Algebraic Number Theory: ex sheet 2

1.(i) Assume $y \notin K(x)$, $y = y_1, \dots y_n$ the conjugates of y over K. Then $\{\sigma \in \mathrm{Gal}(\overline{K}/K(x)) \mid \sigma(y) = y\}$
$= \mathrm{Gal}(\overline{K}/K(x,y)) \subsetneq \mathrm{Gal}(\overline{K}/K(x))$, so for some σ,
$\sigma(x) = x$ and $\sigma(y) = y_j \neq y$. As $|\cdot|$ on $\overline{K}$ is the unique extension of the AV from K, it is invariant under σ (as $|\sigma(-)|$ is also an AV) so

$$|x - y| = |\sigma(x-y)| = |x - \sigma(y)|, \text{ contradiction.}$$

(ii) $y_m = \sum_{i=0}^{m} p^i \zeta_{p^i}$. If $|y_m - x|_p \leq p^{-m-1}$ then

$$x = y_m + p^{m+1} z, \quad |z|_p \leq 1.$$

Let $y_m' \neq y_m$ be a conjugate of y_m; so $y_m' = \sum_{i=1}^{m} p^i \zeta_{p^i}^a$, $a \in \mathbb{Z}$, $(p, a) = 1$

as $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p) = (\mathbb{Z}/p^m\mathbb{Z})^*$.

WLOG $a \not\equiv 1 \bmod p^r$, $a \equiv 1 \bmod p^{r-1}$ for $0 < r \leq m$,

If $i < r$ then $\zeta_{p^i}^{a-1} = 1$, so $\quad a - 1 = p^{r-1} b$, $(b, p) = 1$

$$|y_m' - y_m| = \left| \sum_{i=r}^{m} p^i \zeta_{p^i} (\zeta_{p^i}^{a-1} - 1) \right|$$

$$= \left| \sum_{i=r}^{m} p^i (\zeta_{p^{i+1-r}}^{b} - 1) \right| = |p^r (\zeta_p^b - 1)| = p^{-r - \frac{1}{p-1}}$$

$$\therefore |x - y_m'| = |p^{m+1} z + (y_m - y_m')| = |y_m - y_m'| > |x - y_m|.$$

$\therefore \mathbb{Q}_p(x) \supset \mathbb{Q}_p(y_m)$ by Krasner's lemma.

Suppose now that $y_m \to x \in \bar{\mathbb{Q}}_p$. Then

$$x = \sum_{i=0}^{\infty} p^i \zeta_{p^i} \Rightarrow |x - y_m| \leq p^{-m-1}$$

$$\Rightarrow \quad \forall m, \quad y_m \in \mathbb{Q}_p(x)$$

$$\Rightarrow \forall m, \quad \zeta_{p^m} = \frac{y_m - y_{m-1}}{p^m} \in \mathbb{Q}_p(x).$$

– contradiction.

2 (i) $f = \sum_{i=0}^{n} a_i X^i = g + \sum_{i=0}^{n-1} c_i X^i$, $|c_i| \leq \varepsilon$ say.

Let $g(\beta) = 0$, $\{\alpha_j\}$ = roots of f, and assume that $\forall j \neq 1$, $|\alpha_j - \beta| \leq |\alpha_1 - \beta|$.

$$\Rightarrow f(\beta) = \prod_{j=1}^{n} (\beta - \alpha_j) = \sum_{0}^{n} c_i \beta^i$$

$$\Rightarrow \quad \prod_j |\beta - \alpha_j| \leq \max |c_i \beta^i| \leq \varepsilon$$

$$\Rightarrow \quad |\beta - \alpha_1| \leq \varepsilon^{1/n}$$

Choose $\varepsilon > 0$ s.t. $|\alpha_1 - \alpha_j| > \varepsilon^{1/n} \quad \forall j \neq 1$

$$\Rightarrow |\beta - \alpha_j| = |(\beta - \alpha_1) + (\alpha_1 - \alpha_j)| > \varepsilon^{1/n} \quad \forall j \neq 1$$

So $K(\alpha_1) \subset K(\beta)$ by Krasner's Lemma.

$[K(\alpha_1) : K] = n$ as f irred., and $[K(\beta) : K] \leq n$

$\Rightarrow K(\alpha_1) = K(\beta)$. Now $\alpha = \sigma(\alpha_1)$, some $\sigma \in Gal(\bar{K}/K)$

$$\Rightarrow K(\alpha) = K(\sigma(\beta)), \quad g(\sigma(\beta)) = 0$$

(ii) Let $S = \{$all Eisenstein polys. of degree n $/\mathcal{O}_K\}$

$$\xrightarrow{\sim} (\pi_K \mathcal{O}_K)^{n-1} \times \pi_K \cdot \mathcal{O}_K^{\times}$$

by $f = T^n + \sum_0^{n-1} a_i T^i \longmapsto (a_{n-1}, \ldots, a_1 ; a_0)$

For $g \in S$, choose $r_g > 1$ s.t.:

$g_1 \in S$ & $g_1 \equiv g \mod \pi_K^{r_g} \Rightarrow K[T]/(g) \simeq K[T]/(g_1)$.

Let $U_g = \{g_1 \in S \mid g_1 \equiv g \mod \pi_K^{r_g}\} \underset{\text{open}}{\subset} S$.

Now $S = \bigcup_{g \in S} U_g = \bigcup_{g \in S_1} U_g$ for some finite $S_1 \subset S$ by compactness of S.

$\therefore$ every totally ramified L/K is $\cong K[T]/(g)$ for some $g \in S_1$. $\therefore$ only finitely many isom. classes of totally ramified L/K of degree n. As each L has only a finite no. (n) of embeddings into $\overline{\mathbb{Q}}_p$ which are the identity on K, there are only finitely many subfields of $\overline{\mathbb{Q}}_p$ which are totally ram. ext^{n}s of K of degree n.

Suppose $L/\mathbb{Q}_p$ is an ext^n of degree d. Then max^l unramified subfield L_0 is one of a finite set (it is splitting field of $T^{p^r} - 1$ for some $r \mid d$).

And L/L_0 is one of a finite set of exts, by what was just shown. $\therefore$ only finitely many L exist.

(iii) Answer is no. As $\mathrm{char}(K) = p > 0$, every separable cyclic extension $L|K$ of degree p is of the form

$$L = K(x), \quad x^p - x = f \in K.$$

f and g determine same extension iff $f = g + h^p - h$ for some $h \in K$.

So enough to show that $K/\{h^p - h \mid h \in K\}$, where $K = \mathbb{F}_p((T))$, is infinite-dimensional over $\mathbb{F}_p$. But

$$(T^i)_{(i,p)=1}$$ is a linearly independent family.

3(i) $K \supset K_0 \supset \mathbb{Q}_p$ K/K_0 tot. ram,

$K_0/\mathbb{Q}_p$ unramified, $[K_0 : \mathbb{Q}_p] = f$.

Coroll 3.8 $\Rightarrow K_0 = \mathbb{Q}_p(\zeta_{p^f-1})$

(ii) Suppose $\zeta \in K$ is a primitive n^{th} root of unity.

If $(n,p) = 1$ then as $T^n - 1 \bmod p$ is separable

$\Rightarrow K_1 = \mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ unramified, $\mathcal{O}_{K_1} = \mathbb{Z}_p[\zeta_n]$ and

k_{K_1} = splitting field of $T^n - 1$ over $\mathbb{F}_p$ $\subset k_K = \mathbb{F}_{p^f}$

$\Rightarrow n \mid p^f - 1$.

STP $\zeta_p \notin K$. But $e(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = p-1$.

4. Obviously $m_L \supset m_K \mathcal{O}_L$. If $x \in m_L$, then $|x| > 0$. As AV on K is not discrete, $\exists y \in K$ with $|x| \geq |y| > 0 \Rightarrow y \in m_K$, $x/y \in \mathcal{O}_L$ so $x \in m_K \mathcal{O}_L$.

Next let, suppose $\mathcal{O}_L = \sum_{i=1}^{d} \mathcal{O}_K . e_i$, d minimal.

Then $\mathcal{O}_L = \bigoplus \mathcal{O}_K e_i$, for if not, $\sum x_i e_i = 0$ for some $x_i \in \mathcal{O}_K$, not all zero. Dividing by the x_j with $v(x_j)$ minimal, WLOG some $x_j = 1$

and then $e_j = -\sum_{i \neq j} x_i e_i$ ie. d not minimal.

Therefore sum is direct and $d = [L:K]$

So $m_L = m_K \mathcal{O}_L \Rightarrow m_L = \bigoplus m_K . e_i$

$\Rightarrow k_L = \bigoplus k_K \bar{e}_i$ ie. $[k_L : k_K] = d = [L:K]$

Let $L = \bigcup L_n$ $\quad L_n = \mathbb{Q}_p(\zeta_{p^n})$ $\quad [L_n : \mathbb{Q}_p] = (p-1)p^{n-1}$.

$K = \bigcup K_n$, $[L_n : K_n] = p-1$

Then $k_L = \mathbb{F}_p = k_K$ but $[L:K] = p-1$.

5 (i). $e(M/K) = e(M/L)\,e(L/K)$.

(ii) $L = L_1 L_2 \supset K$; let L_0 = max^{l} sub field unram $/K$.

Then L_1, L_2 unram. $\Rightarrow L_1, L_2 \subset L_0$. $\Rightarrow L_1 L_2 \subset L_0$.

(iii) $L_i = K(\pi_i)$ π_i unif. of L_i.

Say $n_i = [L_i : K]$. $(n_1, n_2) = 1$

$\Rightarrow \exists\, c_i \in \mathbb{Z},\ c_2 n_1 + c_1 n_2 = 1$.

$v_K(\pi_i) = \frac{1}{n_i} \Rightarrow v_K(\pi_1^{c_1}\pi_2^{c_2}) = \frac{1}{n_1 n_2}$

$\Rightarrow e(L_1 L_2/K) \geq n_1 n_2 \Rightarrow e(L_1 L_2/K) = n_1 n_2$

& ext^n is totally ram.

However, suppose $a \in \mathbb{Z}_p^*$ not a square, $p > 2$.

$L_1 = \mathbb{Q}_p(\sqrt{p})$, $L_2 = \mathbb{Q}_p(\sqrt{ap})$

Then $L_1 L_2 \supset \mathbb{Q}_p(\sqrt{a})$ which is unramified $/\mathbb{Q}_p$

6 (i) $L = \mathbb{Q}_3(\sqrt[3]{2}, \zeta_3)$

$$\begin{tikzcd}[column sep=small]
 & L = \mathbb{Q}_3(\sqrt[3]{2}, \zeta_3) \arrow[dl, dash, "3"'] \arrow[dr, dash, "2"] & \\
F = \mathbb{Q}_3(\zeta_3) \arrow[dr, dash, "2"'] & & K = \mathbb{Q}_3(\sqrt[3]{2}) \arrow[dl, dash, "3"] \\
 & \mathbb{Q}_3 &
\end{tikzcd}$$

$\pi_K = \sqrt[3]{2} + 1$ has min. poly $(T-1)^3 - 2$
$= T^3 - 3T^2 + 3T - 3$
so $K/\mathbb{Q}_3$ totally ramified, uniformiser $= \pi_K$.

$\pi_F = \zeta_3 - 1$ is a uniformiser of F.

So if $\pi_L = \pi_F / \pi_K$, $v_3(\pi_L) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

ie π_L is a uniformiser of L & $L/\mathbb{Q}_3$ totally ramified.

$L/\mathbb{Q}_3$ is Galois, $\mathrm{Gal}(L/\mathbb{Q}_3) = G = I$ as tot. ramified

and $P = G_1$ is Sylow 3-sgp $\Rightarrow P = \mathrm{Gal}(L/F) = \langle \sigma \rangle$

$\sigma : \sqrt[3]{2} \mapsto \zeta_3 . \sqrt[3]{2}$

$$\sigma(\pi_L) - \pi_L = \frac{\pi_F}{\sigma(\pi_K)} - \frac{\pi_F}{\pi_K} \quad \text{as } \sigma(\pi_F) = \pi_F$$

$$= \frac{\pi_F}{\pi_K . \sigma(\pi_K)} (\pi_K - \sigma(\pi_K))$$

$$= \pi_F . \pi_K^{-1} . \sigma(\pi_K)^{-1} . \sqrt[3]{2} . (1 - \zeta_3)$$

$\Rightarrow v_L(\pi_L - \sigma(\pi_L)) = 2$ as $v_L(\sqrt[3]{2}) = 0$.

$\Rightarrow \sigma \notin G_2$ ie. $G = G_0 \supset G_1 = \langle \sigma \rangle \supset G_2 = \{1\}$.

(ii) Clearly $\mathbb{Q}_2^\times = \langle 2\rangle \times \langle -1\rangle \times (1+4\mathbb{Z}_2)$

(since $\mathbb{Z}_2^\times = (1+4\mathbb{Z}_2) \amalg (-1+4\mathbb{Z}_2)$) and by

part (ii) of Qn 9, $(1+4\mathbb{Z}_2, \times) \simeq (\mathbb{Z}_2, +)$ as ab. group

$\therefore\ \mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

and the corresponding $(\mathbb{Z}/2)^3$-extension of $\mathbb{Q}_2$ is

$L = \mathbb{Q}_2(\sqrt{2}, \sqrt{-1}, \sqrt{5}) = \mathbb{Q}_2(\zeta_8, \sqrt{5})$

since $\zeta_8 = \dfrac{1+\sqrt{-1}}{\sqrt{2}}$. Now $\mathbb{Q}_2(\zeta_8)/\mathbb{Q}$ is totally ramified.

$L_0 = \mathbb{Q}_2(\sqrt{5})/\mathbb{Q}_2$ is <u>unramified</u> since $\mathcal{O}_{L_0} \ni \dfrac{1+\sqrt{5}}{2}$

min. poly $T^2 - T - 1$.

$$\begin{array}{l} L = L_0(\zeta_8) \\ \therefore\ \Big|\, 4 \text{ totally ram.} \\ L_0 \\ \Big|\, 2 \\ \mathbb{Q}_2 \end{array}$$

$I = \mathrm{Gal}(L/L_0) \simeq (\mathbb{Z}/8\mathbb{Z})^\times \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$

$(\sigma_a : \zeta_8 \mapsto \zeta_8^a) \longleftarrow\!\shortmid\ a$

$\pi_L = \zeta_8 - 1$ uniformiser of L.

$v_L(\sigma_a(\pi_L) - \pi_L)$

$= v_L(\zeta_8^{a-1} - 1)$

$$= \begin{cases} \infty & a = 1 \\ v_L(\zeta_4 - 1) = 2 & a = 3 \\ v_L(\zeta_2 - 1) = 4 & a = 5 \\ v_L(\zeta_4^{-1} - 1) = 2 & a = 7 \end{cases}$$

$\therefore\ I = G_0 \simeq (\mathbb{Z}/8\mathbb{Z})^\times$

$$\begin{array}{c} G_0 \\ \parallel \\ G_1 \\ \cup \\ G_2 = G_3 = \langle \sigma_5 \rangle \\ \cup \\ G_4 = \{1\} \end{array}$$

7 (i) $\pi_K = \zeta_p - 1$. Then

$$\frac{1-\zeta_p^i}{1-\zeta_p} = \frac{(\pi_K+1)^i - 1}{\pi_K}$$

$$= \pi_K^{i-1} + i\pi_K^{i-2} + \dots + \binom{i}{2}\pi_K + i$$

$$\equiv i \pmod{\pi_K}$$

$$\Rightarrow \prod_{i=1}^{p-1}\left(\frac{1-\zeta_p^i}{1-\zeta_p}\right) \equiv (p-1)! \equiv -1 \pmod{\pi_K}$$

and $\displaystyle\prod_{i=1}^{p-1}(1-\zeta_p^i) = \prod_{i=1}^{p-1}(T-\zeta_p^i)\Big|_{T=1} = (T^{p-1}+\dots+T+1)\Big|_{T=1}$

$$= p$$

$$\Rightarrow (1-\zeta_p)^{p-1} \equiv -p \mod (\pi_K p)$$

ie. $(1-\zeta_p)^{p-1} = -pu$, $u \in 1+\pi_K \mathcal{O}_K$.

(ii) $f(T) = T^{p-1} - u$; $f(1) \equiv 0 \mod \pi_K$

$f'(1) = p-1 \not\equiv 0 \mod \pi_K$.

So by Hensel $\exists v \equiv 1 \mod \pi_K$ with $v^{p-1} = u$.

(iii) (i) + (ii) $\Rightarrow$ $-p = \left(\frac{1-\zeta_p}{v}\right)^{p-1}$ so $K \supset \mathbb{Q}_p(\sqrt[p-1]{-p})$.

But as $T^{p-1}+p$ is irreducible, $[\mathbb{Q}_p(\sqrt[p-1]{-p}) : \mathbb{Q}_p] = p-1$

$$\Rightarrow K = \mathbb{Q}_p(\sqrt[p-1]{-p})$$

8. $\mathcal{O}_L = \mathcal{O}_K[\pi_L] = \mathcal{O}_K[T]/(f)$ where

f = min. poly of $\pi_L = \prod_{\sigma \in G} (T - \sigma(\pi_L))$

$\therefore \mathcal{D}_{L/K} = (f'(\pi_L)) = \prod_{1 \neq \sigma \in G} (\sigma(\pi_L) - \pi_L)$.

$\Rightarrow v_L(\mathcal{D}_{L/K}) = \sum_{1 \neq \sigma \in G} i_{L/K}(\sigma)$ (this is the def^{n} of $i_{L/K}(\sigma)$).

On other hand, if $\sigma \in G$ then

$i_{L/K}(G) = \max\{i \mid \sigma \in G_i\} + 1$ by def^{n} of G_i.

$= \#\{i \geq 0 \mid \sigma \in G_i\}$.

$$\therefore \sum_{1 \neq \sigma \in G} i_{L/K}(\sigma) = \sum_{\substack{i \geq 0 \\ 1 \neq \sigma \in G}} \begin{pmatrix} 1 & \text{if } \sigma \in G_i \\ 0 & \sigma \notin G_i \end{pmatrix}$$

$$= \sum_{i \geq 0} (\# G_i - 1).$$

9. (i) $v_p\left(\frac{x^n}{n}\right) = n \cdot v_p(x) - v_p(n)$

$0 \le v_p(n) \le \frac{\log n}{\log p}$. $\Rightarrow \sum \pm \frac{x^n}{n}$ cvges iff $|x| < 1$.

$v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$ ("Sieve of Eratosthenes")

$\le n \cdot \left(\frac{1}{p} + \frac{1}{p^2} + \dots\right) < \frac{n}{p-1}$.

So $v_p(x^n/n!) > n\left(v_p(x) - \frac{1}{p-1}\right)$.

$\therefore \sum \frac{x^n}{n!}$ cvges if $v_p(x) > \frac{1}{p-1}$.

$v_p(x) \le \frac{1}{p-1}$: take $n = p^r$. Then $v_p(n!) = p^{r-1} + \dots + p + 1$
$= (p^r - 1)/(p-1)$.

$\Rightarrow v_p\left(x^{p^r}/(p^r)!\right) \le 0$. $\Rightarrow \sum \frac{x^n}{n!}$ doesn't converge.

(ii) Suppose p odd. Then if $v_p(x) \ge 1$,

$\forall n \ge 1$, $v_p\left(\frac{x^n}{n!}\right) > 0$ and $v_p\left(\frac{x^n}{n}\right) > 0$ (by above)

so $\log : 1 + p\mathbb{Z}_p \longrightarrow p\mathbb{Z}_p$, $\exp : p\mathbb{Z}_p \longrightarrow 1 + p\mathbb{Z}_p$.

We need to show $\log(\exp(x)) = x$, $\exp(\log(1+x)) = 1+x$

and $\exp(x+y) = \exp x \cdot \exp y$. As the power series

are all convergent, it's enough to to know that

the identities hold for formal power series.

For this, either prove directly

$$\left[\text{e.g. } \exp(X+Y) = \sum_{n\geq 0} \frac{(X+Y)^n}{n!} = \sum_{n\geq 0} \sum_{i+j=n} \frac{X^i Y^j}{n!}\binom{n}{i} \right.$$

$$\left. = \sum_{i\geq 0} \frac{X^i}{i!} \sum_{j\geq 0} \frac{Y^j}{j!} = \exp X \exp Y \right]$$

or use fact that the identities hold for

$x, y \in \mathbb{C}$ with $|x|, |y| < 1$ to deduce the identity

for formal power series.

$p=2$: then $v_p(x) \geq 2 \Rightarrow \forall n \geq 1$,

$v_p\left(\frac{x^n}{n!}\right) > n\left(2 - \frac{1}{p-1}\right) > n$ So $x \in 4\mathbb{Z}_p \Rightarrow \frac{x^n}{n!} \in 4\mathbb{Z}_p$

and rest is the same.

(iii) If $x \in \mathbb{Q}_p^*$ and $|v_p(x)| = n > 0$ then x is not an m-th power for any $m > n$.

If $x \in \mathbb{Z}_p^*$ and $x = y^{p-1}$ then $x \equiv 1 \pmod p$.

So the subgroup of prime-to-p-divisible elements of $\mathbb{Q}_p^*$ is contained in $1 + p\mathbb{Z}_p$.

Now as $\log : (1+p\mathbb{Z}, \times) \xrightarrow{\sim} (p\mathbb{Z}_p, +)$ and $\mathbb{Z}_p$ is prime-to-p-divisible, we have equality.

(iv) First suppose $p > 2$. Any automorphism σ of $\mathbb{Q}_p$ preserves the sgp. of prime-to-p-divisible elements of $\mathbb{Q}_p^*$, so $\sigma(1+p\mathbb{Z}_p) = 1+p\mathbb{Z}_p$.

$$\Rightarrow \quad \sigma(p\mathbb{Z}_p) = p\mathbb{Z}_p$$

$$\Rightarrow \quad \sigma(p^n\mathbb{Z}_p) = p^n\mathbb{Z}_p \quad \forall n > 0.$$

So σ is continuous, and as $\mathbb{Q} \subset \mathbb{Q}_p$ is dense, $\sigma = \mathrm{id}$

Case $p=2$: same argument as in (iii) shows that

$1+4\mathbb{Z}_2 \subset \{$prime-to-2-divisible elements of $\mathbb{Q}_2^*\} \subset \mathbb{Z}_2^*$

and since $\mathbb{Z}_2^* = \{\pm 1\} \times (1+4\mathbb{Z}_2)$ and -1 is prime-to-2-divisible, the 2nd "$\subset$" is an "$=$".

So $\sigma \in \mathrm{Aut}(\mathbb{Q}_2)$ preserves $\mathbb{Z}_2^* = 1+2\mathbb{Z}_2$.

& rest of argument is the same.

10. (i) $x \equiv y \pmod{\pi}$. Claim $x^{p^n} \equiv y^{p^n} \pmod{\pi^{n+1}}$.

Induction: say $x^{p^{n-1}} = y^{p^{n-1}} + \pi^n z$, some $n \geq 1$

$$\Rightarrow x^{p^n} = (y^{p^{n-1}} + \pi^n z)^p = y^{p^n} + p \cdot \pi^n (y^{(p-1)p^{n-1}} \cdot z + \dots\dots) + \pi^{np} z^p \quad \text{(binomial expansion)}$$

$$\equiv y^{p^n} \bmod \pi^{n+1}.$$

(ii) Let $a_n = \widehat{x^{1/p^n}}$ be some lift of x^{1/p^n} to R.

Need to show that $(y_n) = (a_n^{p^n})$ is Cauchy: so as non-archimedean, STP $|y_{n+1} - y_n| \to 0$.

Well, $(x^{1/p^{n+1}})^p = x^{1/p^n} \Rightarrow a_{n+1}^p \equiv a_n \bmod \pi$.

$\Rightarrow a_{n+1}^{p^{n+1}} \equiv a_n^{p^n} \bmod \pi^{n+1}$. So sequence is Cauchy. Also if a_n' is another lift, $a_n' \equiv a_n \pmod{\pi}$

$\Rightarrow y_n' := (a_n')^{p^n} \equiv a_n^{p^n} = y_n \bmod \pi^n$

$\Rightarrow \lim(y_n) = \lim(y_n')$

Clearly if $x = 0$ may take $a_n = 0$ $\forall n$ ie. $[0] = 0$

Also, if $\begin{cases} a_n \text{ lifts } x^{1/p^n} \\ b_n \text{ lifts } y^{1/p^n} \end{cases} \Rightarrow a_n b_n$ lifts $(xy)^{1/p^n}$

ie $[x][y] = \lim(a_n^{p^n}) \lim(b_n^{p^n}) = \lim(a_n b_n)^{p^n} = [xy]$.

(iii) As $\pi^n \mathcal{O}_K$ is the disjoint union of cosets $\pi^n a + \pi^{n+1}\mathcal{O}_K$, a running over $\mathcal{O}_K/\pi\mathcal{O}_K$, STP. that $\forall x \in \mathcal{O}_K/\pi\mathcal{O}_K$, there is a unique $[x']$ st. $[x'] \bmod \pi\mathcal{O}_K = x$. But obviously $[x] \bmod \pi = x$.

11. Let $y = x + \pi$, and $\bar{\ } : \mathcal{O}_L \longrightarrow k_L$ the reduction map. The residue field of $K(y)$ contains $\bar{y} = \bar{x}$.

Now $k_L = k_{L_0} = k_K(\bar{x})$, so $k_{K(y)} = k_{L_0}$ and therefore $L_0 \subset K(y)$. So $x \in K(y) \Rightarrow \pi_L = y - x \in K(y)$

$\Rightarrow K(y) = K(x, \pi_L) = L_0(\pi_L) = L$.

In particular, $\mathcal{O}_K[y] = \bigoplus_{i=0}^{n-1} \mathcal{O}_K . y^i \subset \mathcal{O}_L$,

and $\mathcal{O}_L = \bigoplus \mathcal{O}_K e_i$, $e_i \in L = K[y]$ (some basis)

$\Rightarrow \mathcal{O}_L \subset \frac{1}{\pi_K^n} . \mathcal{O}_K[y]$ for some n

$\therefore \mathcal{O}_K[y] \subset \mathcal{O}_L$ is open, hence also closed.

Let f = min. poly of x over $\mathcal{O}_K$. As L_0/K is unramified, $\bar{f}$ is separable, so $f(T) = (T-x)\, g(T)$, $\overline{g(x)} \neq 0$.

Let $\pi' = f(x+\pi_L) = \pi_L \cdot g(x+\pi_L)$.

Then $\overline{g(x+\pi_L)} = \overline{g(\bar{x})} \neq 0 \Rightarrow v_L(\pi') = 1$

Then as $\mathcal{O}_K[y] \hookrightarrow \mathcal{O}_L \to k_L = k_K[\bar{x}]$ is $\underline{\text{surjective}}$,

$$\mathcal{O}_L = \pi_L \mathcal{O}_L + \mathcal{O}_K[y]$$
$$= \pi' \mathcal{O}_L + \mathcal{O}_K[y]$$
$$= \pi'(\pi' \mathcal{O}_L + \mathcal{O}_K[y]) + \mathcal{O}_K[y]$$
$$= \pi'^2 \mathcal{O}_L + \mathcal{O}_K[y] \qquad (\text{since } \pi' \in \mathcal{O}_K[y])$$
$$= \pi'^2(\pi' \mathcal{O}_L + \mathcal{O}_K[y]) + \mathcal{O}_K[y]$$
$$\vdots$$
$$= (\pi')^n \mathcal{O}_L + \mathcal{O}_K[y] \qquad \forall n \geq 1.$$

$\therefore \mathcal{O}_K[y] \twoheadrightarrow \mathcal{O}_L / \pi^n \mathcal{O}_L \quad \forall n.$

$\Rightarrow \mathcal{O}_K[y] \subset \mathcal{O}_L$ is dense. As it is closed,

$$\mathcal{O}_K[y] = \mathcal{O}_L.$$

12. (i) Enough to find a nbd. U of 0 in $\mathbb{A}_K$ st. $U \cap K = \{0\}$

Let $U = \prod_{v|\infty} \{x \in K_v \mid |x|_v < 1\} \times \prod_{v \nmid \infty} \mathcal{O}_v$.

– this is open, and $x \in K$, $x \in U \Rightarrow |x|_v \leq 1 \ \forall v \nmid \infty$

$\Rightarrow x \in \mathcal{O}_K$. Then $|x|_v < 1 \Rightarrow |N_{K/\mathbb{Q}}(x)| < 1$

$\Rightarrow x = 0$ as $N_{K/\mathbb{Q}}(\mathcal{O}_K) \subset \mathbb{Z}$

ii) $(p_i) \to \infty$, $x_{(i)} = (1, \ldots, \underset{\substack{\uparrow \\ v = p_i}}{p_i}, 1 \ldots \ldots)$

$$\Rightarrow \quad x_{(i)} - 1 = (0, \ldots\ldots, 0, \underset{\substack{\uparrow \\ v = p_i}}{p_i - 1}, 0, \ldots\ldots)$$

Let $U \subset \mathbb{A}_\mathbb{Q}$ be any nbd. of 0. Then

$\exists$ finite set S of places of $\mathbb{Q}$ and $\varepsilon > 0$ st.

$$U \supset \prod_v U_v, \quad U_v = \begin{cases} \{|x|_v \leq 1\} & \text{if } x \notin S \\ \{|x|_v < \varepsilon\} & \text{if } x \in S \end{cases}$$

by defn of product topology. Then for i sufficiently large, $p_i \notin S$ and so $x_{(i)} - 1 \in U$

$\Rightarrow x_{(i)} - 1 \to 0$ in $\mathbb{A}_\mathbb{Q}$.

But $\mathbb{R}^* \times \prod_p \mathbb{Z}_p^* = U$ is an open nbd. of 1 in $J_{\mathbb{Q}}$, which clearly contains no $x_{(i)}$

$$\therefore \; x_{(i)} \not\to 1 \text{ in } J_{\mathbb{Q}}.$$

So restriction of topology on $\mathbb{A}_{\mathbb{Q}}$ to $J_{\mathbb{Q}}$ can't be the topology of $J_{\mathbb{Q}}$. ($J_{\mathbb{Q}} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$ is continuous but not a homeomorphism onto its image.)

(iii) $\Theta : J_K \hookrightarrow \mathbb{A}_K^2$, $x \mapsto (x, x^{-1})$.

Now Θ is certainly continuous, so STP that for every $V \subset J_K$ open, $\exists\, U \subset \mathbb{A}_K^2$ open with $\Theta^{-1}(U) = V$.

If $y \in J_K$ then $\mathbb{A}_K^2 \longrightarrow \mathbb{A}_K^2$, $(x, x') \mapsto (xy, x'y^{-1})$ is continuous (since $x_v \in \mathcal{O}_v^*$ for almost all v)

So V is open in the induced topology $\Leftrightarrow$ yV is.

$\therefore$ enough to show that the basic open nbd.

$$V = \prod_{v \in S} \{ |x-1|_v < \varepsilon \} \times \prod_{v \notin S} \mathcal{O}_v^* \qquad \left(\begin{array}{l} S \text{ finite} \\ \supset \Sigma_{K,\infty} \end{array} \right)$$

is open in the induced topology $\qquad 0 < \varepsilon < 1$ say

But $V = \Theta^{-1}(U \times U')$ where

$$U = \prod_{v \in S} \{ |x-1|_v < \varepsilon \} \times \prod_{v \notin S} \mathcal{O}_v$$

$$U' = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v .$$

(iv) More generally, let X_α be an infinite family of (non-empty) non-compact spaces. Then $\prod X_\alpha$ (with product topology) is not locally compact.

Proof: Let $x \in \prod X_\alpha$, V any nbd. of x.

Then $V \supset U = \prod_{\alpha \in S} U_\alpha \times \prod_{\alpha \notin S} X_\alpha$

with S finite, $U_\alpha \subset X$ open nonempty

Then the closure $U^c = \prod U_\alpha^c \times \prod X_\alpha$

is closed in V. If V were compact, so would be U^c. But if $\alpha \notin S$, $U^c \twoheadrightarrow X_\alpha$ is continuous and X_α isn't compact. So V not compact i.e. x has no compact neighborhood.