mid x \in K^* \cap U_{K,m} \}$.

so $\ker \Theta_S = U_{K,m} \cdot (K^* \cap J_{K,m})$.

So have

$$\frac{I_m(S)}{P_m(S)} \xleftarrow[\sim]{\Theta_S} \frac{J_{K,m}}{(K^* \cap J_{K,m}) \cdot U_{K,m}} \xrightarrow{\beta} \frac{J_K}{K^* U_{K,m}}$$

where β is induced by $J_{K,m} \hookrightarrow J_K$. Enough to show β is an isom. Now

$$J_{K,m} \cap K^* U_{K,m} = (J_{K,m} \cap K^*) . U_{K,m}$$

as $U_{K,m} \subset J_{K,m}$

so β is injective. It is surjective by the strong approximation thm: let $x = (x_v) \in J_K$.

We can then find $a \in K$ such that

$\forall$ real $v \in S$, $|x_v - a|_v < |x_v|_v$

(which implies $a/x_v > 0$)

$\forall$ finite $v \in S$, $|x_v - a|_v < q_v^{-m_v} . |x_v|_v$

(which implies $|a/x_v - 1|_v < q_v^{-m_v}$)

So $a \in K^*$ and $a x^{-1} \in J_{K,m}$ ie. $J_K = K^* J_{K,m}$

Hence β surjective.

6 (i). Consider the restriction of χ to $\mathcal{O}_F^*$. Its image is a compact sgp. of $\mathbb{C}^*$, so must be contained in $U(1)$ (otherwise it would be unbounded).

Let $\mathcal{U} = \{ e^{2\pi i x} \mid -\pi < x < \pi \} . \subset U(1)$.

The $\chi^{-1}(\mathcal{U}) \cap \mathcal{O}_F^*$ is a open nbd. of 1, so contains an open sgp. $1 + \pi^n \mathcal{O}_F$ for some $n \geq 1$.

But then $\chi(1+\pi^n \mathcal{O}_F)$ is a sgp. of $U(1)$ contained in $\mathcal{U}$, and obviously the only such subgroup is $\{1\}$.

$\therefore$ $1+\pi^n \mathcal{O}_F \subset \ker \chi$, so $\ker \chi$ is open.

[Same argument shows that if G is any profinite group and $\rho : G \to GL_n(\mathbb{C})$ is a cts. hom$^{\underline{m}}$, then $\rho(G)$ is finite — take $\mathcal{U}$ = open subset of $U(n) \subset GL_n(\mathbb{C})$ consisting of matrices whose eigenvalues λ have $|\arg \lambda| < \pi/2$.]

(ii) As $|\cdot|$ is cts, $\forall s \in \mathbb{C}$, $x \mapsto |x|^s$ is an unramified character. Conversely, if χ is unramified, pick $s \in \mathbb{C}$ s.t. $q^{-s} = \chi(\pi)$. Then if $\underset{\in F^*}{x = \pi^n u}$, $u \in \mathcal{O}_F^*$,

$$\chi(x) = \chi(\pi)^n \chi(u) = q^{-ns} = |x|^s.$$

(iii) Let $\chi : J_K \longrightarrow \mathbb{C}^*$ be continuous.

The group $\prod_{v \nmid \infty} \mathcal{O}_v^*$ is profinite, so same argument as (i) shows that $\ker \chi$ contains an open sgp. of $\prod \mathcal{O}_v^*$; hence contains $W = \prod_{v \in S} (1+\pi_v^N \mathcal{O}_v) \times \prod_{v \notin S} \mathcal{O}_v^*$ for some finite S & $N \geq 1$

Let χ_v be the composite $K_v^* \hookrightarrow J_K \xrightarrow{\chi} \mathbb{C}^*$.

Then χ_v is cts $\forall v$, and for all finite $v \notin S$, $\chi_v(\mathcal{O}_v^*) = 1$. $\Rightarrow$ (a).

Therefore the homomorphism

$$\chi^f : J_K \longrightarrow \mathbb{C}^*, \quad \chi^f((x_v)) = \prod_{v \text{ finite}} \chi_v(x_v)$$

is well-defined (as for every $(x_v) \in J_K$, $\chi_v(x_v) = 1$ for a. all v by (a)). It is continuous as its kernel contains the open sgp. $\prod_{v|\infty} K_v^* \times W \subset J_K$

Obviously the map $\chi^\infty : (x_v) \longmapsto \prod_{v|\infty} \chi_v(x_v)$ is also continuous, so the product

$$\chi^\infty \chi^f : J_K \longrightarrow \mathbb{C}^*, \quad (x_v) \longmapsto \prod_{\text{all } v} \chi_v(x_v)$$

is continuous. On the dense subgroup $\bigoplus_{\text{all } v} K_v^* \subset J_K$, $\chi^\infty \chi^f$ and χ are equal, so by continuity, $\chi = \chi^\infty \chi^f$ (i.e. (c)) and hence (b) as $\chi(K^*) = 1$, and the local components (χ_v) determine χ uniquely

Conversely, let (χ_v) be given satisfying (a) – (c). Define χ^f, χ^∞ as above. By (a) and part (i), $\exists$ finite set S and N as above such that $\chi^f(W) = 1$, so χ^f is cts. and $\chi = \chi^\infty \chi^f$ is a Hecke character.

(iv) We know $\ker(\chi) \supset$ open sgp W of $\prod_{v \nmid \infty} \mathcal{O}_v^\times$,

and the quotient $\prod \mathcal{O}_v^\times / W$ is finite, of order d say.

Then if $\varepsilon \in \mathcal{O}_K^\times$, the image of ε^d in $\prod \mathcal{O}_v^\times$ lies

in W. So $\chi_v(\varepsilon^d) = 1$ $\forall$ finite d, and as $\chi(\varepsilon) = 1$,

we get $\prod_{v|\infty} \chi_v(\varepsilon)^d = 1$

Suppose K totally real of degree n. Then $\mathcal{O}_K^\times$ has rank

$n-1$. Let $\varepsilon_1, \dots, \varepsilon_{n-1}$ be independent elts. of $\mathcal{O}_K^\times$;

replacing ε_i by ε_i^{2d} may assume $\prod_{v|\infty} \chi_v(\varepsilon) = 1$

and ε_i is positive for every embedding $K \hookrightarrow \mathbb{R}$.

Then $\chi_v(\varepsilon_i) = |\varepsilon_i|_v^{m_v}$ $\forall v \in \Sigma_{K,\infty}$, and so

$$\sum_{v|\infty} m_v \log|\varepsilon_i|_v = 0$$

But the matrix $(\log|\varepsilon_i|_v)$ has rank $(n-1)$

by unit theorem, and the only linear relation

between its rows is that their sum is 0

(by product formula): so $(m_v)_v$ is a multiple

of $(1, \dots, 1)$.

$\boxed{13}$ (v) The units of K are $\{\pm 1\}$, and $\left(\frac{-1}{7}\right) = -1$.

So if $\alpha \in \mathcal{O}_K \setminus (\sqrt{-7})$, exactly one of $\left(\frac{\alpha}{7}\right), \left(\frac{-\alpha}{7}\right)$ is $+1$; applying this to $P_v = (\alpha)$ gives the first statement, since $\mathcal{O}_K$ is a P.I.D.

Let ∞ be the unique complex place of K.

Then we are required to have:–

$$\chi_v(x) = \pi_v^{v(x)} \quad \forall x \in K_v^*, \text{ if } v \notin \{w, \infty\}$$

$$\chi_w(x) = \left(\frac{x}{7}\right) \quad \forall x \in \mathcal{O}_w^*$$

In particular, $\chi_v(\pi_v) = 1$, and if $v' \neq v$ is any other finite place, $\pi_v \in \mathcal{O}_{v'}^*$ so $\chi_{v'}(\pi_v) = 1$

So if a Hecke character χ as stated exists, then $\chi(K^*) = 1$ implies $\chi_\infty(\pi_v) = \chi_v(\pi_v)^{-1} = \pi_v^{-1}$ $\forall$ finite $v \neq w$. Hence $\chi_\infty(x) = x^{-1}$ for every x in the subgroup $H = \langle \pi_v \mid v \neq w \rangle \subset K^* \subset K_\infty^* = \mathbb{C}^*$.

<u>Claim:</u> H is dense in $\mathbb{C}^*$.

<u>Proof</u> (naive) Note that $\frac{1 \pm \sqrt{-7}}{2}, 2 \pm \sqrt{-7} \in \{\pi_v\}$

so H contains their norms, 2 & 11; and the subgroup $\langle 2, 11 \rangle$ is dense in $\mathbb{R}_{>0}^*$ $(\cong \mathbb{R})$. So the closure of H contains $\mathbb{R}_{>0}^*$. But $\frac{1+\sqrt{-7}}{1-\sqrt{-7}} = x \in H$ has $|x| = 1$, and x is not a root of 1. So $\langle x \rangle$ is dense in $U(1)$, hence H is dense in $\mathbb{C}^*$. $\square$

So $\chi_\infty(x) = x^{-1}$ $\forall x \in \mathbb{C}^*$. But then

$$\chi_w(\sqrt{-7}) = \prod_{\substack{v \neq w \\ v \text{ finite}}} \chi_v(\sqrt{-7})^{-1} \cdot \chi_\infty(\sqrt{-7})^{-1}$$

$$= \chi_\infty(\sqrt{-7})^{-1} = \sqrt{-7}$$

so there is at most 1 choice for the remaining 2 factors χ_∞, χ_w.

To check these choices really define a Hecke character, we need to verify that

$$\forall x \in K^*, \prod_{\text{all } v} \chi_v(x) = 1. \qquad (*)$$

The choice of χ_∞ ensures $(*)$ holds for $x = \pi_v$, $v \neq w$.

The choice of χ_w ensures $(*)$ holds for $x = \sqrt{-7}$.

Now $K^* = \langle -1, \sqrt{-7}, \{\pi_v\}_{v \neq w} \rangle$ by unique factorisation, so just need to check $x = -1$. But

$\chi_\infty(-1) = -1 = \left(\frac{-1}{7}\right) = \chi_w(-1)$, $\chi_v(-1) = 1$ $\forall v \neq w, \infty$

So $\chi(K^*) = 1$ and χ is a Hecke character. !!