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The Simplest Case of Ramsey's Theorem

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This is a somewhat personalized, and far from comprehensive, essay on the "simplest case" of Ramsey's theorem, namely, the two-coloured graph case.

1. Erdős and Ramsey's Theorem

The association of Paul Erdős with Ramsey's theorem goes right back to his youth, to his famous paper with Szekeres [43]. Somewhat later, his equally famous proof of an exponential lower bound for Ramsey numbers [23] may be regarded as introducing probabilistic methods to graph theory, though the paper itself makes no mention of probability. Despite its antiquity, the Erdős–Szekeres proof of Ramsey's theorem is still the proof of choice. Moreover, though in recent decades innumerable powerful and beautiful combinatorial results have been obtained by probabilistic methods, Erdős's first application remains perhaps the most striking, because of its simplicity and the extent to which the result he obtained is still unrivalled by any other technique.

Many people can look back to the effect of Erdős at a critical point in their lives, myself included. When I was an undergraduate taking a course in graph theory, Erdős came to visit Béla Bollobás for a month or two. Hallard Croft, who was responsible for directing my studies, somehow arranged that Erdős would see a few of us once a fortnight to talk about the course. Naturally, we had no idea of the privilege accorded us. Naturally, Erdős took no notice of what was in the course. One week, he defined for us a projective plane, and asked us to go away and construct a plane of order ten. We failed. Then he sent us away to find the Ramsey number $r(5)$. This time we did a bit better — we beat the upper bound given to us in the lectures. It turned out that Walker [103] had already improved the bound, but the bug had bitten.

The subject of this article is Ramsey's theorem in its "simplest case", as Ramsey himself calls it [77, p. 269], asserting that there exists a smallest number $r(s)$ so that, whenever the edges of the complete graph $K_{r(s)}$ are coloured red and blue, a red or a blue K_s will appear. The paper is not really a survey — there are several good surveys covering this material and much more besides (such as Graham, Rothschild and Spencer [54], Nešetřil [71], Nešetřil and Rödl [72], Rödl [83] and, for actual numbers, Radziszowski [76]) — it is more in the way of a brief essay written as a token to acknowledge Erdős's decisive effect on me. Limitations of time and space prevent the mentioning here of much that Erdős has done, and that he has inspired others to do, concerning the general Ramsey theorem and in the wider field of Ramsey theory.

2. Basic Bounds

It is remarkable that so little progress has been made, over so many years, in improving either the upper or the lower bounds for Ramsey numbers. It would seem that making progress is not easy. Indeed, it has been written that *"perhaps the first question which one is tempted to consider is the problem of the actual size of a set which guarantees the validity of Ramsey's (and Ramsey type) theorem. One should try to resist the temptation since it is well known that Ramsey numbers are difficult to determine and even good asymptotic estimates are difficult to find (and improve)"* (Nešetřil [71, p. 346]). But resisting temptation is not always easy either.

2.1. Classical upper bounds

Ramsey [77], when introducing the theorem that now bears his name, proved first the infinite case, and then adapted his argument to the finite case. His argument, when restricted to the two-colour graph case, can easily be described. It shows that $r(s) \le 2^{\binom{s}{2}}$ and, more generally, that if $n \ge 2^{\binom{s}{2}}k$

and K_n is two-coloured then a monochromatic $K_{s-1} + \overline{K}_k$ appears. The proof is by induction on s. We may assume there is a red $K_{s-2} + \overline{K}_m$ where $m = 2^{s-1}k$. Let M be the set of m vertices joined to the red K_{s-2} by red edges. Select $v_1 \in M$, and some $2^{s-2}k$ other vertices of M joined to v_1 by edges of the same colour. If these edges are red we have our $K_{s-1} + \overline{K}_k$, so we may assume they are blue. Next, amongst those $2^{s-2}k$ vertices, pick v_2 and $2^{s-3}k$ others joined to v_2 by edges of the same colour, which we may again assume to be blue. Proceeding in this way we get a set $\{v_1, \ldots, v_{s-1}\}$ spanning a blue K_{s-1} and joined to a further set of k vertices by blue edges, which is a blue $K_{s-1} + \overline{K}_k$, as claimed.

Ramsey proved his theorem whilst investigating a problem in logic, in pursuit of an algorithm for decidability. He states *"we should at the same time like to have information as to how large"* Ramsey numbers actually are. He notes that his general estimates are inefficient in the two-colour graph case, for, since it may be assumed that M contains no vertex of red degree k, the same argument works with $s!k$ in place of $2^{\binom{s}{2}}k$, so $r(s) \leq s!$. Remarkably, he went so far as to observe that there can be no $(k-1)$-regular graph of order $s!k - 1$ when k is even, so the value $s!$ can be lowered slightly further, but *"this value is, I think, still much too high"*. He was right.

Definition 2.1. We call a colouring of K_s *orderable* if its vertices can be ordered so that all edges with the same first vertex have the same colour.

(An orderable K_s is called *good* by Nešetřil [71].) It is easily seen that a two-coloured K_{2^t} contains an orderable K_{t+1}; as in Ramsey's argument, pick any first vertex, select 2^{t-1} other vertices joined to it by the same colour, and repeat the argument within these vertices. Now associate with each vertex the colour of those edges of which it is the first vertex. The pigeonhole principle, applied to the vertex colours, shows at once that an orderable K_{2s-2} contains a monochromatic K_s, and hence $r(s) \leq 2^{2s-3}$.

The reason that Ramsey's bound is so poor, and the reason his proof is frustrating to read for the modern reader weaned on the Erdős–Szekeres proof, would seem to be his failure to apply the pigeonhole principle in the last step! Ramsey, in effect, finds an orderable $K_{\binom{s}{2}+1}$ and then argues as follows. The first vertex colour may as well be red. Then, either some vertex amongst the next two is red, or both of them are blue, so after three vertices we have two of the same colour, say red. After that, either some one of the next three is red, or all three are blue, so after six vertices we have three of the same colour. Continuing in this way, we need $\binom{s}{2}$ vertices to find $s-1$ of the same colour, which together with the last vertex make a monochromatic K_s.

When considering the argument above, that shows $r(s) \leq 2^{2s-3}$, it is easy to spot that it should be possible to improve it because, when building up our orderable K_{2s-2} with a view to finding a monochromatic K_s, if the first few vertex colours are red, then we are closer to a red K_s than to a blue K_s, and so we could afford to choose for the next vertex one whose red degree is smaller than its blue degree. Put another way, if at each stage we cannot find a vertex joined to more than half the rest by the same colour, then we may as well choose red edges each time, and so get home in s steps rather than $2s-2$. Pursuing this thought leads naturally to the simple, but important, idea that one should consider situations where *the red goal graph and the blue goal graph may differ.* So we define $r(s,t)$ to be the smallest number such that, whenever the edges of the complete graph $K_{r(s,t)}$ are coloured red and blue, a red K_s or a blue K_t will appear. By now, a moment's reflection will show that the end result of this argument is no more than the famous recurrence

$$r(s,t) \leq r(s-1,t) + r(s,t-1) \quad \text{whence} \quad r(s,t) \leq \binom{s+t-2}{t-1}.$$

This recurrence is, of course, that of Erdős and Szekeres [43]; Erdős [23] attributes it to Szekeres. Sadly, the gain it gives over the bound $r(s) \leq 2^{2s-3}$ is small (though see the remark in §4.2), because $\binom{2s-2}{s-1} \sim \left(2/\sqrt{\pi s}\right) \times 2^{2s-3}$.

2.2. More recent upper bounds

The Erdős–Szekeres bound has been improved, but barely. Frasnay [50] showed that $r(s) \leq (8/9)\binom{2s-2}{s-1}$; he did this by calculating the gain implied by the fact that $r(s,t) \leq r(s-1,t) + r(s,t-1) - 1$ whenever both $r(s-1,t)$ and $r(s,t-1)$ are even. The first to establish that $r(s,t) = o\left(\binom{s+t-2}{t-1}\right)$ was Rödl [82], who proved that

$$r(s,t) \leq \frac{A}{\left(\log(s+t)\right)^B} \binom{s+t-2}{t-1} \quad \text{for} \quad s,t \geq 3,$$

where A and B are absolute constants (for a proof of the weaker bound $r(s,t) \leq 6\binom{s+t-2}{t-1}/\log\log(s+t-2)$, see Graham and Rödl [55]). Another upper bound [100], which is an improvement when $t/\sqrt{\log t} \ll s \leq t$, is

$$r(s,t) \leq t^{-s/2t+A/\sqrt{\log t}} \binom{s+t-2}{t-1} \quad \text{for} \quad 1 \leq s \leq t.$$

The methods used in the proofs are similar, both being based on counting the number of monochromatic triangles (as in §5.1); if there are k triangles sharing a common edge then we have a monochromatic (say red) $K_2 + \overline{K}_k$, and if $k \geq r(s-2,t)$ then we get a red K_s or a blue K_t. But the improvement achieved this way is small in comparison with the work put in, and the calculations are quite delicate — an initial estimate, ignoring the smaller error terms, suggested a stronger result might be attainable [99, Theorem 6.11], but this turned out to be over-optimistic. The method offers no hope of deciding whether $r(s) < (4-\varepsilon)^s$ for some absolute $\varepsilon > 0$, which remains an outstanding question (but see §4.2).

In the extreme asymmetric case when s is fixed and t is large, Graver and Yackel gave a long argument for the bound $r(s,t) = O\big(t^{s-1}(\log\log t)/\log t\big)$. A breakthrough was made by Ajtai, Komlós and Szemerédi [2], who showed

$$\text{for } s \text{ fixed}, \quad r(s,t) = O\big(t^{s-1}/(\log t)^{s-2}\big).$$

In particular they established the bound $r(3,t) = O(t^2/\log t)$ (which is best possible; see §2.3). Their method was cleaned and polished by Shearer [86], obtaining $r(3,t) \leq t^2/\log(t/e)$.

Finally, the achievement of McKay and Radziszowski [68], in proving

$$r(5,5) \leq 49,$$

must be mentioned. Their approach greatly extends the triangle counting method by considering many carefully chosen combinations of small subgraphs, and it requires a large amount of computation.

2.3. Lower bounds

It is not easy to think of a better lower bound for $r(s)$ than that obtained from Turán's colouring of $s-1$ disjoint red K_{s-1} joined by blue edges, namely the bound $r(s) > (s-1)^2$. In view of this, Erdős's proof in 1947 [23] that

$$r(s) > 2^{s/2} \quad \text{for} \quad s \geq 3 \cdot$$

is breathtaking. As mentioned earlier, the paper never uses the language of probability, but the argument is simply that the expected, or average, number of monochromatic K_s, over all colourings of K_n, is $2\binom{n}{s}2^{-\binom{s}{2}}$, so if this number is less than one, which it is if $n \leq 2^{s/2}$, then there is a colouring of K_n with no monochromatic K_s. The argument in fact yields $r(s) > 2^{s/2}\left(s/e\sqrt{2}\right)$.

In introducing this argument, namely, proving existence by probabilistic means, Erdős had an incalculable effect on the future course of combinatorics, and it is an idea that will be forever associated with his name. The idea took a little while to mature, and perhaps it did not really take off until his probabilistic proofs, in 1959 [25] of the existence of graphs of large girth and large chromatic number, and in 1961 [26] that $r(3,t) > c(t/\log t)^2$. In his 1962 paper [27], which we shall discuss in §5.2, he notes that the number of monochromatic K_3s in a random colouring is more or less minimal, and he states *"it is perhaps surprising that a crude probabilistic argument gives a result which for $k = 3$ is so close to the correct one. This phenomenon can often be observed in this subject* [ref]", where the reference [ref] is to the above cited papers [23, 25, 26]. It is noteworthy that Erdős chose identical titles for the latter two papers.

All the best lower bounds for Ramsey numbers use the probabilistic method, or a clever combination of probability and determinism, as in [26]. When Erdős and Lovász [36] introduced the Local Lemma to treat almost independent events, Spencer [94] applied it to improve the lower bound for $r(s)$. Despite the power of the lemma, the improvement is only a factor of two; it gives

$$r(s) > \frac{\sqrt{2}s}{e}2^{s/2}.$$

This remains the best bound for the diagonal case, though McDiarmid and Steger [67] have shown that the same bound can be also achieved by colourings that are regular, self-complementary and pseudo-random (in the sense of §4.2). Spencer [95] likewise obtained the lower bound

$$\text{for } s \text{ fixed,} \quad r(s,t) > c_s(t/\log t)^{(s+1)/2}$$

and the same result was obtained by Krivelevich [63] using large deviation inequalities.

It is fitting that the Ramsey number $r(3,t)$ has been sharply estimated from both above and below, in each case by probabilistic means. In fact, the method of repeatedly selecting random subsets, used by Ajtai, Komlós and Szemerédi [2] to prove $r(3,t) = O(t^2/\log t)$ and used spectacularly by Rödl [81], was the same as that used by Kim [61] in proving his celebrated lower bound

$$r(3,t) > c\frac{t^2}{\log t}.$$

Of course, the latter proof requires probabilistic tools not available until recently, including some developed specifically for the job.

2.4. Problems and conjectures

Let us note some problems, posed by Erdős about the Ramsey numbers $r(s,t)$, which are very simple in appearance, but which seem to be out of reach.

- Is $\limsup r(s)^{1/s} < 4$?
- Is $\liminf r(s)^{1/s} > \sqrt{2}$?
- Does $\lim r(s)^{1/s}$ exist?
- Does there exist $\varepsilon > 0$ such that $r(s+1,s) > (1+\varepsilon)r(s,s)$?
- Does $r(3,t+1) - r(3,t) \to \infty$?
- Is $r(3,t+1) - r(3,t) = o(t)$?
- Does $r(4,t) > t^{3-\varepsilon}$ provided t is large, for every $\varepsilon > 0$?

These questions illustrate how little is known even about two-colour graph Ramsey numbers. Some of these questions arose out of Erdős's work with with Sós and with Faudree, Rousseau, and Schelp; such questions, and related conjectures, appear in many of Erdős's lists of problems, such as [28, 30].

A naive heuristic argument in [97] led to the conjecture that $r(s,t) \le r^*(s,t)$, where $r^*(s,t)$ was given by a certain formula. In the case $s = t$ the conjecture became $r(s) \le 2^{s-1}(s-2)+2$, which is exact for $s \le 4$ and is true for $s = 5$ ([68], see §2.2). In fact, of the 9 pairs $3 \le s \le t$ for which $r(s,t)$ is known exactly, in 6 cases the value is $r^*(s,t)$. Some support for the conjecture came from the fact that a similar heuristic led to the proof in [97] of the conjecture of Beineke and Schwenk [13], that every two-colouring of the complete bipartite graph $K_{n,n}$ contains a monochromatic

$K_{s,t}$ if $n > 2^s(t-1)$. (For large s, t, the bound implied by Füredi [51] is better than this.)

On the other hand, the same heuristic implies lower bounds on the number of monochromatic K_s in a large two-coloured K_n that are false if $s > 3$ (see §5.2). It also implies that K_n contains a monochromatic $K_s + \overline{K}_k$ if $n = 2^s(s+k-2)+2$, which is true if $s \le 2$ (see §5.1, and Rousseau and Sheehan [84] for a study), but it is false for $s = 3$ (see §5.3). Most fatally for the conjecture, Kim's lower bound for $r(3,t)$ (see §2.3) shows it to be false in this extreme off-diagonal case, because $r^*(3,t) = O\big(t^2/(\log t)^2\big)$. The shakiest small case is $r^*(3,10) = 40 \le r(3,10)$, shown by Exoo [45].

2.5. Constructive lower bounds

Probabilistic arguments offer good lower bounds for Ramsey numbers, but they offer little help in displaying good colourings. So it is natural to ask for explicit colourings having no monochromatic K_s. However, even beating the Turán bound $r(s) > (s-1)^2$ in this way is not straightforward. In 1972 Abbott [1] proved $r(s) > s^{2+\varepsilon}$ constructively, and Nagy [70] proved $r(s) > \binom{s-1}{3}$. Nagy's construction is to take, as vertices, all 3-element subsets of a set of $s-1$ elements, two being joined by a red edge if they intersect in exactly one element. In fact, two copies of this colouring, joined entirely by red edges, shows $r(s) > 2\binom{s-1}{3}$ for $s \ge 15$. The properties of the colouring are best verified by an *ad hoc* argument. However, some information can be derived from the generalization of Fisher's inequality proved by Ray-Chaudhuri and Wilson [78], stating that if A is a family of ℓ-subsets of an h-set then $|A| \le \binom{h}{t}$, where $t = \Big|\big\{\,|a \cap b| \,:\, a, b \in A,\ a \ne b\big\}\Big|$.

Frankl's construction [48] of a super-polynomial lower bound for $r(s)$ was a great advance. The best construction to date remains that of Frankl and Wilson [49], giving the lower bound

$$r(s) > \exp\Big\{\big(1+o(1)\big)\log^2 s/4\log\log s\Big\}.$$

The example of Frankl and Wilson is also defined by subset intersections, and arises out of their elegant theory of set intersections modulo a prime. One of their theorems is that, if A is a family of ℓ-subsets of an h-set, such that $|a \cap b| \not\equiv \ell \pmod q$ where q is a prime power, then $|A| \le \binom{h}{q-1}$. Now take, as the vertices of a complete graph, all (q^2-1)-element subsets of some h-element set, and colour ab red if $|a \cap b| \equiv -1 \pmod q$. The theorem just stated shows that any blue K_s satisfies $s \le \binom{h}{q-1}$. The Ray–Chaudhuri–Wilson theorem gives the same bound for a red K_s, and taking $h = q^3$ gives the claimed lower bound for $r(s)$.

Recently, Grolmusz [58] has given a different colouring that achieves almost the same lower bound for $r(s)$ as the Frankl–Wilson colouring, the constant 4 being replaced by a somewhat larger one. His colouring is, in some sense, antithetical to the previous example, insofar as it is based on constructions of set systems showing that the Frankl–Wilson theorem very strongly fails to hold for non-prime power moduli. The colouring is of the complete graph K_n on $n = k^k$ vertices, with vertex set $\{1,\dots,k\}^k$. For each subset $T \subset \{1,\dots,k\}$, a non-negative integer a_T is chosen. Given two vertices $x = (x_1,\dots,x_k)$ and $y = (y_1,\dots,y_k)$, let $a_{xy} = \sum_{T\subset S} a_T$, where $S = \{i : x_i = y_i\}$. Colour xy red if a_{xy} is odd. Now if we can choose the numbers a_T so that $a_{xx} \equiv 0 \pmod 6$ but $a_{xy} \not\equiv 0 \pmod 6$ if $x \ne y$, then the Frankl–Wilson theorem implies that a monochromatic K_s must satisfy $s \le \binom{h}{2}$, where $h = \sum_T a_T k^{|T|}$.

To find such numbers a_T, take a polynomial Q in k variables, with integer coefficients, that represents NAND (modulo 6); that is, if $z = (z_1,\dots,z_k) \in \{0,1\}^k$, then $Q(z) \equiv 0 \pmod 6$ if and only if $z_1 = \cdots = z_k = 1$. Since we are interested only in the values of $Q(z)$ modulo 6 and only in $z \in \{0,1\}^k$, we may reduce Q to a sum of monomials, namely, $Q(z) \equiv \sum_T a_T z_T$, where the monomial z_T is the product of the variables indexed by T, and $0 \le a_T \le 5$. Such a choice for a_T works (take $z_i = \delta_{x_i y_i}$), and gives the bound $h \le 10k^{2d}/d!$, where d is the degree of the polynomial Q. One possibility for Q is $1 - z_1 z_2 \dots z_k$, which has degree k, but the smaller the degree, the better the end result.

A good choice for Q is $Q(z_1,\dots,z_k) = P(1-z_1,\dots,1-z_k)$, where $P(z)$ is the polynomial representing OR (modulo 6) found by Barrington, Beigel and Rudich [12]. They take integers e and f with $2^e 3^f > k$, and define

$$G_2(z) = \sum_{0<|T|<2^e} (-1)^{|T|+1} z_T \quad \text{and} \quad G_3(z) = \sum_{0<|T|<3^f} (-1)^{|T|+1} z_T.$$

Now $G_2(z) \equiv 0 \pmod 2$ if and only if $\big|\{i : z_i = 1\}\big| \equiv 0 \pmod{2)^e}$, and $G_3(z)$ has a similar property. Thus $P(z) = 3G_2(z) + 4G_3(z)$ satisfies $P(z) \equiv 0 \pmod 6$ if and only if $z_1 = \cdots = z_k = 0$, as desired. In this way, the degree d of Q can be made at most $\sqrt{6k}$. Then, from the above observations, the bound $r(s) > \exp\big\{ \big(1+o(1)\big) \log^2 s/108 \log\log s \big\}$ follows.

In an attempt to quantify the difficulty in finding good constructive lower bounds for Ramsey numbers, Babai [10] conjectured that such colourings cannot be defined by real polynomials. To be precise, he conjectured that there is an absolute positive constant ε and, for each real polynomial $P(x,y,z)$, a positive constant c_P, such that the colouring on vertex set $\{1,\ldots,n\}$, defined by ij being red if and only if $P(i,j,n) > 0$, contains a monochromatic K_s where $s > c_P n^{\varepsilon}$. Note that the polynomial $(i-j)^2 - n$ gives a colouring akin to the Turán colouring. Alon [3] proved a weakened form of Babai's conjecture, with $c_P n^{\varepsilon}$ replaced by $c\sqrt{2}^{\sqrt{\log n}}$.

It is thought that good constructive lower bounds for $r(s)$ might prove helpful in complexity theory, or in derandomization procedures. There are techniques for producing colourings on $2^{s/2}$ vertices without using as many random bits as does a random colouring. Razborov [79] gives an economical method. He proves that there is a sequence of Boolean formulæ f_s of s variables, such that the colouring on $2^{s/2}$ vertices, defined by ab being red if $f_s(a,b) = 1$, has no monochromatic K_s, and moreover f_s is computable by a circuit of depth 3 comprising only $O(s^5 \log s)$ randomly chosen binary $\wedge$-gates and $\oplus$-gates. For a connection between constructive bounds (such as the Frankl–Wilson bound) and Shannon capacity, see Alon [6].

2.5.1. Paley graphs. The only two values of $r(s)$ known are $r(3) = 6$ and $r(4) = 18$, and the colourings of K_5 and K_{17} giving the lower bounds are unique, as Greenwood and Gleason [57] discovered. The colourings are symmetric in red and blue; the red graphs are the *Paley* graphs P_5 and P_{17}. There is a Paley graph P_q of order q if q is a prime power with $q \equiv 1 \pmod 4$. Its vertices are the elements of the finite field of order q, and ab is an edge if $a - b$ is a square.

There is a long history of association between Paley graphs and Ramsey's theorem. This is partly because Paley graphs give the best lower bounds for $r(3)$, $r(4)$ and $r(3,5)$, and modifications of them give good lower bounds for many other values (see Radziszowski [76]). But it is also because Paley graphs mimic random graphs very well in some ways (see §4.2).

However, it is known that not all Paley colourings can perform as well as random ones. To see this, observe that, if q is a prime, then the clique number of P_q is at least as large as the smallest non-square (modulo q). Now all the numbers up to x are squares if, and only if, all the primes up to x are squares. There are about $x/\log x$ of these primes, and if each prime is a square 'independently with probability 1/2' then, with 'probability at least $1/q$', all the primes up to $\log q \log\log q$ are squares. This 'implies' that P_q contains a clique of size $\log q \log\log q$ infinitely often, whereas a random colouring of K_q has no clique larger than $2\log_2 q$. Of course, this discussion is purely heuristic, but the statements can easily be justified if one assumes the Generalized Riemann Hypothesis, as Montgomery [69] proved. Moreover, Graham and Ringrose [55] have given an unconditional proof that P_q contains a clique of size $c \log q \log\log\log q$ infinitely often.

The previous paragraph is not fatal for the Paley graphs — it may be that they do give good colourings infinitely often, or even most of the time. But an analysis of cliques in Paley graphs requires estimates for character sums, and such estimates are notoriously difficult. From the character sum point of view, though, a more natural sequence than the sequence of all primes is the sequence of powers of a fixed prime. Perhaps the graphs P_q for $q = 5^{2k+1}$ offer good examples (we leave out those q that are squares, since the subfield of order $\sqrt{q}$ spans a complete subgraph). But the technical problems are severe.

More information about Paley colourings is given in §4.2.

2.5.2. Off-diagonal constructions. As for off-diagonal constructions, Erdős [24] described graphs showing that $r(3,t) = \Omega(t^{1+\varepsilon})$ and later [28] that $r(3,t) = \Omega(t^{1.13})$ for large t. This was improved by Cleve and Dagum [20] to $r(3,t) = \Omega(t^{1.26})$ and together with Chung [16] to $r(3,t) = \Omega(t^{1.29})$, and Alon [5] obtained $r(3,t) = \Omega(t^{4/3})$. The best bound to date was achieved by Alon [4], namely

$$r(3,t) = \Omega(t^{3/2}).$$

Another $\Omega(t^{3/2})$ construction is given by Codenotti, Pudlák and Resta [21]. The vertices of Alon's graph are the elements of the field of order 2^{3k}, where k is not divisible by 3. Two vertices are adjacent if their sum, represented as a binary vector, has the form $(w_0, w_0^3, w_0^5) + (w_1, w_1^3, w_1^5)$, where w_0 and w_1 are regarded as elements of the field of order 2^k, and are such that the leftmost bits of w_0^7 and w_1^7 are 0 and 1 respectively. The graph is a Cayley graph, and its properties are derived from certain BCH codes.

Until recently there was no constructive proof that $r(s,t) \geq t^2$ for any fixed s and all large t. But, developing the properties of the norm graphs of Kollár, Rónyai and Szabó [62], Alon and Pudlák [9] have now constructed examples to show that

$$r(s,t) \geq t^{c\sqrt{\log s/\log\log s}},$$

where c is an absolute constant. They begin with the norm graph $N(q,t)$ whose vertices are the elements of the field of order q^t, two vertices a and b being adjacent if $\|a+b\| = 1$, where the norm $\|x\|$ of x equals $x^{1+q+q^2+\cdots+q^{t-1}}$. The construction required is then the clique graph, whose vertices are the cliques of order $\lceil t/2 \rceil$ in $N(q,t)$, two cliques K and L being adjacent if there is an edge uv of $N(q,t)$ with $u \in K \setminus L$ and $v \in L \setminus K$. The verification of the properties of this graph involves rather deep methods.

3. Canonical Ramsey Theory

In §2.1 we defined an orderable colouring of K_s. We might define $\rho(s)$ to be the smallest n such that every two-colouring of K_n yields an orderable K_s. From the argument of §2.1 it can be seen that $\rho(s) \le r(s) \le \rho(2s-2)$, and thus the study of $\rho(s)$ is intimately bound up with that of $r(s)$.

Now the definition of orderability holds good even if we allow more than two colours. So for the purposes of this section we shall permit ourselves to wander from our brief of studying only the simplest case of Ramsey's theorem, and we shall think about more general situations.

Consider colourings of K_n in which an unrestricted number of colours may be used. Richer [80] has studied the parameter $CR(s,t)$, the smallest n such that any colouring of K_n yields either an orderable K_s or a *distinctly coloured* K_t, the latter meaning that the edges of K_t all have different colours. Richer proves that

$$\left(\binom{t}{2} - 2\right)^{s-2} + 1 \le CR(s,t) \le 3^{3-s} t^{4s-4}.$$

In particular, $CR(s,t) = 2^{\Theta(s \log t)}$. It is worth remarking, in the light of §2.3, that although a similar lower bound can be obtained probabilistically, the bound stated is proved deterministically, and the constants are better.

The definition of $CR(s,t)$ is related to, and is, of course, to a great extent motivated by, the canonical Ramsey theorem of Erdős and Rado [38]. In the graph case, this states that there is a number $ER(2,s)$ such that, if $n \ge ER(2,s)$ and the graph on vertex set $\{1,\ldots,n\}$ is coloured with any number of colours, there will be a K_s coloured in one of four specific ways: either it is monochromatic, or the colour of an edge is determined by its first vertex, or the colour of an edge is determined by its last vertex, or it is distinctly coloured. Note that, in the second and third cases, the colours must be distinct for each vertex, so the requirement is stronger than orderability.

Clearly $CR(s) \leq ER(2,s)$. The Erdős–Rado numbers $ER(2,s)$ have been studied by Lefmann and Rödl [64, 65]; they prove that

$$2^{c_1 s^2} \leq ER(2,s) \leq 2^{c_2 s^2 \log s}$$

for some constants c_1 and c_2.

In the general case of the Erdős–Rado theorem, where k-sets are coloured, there are 2^k canonical colourings rather than just four, namely, for each subset $S \subset \{1,\ldots,k\}$, a colouring in which the colour of an edge is determined by the $|S|$-subset of its vertices indexed by S. For the corresponding numbers $ER(k,s)$, Lefmann and Rödl [65] gave a lower bound which was a tower of twos of height $k-1$ topped by $c_k s^2$, and an upper bound which was a tower of height k. The gap was closed by Shelah [88], who showed that $ER(k,s)$ is at most a tower of $k-1$ twos topped by a polynomial in s.

It follows that the Erdős–Rado numbers are generally no bigger than the ordinary Ramsey numbers, in which l colours are used and we ask for a monochromatic s-subset, for it is known that these numbers too are bounded below by tower functions, which for $l \geq 4$ have height $k-1$ (Erdős and Rado [39], Erdős, Hajnal and Rado [35], Erdős and Hajnal [31, 32], Duffus, Leffman and Rödl [22]).

For two or three colours, however, the towers in the lower bounds have height only $k-2$ (for $k \geq 3$). The most basic outstanding problem is to decide whether the two-colour Ramsey number for triples ($k=3$) is singly or doubly exponential (see Erdős, Hajnal and Rado [35]). Erdős, Hajnal, Máté and Rado [34] are encylopædic on these matters.

But we have now strayed very far from the simplest case of Ramsey's theorem, and it is time to return to earth.

4. Random-like Colourings

The arguments in §2.1 for the upper bounds on $r(s)$ are tightest when the colouring has the property that every vertex has equal red and blue degree, and within each vertex neighbourhood the same holds, and so on.

Thus, in these colourings, red and blue edges appear to be spread around very uniformly, in a way that is typically found in a random colouring. Moreover, the best lower bounds for $r(s)$, described in §2.3, all come from random colourings.

For these reasons, many have felt that the extremal colourings for $r(s)$ will turn out to be randomlike in appearance. The argument for this is not entirely convincing — for example, random colourings themselves don't come close to optimality in the hypergraph case (see §3), in the multi-colour graph case (see Chung and Grinstead [55]) and in a natural extension of the two-colour graph case (see §5.2 below) — but nevertheless the argument retains a strong appeal.

4.1. Induced subgraphs

One property that random colourings of K_n enjoy is having a large number of non-isomorphic induced subgraphs. Indeed, they have exponentially many. Moreover, they are $c \log n$-*universal,* where k-universal means that every colouring of K_k is a subgraph. The second of these properties is enjoyed by pseudo-random colourings (in the sense studied in [98] and by Chung, Graham and Wilson in [17]) and probably the first is too.

These observations prompted a couple of explorations by Erdős and his co-authors. In one direction, Erdős and Hajnal [32] showed that, if $0 < c < 1/k$, then for $n > n_0(c,k)$ every two-colouring of K_n with no monochromatic K_s, $s = e^{c\sqrt{\log n}}$, is k-universal. Prömel and Rödl [75] then proved that, for any $c_1 > 0$, there is a $c_2 > 0$ such that every colouring with no monochromatic $K_{c_1 \log n}$ is $c_2 \log n$-universal.

To state some results in another direction, suppose we are given a colouring of K_n. Define s to be the order of a largest complete monochromatic subgraph and I to be the number of distinct (non-isomorphic) induced subgraphs. Erdős and Rényi [40] conjectured that, for every $c_1 > 0$, there exists $c_2 > 0$ such that if $s \le c_1 \log n$ then $I \ge 2^{c_2 n}$. In the same vein, at the conference held in Erdős's honour on his 75$^{\text{th}}$ birthday [11], Hajnal conjectured that if $I = o(n^2)$ then $s = \big(1 - o(1)\big)\, n$.

Alon and Bollobás [7] and Erdős and Hajnal [33] independently proved strong forms of Hajnal's conjecture. Erdős and Hajnal [33] show further that, for example, if $s \le n/k$, k fixed, then $I \ge n^{c\sqrt{k}}$. Alon and Hajnal [8] came close to proving the Erdős–Rényi conjecture by showing that $I \ge 2^{n/2s^{20 \log (2s)}}$. Finally, Shelah [89] proved the full conjecture.

4.2. Pseudo-random colourings

It is curious that, in the argument of §2.1, if we are unable to make a gain in just the first two steps, then overall we can display a huge gain. To make this precise, suppose that we have a colouring of K_n in which every red and blue vertex degree is at most $n/2$, and in which the red and blue neighbourhoods of each vertex have red and blue degrees at most $n/4$. Then, by [98, Theorem 1.1] or [99, Theorem 3.1], the colouring is $\left(1/2, \sqrt{n}\right)$*-jumbled,* where a (p, α)-jumbled colouring is one in which every subset of k vertices spans between $p\binom{k}{2} - \alpha k$ and $p\binom{k}{2} + \alpha k$ red edges. The properties of these colourings ([98, Theorem 1.1] or [99, Theorem 6.9]) then imply that there is a blue $K_u + \overline{K}_w$, provided $w = \left\lceil 2^{-u}n - 4\sqrt{n}\right\rceil \geq 1$. Clearly, if $w \geq r(s, s-u)$ then we have a monochromatic K_s. Taking $u = \left\lfloor (1/2)\log_2 n \right\rfloor - 3$ and using the Erdős–Szekeres bound for $r(s, s-u)$, we obtain $s \geq 0.64 \log_2 n$ for large n, whereas the argument of §2.1 gives only $s \geq 0.5 \log_2 n$.

To put it another way, if it were true that extremal colourings for the Ramsey numbers $r(s)$ had a uniform distribution of colours at the first two steps, as described, then it would be true that $r(s) < 3^s$. Paley graphs (see §2.5.1) give excellent examples of colourings with such a distribution; hence the Paley colouring P_q contains a monochromatic $K_{0.64 \log_2 q}$ and no Paley colouring can ever show $r(s) > 3^s$.

Of course, if we relax the uniformity condition very slightly we still get a better bound than that of Erdős–Szekeres. But the rate at which the gain achieved diminishes with the relaxation is disappointingly rapid, and so the above observations for uniform colourings, when combined with the argument of §2.1 for non-uniform colourings, fail to give an improvement on the upper bounds for $r(s)$ given in §2.2.

5. The Number of Monochromatic Subgraphs

It follows from Ramsey's theorem that a two-colouring of K_n contains many monochromatic K_s if n is large. Indeed, as Erdős [27] observed, a "crude estimate" shows there are at least $\binom{n}{s}\binom{r(s)}{s}^{-1}$ of them. In this section we shall consider just how many there must be.

Definition 5.1. Let K_n be two-coloured with red graph G. Then $k_s(G)$ denotes the number of monochromatic K_s in the colouring. Moreover, let

$$k_s = \lim_{n\to\infty} \min_{|G|=n} \frac{k_s(G)}{\binom{n}{s}}.$$

So k_s is the limiting minimum density of subgraphs of order s that are monochromatic, taken over all two-colourings. (The limit does indeed exist, since the minimum increases with n.)

5.1. Monochromatic triangles

Clearly $k_1 = k_2 = 1$. Let us consider $s = 3$. If K_n is two-coloured with red graph G, and if r_i, $1 \le i \le n$ are the red degrees and $b_i = n - 1 - r_i$ the blue degrees, then counting pairs of incident red and blue edges, and noting that a triangle contains exactly two such pairs if it is not monochromatic, we have

$$k_3(G) = \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i\, b_i.$$

So the number of monochromatic triangles depends only on the degree sequence, and it is minimized when the red and blue degrees are all the same; this can be achieved exactly if $n \equiv 1 \pmod 4$. Thus

$$k_3(G) \ge \frac{1}{24}n(n-1)(n-5) \quad \text{and} \quad k_3 = \frac{1}{4}.$$

This lower bound for $k_3(G)$ is due to Goodman [53]. Sauvé [85] and Lorden [66] offered "simple proofs". All three state the degree formula for $k_3(G)$ in one form or another (it is Goodman's Lemma 1). It is customary to credit the formula to either Sauvé or Lorden, but, in fact, it is mainly their minimization arguments that makes their proofs simple; Goodman, perhaps because his formulation was less transparent, makes rather heavy weather of the job. Lorden also cites Sauvé, though in another context (see §5.5).

It follows from the lower bound that some edge lies in at least $(n - 5)/4$ monochromatic triangles, so every two-colouring of K_{4k+2} contains a monochromatic $K_2 + \overline{K}_k$ (this is often tight; if $q = 4k + 1$ is a prime power, the Paley graph P_q contains no $K_2 + \overline{K}_k$). In general, if K_n contains no monochromatic K_s then $3k_3(G) \le \binom{n}{2}\big(r(s-2, s) - 1\big)$. Moreover, no set of

$r(s-1)$ vertices can be joined by red to one vertex and by blue to another, so $\sum_{i=1}^{n} r_i b_i \le n(n-1)\bigl(r(s-1)-1\bigr)$. In this way, the recurrences

$$r(s) \le 4r(s-2,s)+2 \quad \text{and} \quad r(s) \le r(s-2,s)+3r(s-1)-1$$

were derived by Walker [103] and by Huang [59]. As noted in §2.2, recurrences of this kind underlie the upper bounds for $r(s)$ described in that section.

5.2. Uncommon graphs

Goodman [53] suggested that his methods should extend to the minimization of the number of *"full and empty quadrilaterals, and figures of a higher number of sides"*. Erdős in [27] takes up this problem. He observes that $k_s \le 2^{1-\binom{s}{2}}$, since random colourings give this bound. After making the observation about probabilistic methods cited in §2.3, he remarks that it seems *"Goodman's problem will be much more difficult for $k>3$ than for $k=3$, since it does not seem easy to find graphs which give values as small"* as random examples. It is rather easier nowadays to find such examples, since any pseudo-random colouring, such as Paley colourings and others listed in [98], will do. The number of monochromatic K_4s in the Paley colouring P_q was, in fact, computed exactly by Evans, Pulham and Sheehan [44] for prime q, and for all q in [97], namely

$$k_4(P_q) = \frac{1}{32}\cdot\frac{1}{4!}q(q-1)\bigl\{\,(q-5)(q-17)+4(a^2-1)\bigr\}$$

where $q=a^2+b^2$, a is odd and is coprime to $q=p^l$ if $p\equiv 1 \pmod 4$. Franek and Rödl [46] showed that $k_4(G) \ge \bigl(1/32+o(1)\bigr)\binom{n}{4}$ holds also if G is a small perturbation of a pseudo-random graph.

Erdős suggests *"it seems likely that"* $k_s = 2^{1-\binom{s}{2}}$, and later in the paper he refers to this as a conjectured value for k_s. This conjecture accords with the feeling, referred to in §4, that extremal colourings for monochromatic complete graphs are random-like. For more than two colours, though, this is clearly not the case — for example, the density of monochromatic triangles in the 3-colouring described by a 'blow-up' of a two-coloured P_5 is only $1/25$. On the other hand, for the somewhat related problem of minimizing the number of transitive subtournaments in a large tournament, random tournaments are indeed best. For let t_s be the limiting minimum density of transitive subtournaments of order s; then for a tournament with outdegree and indegree sequences (d_i^+) and (d_i^-), we have

$$t_s\binom{n}{s} \geq \frac{1}{2}\sum_{i=1}^{n}\left[t_{s-1}\binom{d_i^+}{s-1} + t_{s-1}\binom{d_i^-}{s-1}\right] \geq (n+o(n))\frac{t_{s-1}}{2^{s-1}}\binom{n}{s-1},$$

giving $t_s \geq s!2^{-\binom{s}{2}}$, which is exactly the random density.

Burr and Rosta [15] took this notion a step further: they conjectured that, for every graph G, the minimum density of monochromatic copies of G in large two-coloured complete graphs would be attained by random colourings. This conjecture was a step too far. Sidorenko [90] showed that it failed if G is a triangle with a pendant edge; the density of G in the colouring consisting of a complete bipartite red graph, with just a faint random sprinkling of red edges within the two vertex classes, is less than 1/8. Later, Clark [19] gave examples of G for which the ratio of the minimum density to the random density is arbitrarily small.

In fact, Erdős's conjecture itself is false for all $s \geq 4$, as shown in [101] and described below in 5.3. This prompts the following slightly loose, but hopefully clear, definition.

Definition 5.2. A graph G is called *common* if the limiting minimum density of monochromatic copies of G in two-coloured complete graphs is attained by random colourings.

(Common graphs are called *randomness friendly* by Sidorenko [93], who gives an interesting study of this and related properties.) Various graphs are known to be common, such as cycles (Sidorenko [90]), tree-like structures built from triangles (Sidorenko [92]) and even-spoked wheels (Jagger, Šťovíček and Thomason [60]).

Every non-bipartite graph can be made uncommon by appending a large enough path ([60, Theorem 4]). On the other hand, it is thought that every bipartite graph G is common. In fact, this is a phenomenon of extremal graph theory, rather than Ramsey theory. Indeed, Erdős and Simonovits [41, 42], in their studies of supersaturation, conjecture that the graphs of given order and size and *"having roughly the minimum number of copies of G tend to look like random graphs"* [42, p. 205]. A related conjecture has been made by Sidorenko: to state it, we define a homomorphic copy of G in H to be a map $f : V(G) \to V(H)$ such that $f(a)f(b) \in E(H)$ whenever $ab \in E(G)$.

Conjecture 5.3 (Sidorenko [91]). *Let G be a bipartite graph and let H be a graph of order n and average degree pn. Then H contains at least $n^{|G|}p^{e(G)}$ homomorphic copies of G.*

The conjecture has been verified for complete bipartite graphs (Erdős and Moon [37]), trees (Sidorenko [92], see also [60]) and even-length cycles (Sidorenko [91], where a stronger form of his conjecture is made).

The most general result about uncommon graphs is found in [60], where it is shown that every graph G containing K_4 is uncommon. In this sense, almost every graph is uncommon. The proof is described in below. There is no known common graph with chromatic number greater than three; whether the five spoked wheel is common is the outstanding problem in this area.

5.3. Graph products

The original proof in [101], that the colourings given were counterexamples to Erdős's conjecture for $s \geq 4$, is not very transparent. Different examples for the case $s = 4$ were given by Franek and Rödl [47]. Šťovíček saw that the examples of [101] could be analyzed much more cleanly, and his thoughts form the basis of the arguments in [60], where it is proved that the same examples serve to show any graph containing K_4 is uncommon. In fact, it turns out to be possible to give a wide variety of simple examples, as noted in [102]. Here is a brief description.

Let G be a graph on the vertex set $\{1, \dots, s\}$. Let J be a coloured K_j. From J we can produce a colouring of K_{jm} by replacing each vertex $u \in V(J)$ by m copies of itself, colouring blue the edges between the different copies of u, and colouring the edges between these copies and the copies of $v \in V(J)$ the same colour as uv. The number of (labelled) monochromatic G in this colouring is at most $m^s h(J; G)$, where $h(J; G)$ is the number of homomorphic copies $f : V(G) \to V(J)$ of G in the red graph of J (see §5.2), plus the number of homomorphic copies in the blue graph (though in the blue case we allow $f(a) = f(b)$ if $ab \in E(G)$). The random colouring of K_{jm} has about $2^{1-e(G)} j^s m^s$ labelled monochromatic Gs. So we can show that G is uncommon by finding a single fixed J with $h(J; G) < 2^{1-e(G)}|J|^s$, where $|J| = j$, and by letting m grow large. But how do we find such a J?

We can compute $h(J; G)$ by associating with J the $j \times j$ matrix $A(J) = \big(a(u, v)\big)_{u,v \in V(J)}$, where $a(u, v) = -1$ if uv is a red edge, and $a(u, v) = 1$

otherwise; thus the diagonal entries of $A(J)$ are 1. It is readily verified that if

$$\varphi(J;F) = \frac{1}{|J|^s} \sum_{u_1 \in V(J)} \cdots \sum_{u_s \in V(J)} \prod_{ij \in E(F)} a(u_i, u_j)$$

then

$$h(J;G) = 2^{1-e(G)}|J|^s d(J;G) \quad \text{where} \quad d(J;G) = \sum_{F \subseteq G} \varphi(J;F),$$

the sum being over all spanning subgraphs F of G with an even number of edges. Our target has now become to find a J with $d(J;G) < 1$.

What makes these observations useful is that, for certain types of colourings J, it is easy to compute $d(J;G)$, because φ is *multiplicative:* that is,

$$\varphi(J_1 \otimes J_2; F) = \varphi(J_1;F)\varphi(J_2;F)$$

where $J_1 \otimes J_2$ is the product colouring of $K_{|J_1| \times |J_2|}$ on vertex set $V(J_1) \times V(J_2)$, defined by $A(J_1 \otimes J_2) = A(J_1) \otimes A(J_2)$. (Here, as usual, if $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{kl})_{k,l=1}^m$ then $(A \otimes B)_{(i,k)(j,l)} = a_{ij}b_{kl}$.)

Observe that, in this terminology, the m-fold replication of J described above is just the colouring $J \otimes N$, where N is the all-blue colouring of K_m. Since $\varphi(N;F) = 1$ for all F, we always have $d(J \otimes N; G) = d(J;G)$.

It turns out to be easy to find suitable J this way, just by trial and error, at least for small G. As an illustration, let $G = K_4$. The subgraphs of G of even size are the empty graph E, 12 paths P of length 2, 3 matchings M of size 2, 12 triangles with pendant edge T, 3 4-cycles C and one K_4. Now $\varphi(J;E) = 1$ for every J. Table 1 gives the values of $\varphi(J;F)$ for the three colourings $J = R_3$ (the red triangle), $J = M_4$ (two independent red edges in an otherwise blue K_4) and $J = R_4$ (the red K_4). The values in the table can easily be computed by hand.

J	P	M	T	C	K_4
R_3	0.1111	0.1111	−0.1852	0.4074	−0.4815
M_4	0.2500	0.2500	0.1250	0.2500	0.5000
R_4	0.2500	0.2500	−0.1250	0.2500	−0.5000
G_{18}	0.0000	0.0000	0.0000	0.1001	0.1975

Table 1. The values of $\varphi(J,F)$ for various graphs J and for $F \subseteq K_4$

It is immediate from the table that K_4 is uncommon. For consider the colouring $J = R_3^{\otimes k}$, meaning the product of k red triangles, on 3^k vertices. By the multiplicativity of φ we have

$$d(J;K_4) = 1^k+12\times(0.11)^k+3\times(0.11)^k+12\times(-0.18)^k+3\times(0.41)^k+(-0.48)^k$$

which is less than one if k is large and odd!

The colouring G_{18} in the table is the complement of $K_3^{\otimes 2} \otimes \overline{K_2}$; it has order 18 and is 9-regular. The colourings $R_4 \otimes M_4 \otimes G_{18} \otimes N$ have order $n = 288|N|$, are vertex-transitive and are $n/2$ regular. Now $d(R_4 \otimes M_4 \otimes G_{18}; K_4) = 0.9693 < 32/33$. Therefore

$$\frac{1}{46} < k_4 < \frac{1}{33}$$

where the lower bound comes from the ingenious work of Giraud [52].

The drawback of this approach is that, when s is large, it is hard to compute the sum $\varphi(J;F)$ even for small graphs J. However, the colourings M_4 and R_4 of order 4 have algebraic formulations, and it is by exploiting this fact that the values of $\varphi(M_4;F)$ and $\varphi(R_4;F)$ were computed in [60]. Hence the values of $d(T_k^-;G)$ can be estimated, where $T_k^- = R_4 \otimes M_4^{\otimes(k-1)}$, and so it is shown that every graph G containing K_4 is uncommon. The colourings T_k^-, whose red graphs are the strongly regular orthogonal towers, provided the original counterexamples to Erdős's conjecture in [101].

As mentioned in [101], the colourings described there also give counterexamples to the case $s = 3$ of a conjecture mentioned in §2.4, namely, that if $n \geq 8k + 10$ then a two-colouring of K_n contains a monochromatic $K_3 + \overline{K}_k$. This conjecture was made independently by Sheehan [87], whose guess was inaccurate though his title was not. In fact, if $k = 4^\ell + 1$ then there are colourings of K_n with $n = 8k+2\sqrt{k-1}-8$ with no monochromatic $K_3 + \overline{K}_k$.

It is fairly easy to see that, if it could be shown that k_s is very much smaller than $2^{1-\binom{s}{2}}$, then the known lower bounds for $r(s)$ could be improved. Conversely, Rödl has arguments to show that good lower bounds on k_s would improve the known upper bounds for $r(s)$. Unfortunately, the upper bounds for k_s given by the above methods are useless for this purpose, being never smaller than $0.75 \times 2^{1-\binom{s}{2}}$, and if $s \geq 5$ no lower bound for k_s is known other than Erdős's "crude estimate" stated at the beginning of §5.

5.4. Bounds on Ramsey numbers

In an interesting piece of work, Székely [96] gives some tantalizing relationships between the number of monochromatic subgraphs and the Ramsey numbers $R(s,t)$ themselves. Let $G(n)$ be the minimum number of monochromatic subgraphs in a two-colouring of K_n; here, all monochromatic subgraphs are counted, regardless of order. Székely gives the following absolute bounds on $G(n)$, for large n:

$$n^{0.2619 \log n} \leq G(n) \leq n^{0.7214 \log n}.$$

The upper bound comes from a random colouring ($0.7214 = (2\log 2)^{-1}$) but the lower bound requires a bit of work (in fact, Székely obtains 0.2275 in the exponent but a slightly simpler argument, using the average instead of the minimum degree, gives a slightly better constant).

The function $G(n)$ can also be bounded in terms of Ramsey numbers, independently of the above absolute bounds. Let $r()$ be an inverse function to $R()$; that is, $r(n)$ is the maximum value of t such that $R(t) \leq n$. Then Székely shows the following, where $s = r(n)+1$:

$$n^{(1/2+o(1))r(\sqrt{n})} \leq G(n) \leq \frac{1}{(s-2)!} R(s,2)R(s,3)\ldots R(s,s-2)R(s)^2.$$

These bounds come from straightforward counting arguments.

Substantial improvements in any of the four above inequalities would give improved bounds on $R(s)$. For example, reducing the bound $G(n) < n^{0.7214 \log n}$ would increase the bound $R(s) > 2^{s/2}$, both of which bounds come from random colourings. Less obvious is the fact that, as shown in [96], if $G(n) > n^{0.4612 \log n}$ holds then $R(s) < (4-\delta)^s$ holds for some absolute $\delta > 0$.

Székely makes the conjecture that, in an extremal colouring for $R(s)$, most monochromatic subgraphs have around $r(n)$ vertices. More precisely, he conjectures that given $\varepsilon > 0$, if $n = R(s)-1$ is large and G is the red subgraph of a two-colouring of K_n with no monochromatic K_s, then $\sum_{t<(1-\varepsilon)s} k_t(G) \leq \varepsilon \sum_t k_t(G)$. Note that if this conjecture were true then the extremal colourings would not be truly random. The conjecture has an even stronger and more unexpected consequence. It implies that $G(n) \geq n^{(c/2)\log n}$, provided $r(n) > (c+\varepsilon)\log n$ for large n. The absolute upper bound on $G(n)$ then gives $c \leq 1/\log 2$; that is, $R(s) > (2-\varepsilon)^s$ infinitely often.

5.5. Blue-empty colourings

We finish with a look at a related problem, considered by Erdős, but for some reason overlooked until recently. Goodman [53] asked what is the minimum number of red triangles in a colouring of K_n that contains no blue triangles? If n is even then this minimum is the same as the minimum number of monochromatic triangles in any colouring — the blue complete bipartite colouring achieves this. But, as Erdős proved in a letter to Sauvé [85], the two minima differ if $n > 7$ is odd.

The exact minimum number of red triangles in a blue-free colouring was found by Lorden [66]. He showed, by a succinct argument based on the degree formula in §5.1, that for odd $n > 9$ the minimum is attained by the colouring whose blue edges form the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.

Erdős [27] looked at a generalization of this problem. He defined $f(n,k,l)$ to be the minimum number of monochromatic K_ks in a red-blue colouring of K_n containing no blue K_l. Prompted by Lorden's result, he stated that

$$\text{“\textit{perhaps}”} \quad f(n,k,l) = \sum_{i=0}^{l-2} \binom{\lfloor \frac{n+i}{l-1} \rfloor}{k} \quad \text{if} \quad n > n_0(k,l).$$

That is to say, if n is large, perhaps the minimum is attained when the blue graph is the $(l-1)$-partite Turán graph of order n. Goodman's remark and Lorden's theorem confirm the suggestion about $f(n,3,3)$. Erdős stated *“the simplest case which I can not do is* $f(3n,3,4) = 3\binom{n}{3}$”: this case is mentioned by Bollobás [14, Problem 11, p. 361].

Nikiforov [73] has now shown that Erdős's suggestion is, in general, incorrect. Indeed, let $c_{k,l} = \lim_{n\to\infty} f(n,k,l)\binom{n}{k}^{-1}$. It is easily shown that the limit exists, and if Erdős's suggestion were true then $c_{k,l} = (l-1)^{-(k-1)}$ would hold. But Nikiforov has proved that, for $k \geq 3$ and $l \geq 3$,

$$c_{k,l} \leq \frac{(l-1)2^{k-1} - l + 2}{\big(r(3,l) - 1\big)^{k-1}}$$

holds, which is less than $(l-1)^{-(k-1)}$ for all but finitely many pairs k and l, in view of the lower bounds on $r(3,l)$ given by Erdős and by Kim (see §2.3). For $k = 4$ and $l = 3$ Nikiforov's inequality is $c_{4,3} \leq 3/25$; he has shown [74] that if this inequality is strict then the extremal colourings cannot be regular.

The colourings that Nikiforov uses to establish this bound are similar to those of §5.3. Indeed, let J be a colouring with $|J| = r(3,l) - 1$ having no red K_l and no blue triangle. Then the colouring $J \otimes N$, where N is large, has no red K_l and few blue K_ks. Reversing the colours gives the required examples.

For the case $l = 4$ Nikiforov gives the better bound

$$c_{k,4} \leq \frac{4 \cdot 3^k - 8 \cdot 2^k + 5}{17^{k-1}}.$$

This comes via the colourings $P_{17} \otimes N$, where P_{17} is the Paley colouring of K_{17} that is the unique extremal colouring for $r(4)$.

The case highlighted by Erdős, that perhaps $c_{3,4} = \frac{1}{9}$, remains open.

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