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PSEUDO-RANDOM GRAPHS

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We call a graph G (p, α)-*jumbled* if, for every induced subgraph H of G, $\left|e(H)-p\binom{|H|}{2}\right| \leqslant \alpha|H|$ holds; here p and α are real numbers with $0<p<1\leqslant\alpha$, and $e(H)$ is the number of edges in H. We show that a (p, α)-jumbled graph behaves in many ways like a random graph with edge probability p, and some aspects of this similarity are examined.

Since its introduction by Erdös and Rényi, the theory of random graphs has been greatly developed (for a modern treatment of the subject, see Bollobás [2]), and many properties of a random graph have been studied in detail. Random graphs have proved of great interest for various reasons, of which we mention just two: they provide the best known extremal graphs for several extremal problems, such as subcontractions [23], Zarankiewicz's problem [19] and Ramsey's theorem [13], and they offer examples of graphs with certain properties, giving us say expanders [11], graphs of small diameter [9] and parallel sorting algorithms [7]. In all these cases it would be useful to have a criterion by which to decide whether a *specific* graph behaves like a random graph, that is, has the property (of almost all graphs) that interests us. Such a criterion might also be used to describe the class of extremal graphs in the problems mentioned, which may perhaps give useful information about the problems themselves.

The purpose of this paper is to offer such a criterion and to explore some of its consequences. As a result we are able to extend and simplify many earlier results. For instance, results obtained by Bollobás and Thomason [6] using the Riemann hypothesis for algebraic curves over finite fields, by Alon and Milman [1] using eigenvalue methods, and by Gurevich and Shelah [18] using random graph techniques, can be obtained in more general settings by elementary arguments. In the latter two cases, these are described elsewhere in more detail [25], [26].

A graph G is said to be *(p, α)-jumbled* if p, α are real numbers satisfying $0<p<1\leqslant\alpha$ and if every induced subgraph H of G satisfies

$$\left|e(H)-p\binom{|H|}{2}\right|\leqslant\alpha|H|.$$

Here $e(H)$ is the number of edges in H, following [2]. Equivalently, if $d(H)$ is the average degree inside H we may say

$$|d(H)-p(|H|-1)|\leqslant 2\alpha$$

holds for every induced subgraph H. We think of a (p, α)-jumbled graph as behaving somewhat like a random graph where each edge is chosen with probability p. Of course, it is possible to suggest other definitions for a "pseudo-random" graph but it seems they often reduce to this one. Note that if G is (p, α)-jumbled then every induced subgraph is (p, α)-jumbled and the complement of G is $(1-p, \alpha)$-jumbled. Observe too that the clique number of G is at most $1+2\alpha(1-p)^{-1}$ and the independence number is at most $1+2\alpha p^{-1}$.

Naturally every graph of order n is $(p, n/2)$-jumbled, so the definition begins to be interesting when α is small compared to n. Although we require only $\alpha\geqslant 1$, a theorem of Erdös and Spencer [16] shows that α is at least of order $(pn)^{1/2}$. (Their proof is stated only for $p=\frac{1}{2}$ but the extension to other values of p with $pn\to\infty$ and $(1-p)n\to\infty$ is straightforward.) Our results will be stated for all values of p and α but they are best understood by thinking of α of order $(pn)^{1/2}$, and $p>n^{-1/3}$; the latter since many results require $\delta(G)\sim pn$ and $p\delta>\alpha$.

In this paper we show ways in which jumbled graphs behave like random graphs, and illustrate ways in which specific graphs may be shown to have random behaviour. The paper falls into three parts. First we give examples of (p, α)-jumbled graphs with small α; these are typical of the "explicit random graphs" we have in mind. Then we describe two ways of testing whether a graph is jumbled. The first is more or less a degree condition, and can be applied to a specific graph very easily. The second is a global condition. It tells us that if a graph G is *not* (p, α)-jumbled it contains a subgraph H with $\left|e(H)-p\binom{|H|}{2}\right|>\alpha|H|$ and $|H|$ *large*. To this extent it gives information about the extremal graphs in the problems cited earlier. Finally, we derive several properties of jumbled graphs. These properties are chosen mainly because they have been studied frequently in the random graph case and/or because they illustrate the techniques used in dealing with jumbled graphs.

Examples

Here are some examples of jumbled graphs. Some of them exhibit various kinds of pathological behaviour and most will be used later to illustrate particular points and to test the strength of theorems.

(1) Let $G \in \mathcal{G}(n, p)$, that is, the edges of G are chosen at random with probability p. Then G is almost surely $(p, 2(pn)^{1/2})$-jumbled (provided $pn \to \infty$ and $(1-p)n \to \infty$. The constant 2 here is generous).

(2) Choose a graph in $\mathcal{G}(n, p)$, select a subset X of the vertices, with $|X| = \lfloor (pn)^{1/2} \rfloor$, and join each pair of vertices in X. Then G is almost surely $(p, 3(pn)^{1/2})$-jumbled.

(3) As in (2), but with $|X| = \lfloor (pn)^{3/4} \rfloor$. Then G is almost surely $(p, (pn)^{3/4})$-jumbled.

(4) Let $V(G) = \{x_1, \ldots, x_{n/2}, y_1, \ldots, y_{n/2}\}$, where n is an even integer. For each pair $i<j$, join x_j to exactly one of x_i and y_i, chosen at random. Do the same for y_j. The graphs spanned by the x_*'s and the y_*'s are therefore randomly chosen elements of $\mathcal{G}(n/2, \frac{1}{2})$. G itself is $(\frac{1}{2}, n^{1/2})$-jumbled.

(5) Let $V(G)$ be as in (4). For each pair $i<j$, insert at random one of the four paths $x_i x_j y_i$, $x_i y_j y_i$, $x_j x_i y_j$, $x_j y_i y_j$. Add also the edges $x_i y_i$. Then G is $(\frac{1}{2}, n^{1/2})$-jumbled.

(6) Let $k \geqslant 2$ be an integer, let $n = 2kl+1$ be a prime power, and let F_n be the field of order n. Construct a graph G of order n by setting $V(G) = \mathsf{F}_n$ and joining x to y if $x-y$ is a kth power in F_n. If $k=2$ the graph we get, often called the Paley graph of order n, is $(\frac{1}{2}, n^{1/2})$-jumbled. If $k>2$ the graph is $(k^{-1}, 2n^{3/4})$-jumbled. The justification for these assertions will be provided later by Theorem 1.1. (The appropriate values of μ required by the conditions of Theorem 1.1 can be taken as $\mu = \frac{1}{4}$ for $k=2$ and $\mu = n^{1/2}$ for $k>2$. These values can be verified using the elementary theory of characters over finite fields.)

(7) As in (6), but join x to y if $x+y$ is a kth power. Again we get a $(k^{-1}, 2n^{3/4})$-jumbled graph, or $(\frac{1}{2}, 2n^{1/2})$-jumbled if $k=2$. This construction is more natural than that of (6), except when $k=2$, and has an obvious generalisation to hypergraphs. Since we do not need -1 to be a kth power this time, it is enough if $n = kl+1$.

(8) Let q be a prime power and let $V(G)$ be the elements of a vector space of dimension two over F_q; so G has $n = q^2$ vertices. Partition the $q+1$ lines of the space into two sets P and N, where $|P| = k = p(q+1)$, $1 \leqslant k \leqslant q$, and $|N| = q+1-k$. Join x to y if $x-y$ is parallel to a line in P. Then G is $(p, n^{3/4})$-jumbled. This example (when $p = \frac{1}{2}$) is due to Delsarte and Goethals and to Turyn (see [22]). G is in

fact strongly regular with parameters $(k(q-1), (k-1)(k-2)+q-2, k(k-1))$. If $p=\frac{1}{2}$ we can choose the set P so as to obtain the Paley graph of order n.

(9) Let q be a prime and let $V(G)$ be the set F_q. Let t be an integer, $1<t<q$. Join x to y if the fractional part of $(x-y)^2/q$ is at most t/q. By Theorem XIII. 16 of [2] and Theorem 1.1 below, this graph is $(p, n^{3/4}\log n)$-jumbled where $p=t/q$ and $n=q$. An infinite analogue of this graph was shown to have analogous properties by Pinch [21].

(10) Let q be a prime power and let $V(G)$ be the vertices of the projective geometry of dimension k over F_n. Then G has $n=(q^{k+1}-1)(q-1)^{-1}$ elements. Join $\underline{x}=x_0:\ldots:x_k$ to $\underline{y}=y_0:\ldots:y_k$ if $x_0y_0+\ldots+x_ky_k=0$. Then (again by Theorem 1.1) G is $(1/q, 2(n/q)^{1/2})$-jumbled. This graph is sometimes called the Erdös-Rényi graph in the case $k=2$.

(11) The graph of example (10) may be viewed, when $q=2$, as formed by taking as vertices the non-empty subsets of a set of order $k+1$, two vertices being joined if their intersection has even order. Two subgraphs of this graph are particularly interesting. For the first, let G be the graph whose vertices are the non-empty *even* subsets of a set of order $k+1$, where k is even. Join two vertices if their intersection is also even. Then G has order $n=2^k-1$ and is $(\frac{1}{2}, 2n^{1/2})$-jumbled. The reason for choosing k even is just that the graph is fractionally easier to analyse if we do. In fact in this case G is strongly regular with parameters $((n-3)/2, (n-11)/4, (n-3)/4)$.

(12) For the second subgraph, let G be the graph whose vertices are the *odd* subsets of a set of order $k+1$, two vertices being adjacent if their intersection is even. Then G has order $n=2^k$ and is $(\frac{1}{2}, 2n^{1/2})$-jumbled.

(13) Let H be a (p, α)-jumbled graph of order m, and let $k\geqslant 1$ be an integer. For each vertex $x\in H$, let $x_1, \ldots, x_k$ be vertices of the graph G. Join x_i to y_j in G whenever $xy\in H$ and $1\leqslant i, j\leqslant k$. Then G has order km and is $(p, k\alpha+k)$-jumbled.

There are other specific examples of jumbled graphs with less dense edge sets. For instance, in the graph of example (10) we may require more equations to be satisfied before we join two vertices. We do not concentrate on these examples since, as we mentioned earlier, our results are most effective if p is not too small.

§1. Conditions implying a graph is jumbled

We begin by considering two ways of checking whether a given graph is (p, α)-jumbled. The first, Theorem 1.1, is a local approach, the second, Theorem 1.4, is a global approach. We shall make use of the notation $B(x)$, where x is a nonnegative integer, to signify any real number y of absolute value at most x. Thus $y=B(x)$ means $|y|\leqslant x$, and $0\leqslant z\leqslant x$ implies $B(z)=B(x)$. In this sense the notation behaves like Landau's $O(x)$ notation. Using this, we may rewrite the definition of a (p, α)-jumbled graph G in the form

$$e(H)=p\binom{|H|}{2}+B(\alpha|H|)$$

for all induced $H\subset G$.

Theorem 1.1. *Let n be an integer, and let $0<p<1$ and $\mu\geqslant 0$ be real numbers. Let G be a graph of order n with minimum degree pn in which no two vertices have more than $p^2n+\mu$ common neighbours. Then G is (p, α)-jumbled, where $2\alpha=(pn+(n-1)\mu)^{\frac{1}{2}}+p$.*

Proof. Let H be a subgraph of G of order $k\leqslant n$, and let the average degree in H be d. Let $a_1, \ldots, a_k$ be the degree sequence of H, and let $b_1, \ldots, b_{n-k}$ be the numbers of edges between H and each of the $n-k$ vertices of $G-H$. Then

$$\sum_{i=1}^{k} a_i=kd$$

and

$$\sum_{i=1}^{n-k} b_i\geqslant\sum_{i=1}^{k}(pn-a_i)=k(pn-d).$$

Moreover, since no two vertices have more than $p^2n+\mu$ common neighbours, we have

$$\sum_{i=1}^{k}\binom{a_i}{2}+\sum_{i=1}^{n-k}\binom{b_i}{2}\leqslant\binom{k}{2}(p^2n+\mu),$$

so

$$k\binom{d}{2}+(n-k)\binom{k(pn-d)/(n-k)}{2}\leqslant\binom{k}{2}(p^2n+\mu).$$

Rearranging gives

$$(d-pk)^2\leqslant\frac{n-k}{n}[(k-1)\mu+np(1-p)],$$

which is somewhat stronger than the result claimed. □

Obviously, many of our examples (such as (6)–(12)) of concrete random graphs are shown by this theorem to be $(p, c(pn)^{1/2})$-jumbled, for appropriate p and constant c. For random graphs themselves though this theorem isn't so effective, since in this case μ is of order $n^{1/2}$.

It is curious to note that the minimum degree condition in Theorem 1.1 can be dropped if we require every pair of vertices to have $p^2n(1+o(1))$ common neighbours (here we imagine $n\to\infty$). For let v be a vertex of G with degree d. Then each vertex in $G-v$ has $p^2n(1+o(1))$ neighbours in $\Gamma(v)$, so the number of paths of length 2 in $G-v$ with both ends in $\Gamma(v)$ is $(n-1)\binom{p^2n}{2}(1+o(1))$. But this number is $\binom{d}{2}p^2n(1+o(1))$, and we have the minimum degree condition back again.

However, if we weaken the conditions of the theorem to require only every *edge* to be in at most $p^2n+\mu$ triangles, the conclusion fails to hold. Suppose for instance we are given a vertex-transitive pk-regular graph F of order k in which every edge is in p^2k triangles. (We will construct such a graph shortly.) Now for each $v\in F$ take a set $X(v)$ of order m, where m is some integer, and form a graph G with vertex set $\bigcup_{v\in F} X(v)$ with $xy\in G$ if $x\in X(u)$, $y\in X(v)$ and $uv\in F$. Then G has order $n=mk$, is vertex-transitive and pn-regular, and every edge of G is in precisely p^2n triangles. But G is not (p,α)-jumbled for any small value of α since its independence number is at least n/k.

An example of a graph F of the type described can be constructed as follows in the case $p=1/6$ and $k=36$. First take a K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge colour it with colours 0, 1 and 2. Now construct four 9-cycles C^i, $1\leqslant i\leqslant 4$, with vertex sets $\{a^i_j;\ 1\leqslant j\leqslant 9\}$ where a^i_j is joined to $a^i_{j\pm1}$. Construct H from the union of these four cycles by joining a^i_t to a^j_t if $t\equiv c \pmod 3$, where c is the colour of v_iv_j. Then H has order 36, is 3-regular, vertex transitive and has girth 7. Form F from H by setting $V(F)=V(H)$ and joining u to v in F if $d_H(u,v)=2$. Then F is 6-regular, vertex-transitive and every edge is in exactly one triangle.

In contrast to Theorem 1.1 we can show a graph is jumbled if we have only large scale information about it, namely, when we are given the number of edges in subgraphs of some large fixed order.

Lemma 1.2. *Let p, η, m, n be positive real numbers with $0<p, \eta<1$, such that ηn is an integer with $2\leqslant\eta n\leqslant n-2$. Let G be a graph of order n in which for every induced subgraph H of order ηn,*

$$\left|e(H)-p\binom{\eta n}{2}\right|\leqslant m$$

holds. Then

$$\left|e(H)-p\binom{k}{2}\right|\leqslant 80\, m\eta^{-2}(1-\eta)^{-2}$$

for every induced subgraph H of order k.

Proof. Let H be a subgraph of order $k\geqslant\eta n$. If we count the number of edges in each of the $\binom{k}{l}$ subgraphs L of H of order $l=\eta n$, we get

$$e(H)=\binom{k-2}{l-2}^{-1}\sum_{L\subset H}e(L)=\binom{k-2}{l-2}^{-1}\sum_{L\subset H}\left(p\binom{l}{2}+B(m)\right)$$

$$=p\binom{k}{2}+\frac{k(k-1)}{l(l-1)}B(m)=p\binom{k}{2}+B\left(\frac{2k^2m}{l^2}\right)$$

$$=p\binom{k}{2}+B\left(\frac{80m}{\eta^2(1-\eta)^2}\right),\quad \text{as claimed}.$$

Now suppose H is a subgraph of order $k\leqslant\min\{(1-\eta)n, \eta n\}$. Let F be a subgraph of $G-H$ of order ηn, and let L be a subgraph of H of order l, where $1\leqslant l\leqslant k$. Then by the above,

$$e(H\cup F)=p\binom{k+\eta n}{2}+B\left(\frac{2(k+\eta n)^2}{(\eta n)^2}m\right) \tag{a}$$

and

$$e(L\cup F)=p\binom{l+\eta n}{2}+B\left(\frac{2(l+\eta n)^2}{(\eta n)^2}m\right). \tag{b}$$

Holding l fixed and summing over all $\binom{k}{l}$ subgraphs L, we have

$$\sum_L e(L\cup F)=\sum_L\{e(L)+e(L,F)+e(F)\}$$

$$=\binom{k-2}{l-2}e(H)+\binom{k-1}{l-1}e(H,F)+\binom{k}{l}e(F),$$

and dividing by $\binom{k}{l}$ and using (b) gives

$$\frac{l(l-1)}{k(k-1)}e(H)+\frac{l}{k}e(H,F)+e(F)$$

$$=p\binom{l+\eta n}{2}+B\left(\frac{2(l+\eta n)^2}{(\eta n)^2}m\right).$$

By means of (a) and the equation $e(F)=p\binom{\eta n}{2}+B(m)$ we can solve for $e(H)$ to find

$$e(H)=p\binom{k}{2}+B\left[\frac{k-1}{k-l}2m\left(\frac{(k+\eta n)^2}{(\eta n)^2}+\frac{k}{l}\frac{(l+\eta n)^2}{(\eta n)^2}+\frac{k}{2l}+\frac{1}{2}\right)\right].$$

Writing $l=\lambda k$, and since $k\leqslant\eta n$,

$$e(H)=p\binom{k}{2}+B\left(\frac{2m}{1-\lambda}\left(4+\frac{(1+\lambda)^2}{\lambda}+\frac{1}{2\lambda}+\frac{1}{2}\right)\right).$$

Choosing l so that $\frac{1}{3}\leqslant\lambda\leqslant\frac{1}{2}$, which we can if $k\geqslant 2$, we have

$$e(H)=p\binom{k}{2}+B(40m)=p\binom{k}{2}+B\left(\frac{80m}{\eta^2(1-\eta)^2}\right).$$

Finally, suppose $(1-\eta)n\leqslant k\leqslant\eta n$ (this of course happens only if $\eta\geqslant\frac{1}{2}$). Using the result of the previous paragraph we may sum the number of edges in all subgraphs L of order $l=(1-\eta)n$ to get

$$\binom{k-2}{l-2}e(H)=\binom{k}{l}\left[p\binom{l}{2}+B(40m)\right]$$

so

$$e(H)=p\binom{k}{2}+\frac{k(k-1)}{l(l-1)}B(40m)$$

$$=p\binom{k}{2}+B\left(\frac{80m}{\eta^2(1-\eta)^2}\right). \qquad \square$$

Obviously the constant 80 in this lemma could be reduced. However if η is small the argument in the first part of the proof shows that $O(m\eta^{-2})$ is the right bound for the error, and if η is large, an examination of the graph $2K_{n/2}$ shows that $O(m(1-\eta)^{-2})$ is the right order (take $k=n/2$, say). Hence the bound $O(m\eta^{-2}(1-\eta)^{-2})$ cannot in general be improved.

An example of a direct application of Lemma 1.2 is as follows. Suppose we have a sequence $G_1, G_2, \ldots$ where G_n is a graph of order n such that every induced subgraph H of G_n of order $n/2$ satisfies

$$\left|e(H)-p\binom{n/2}{2}\right|\leqslant n^{3/2}.$$

For instance, G_n may be the graph of example (3). Then by Lemma 1.2 we can say G_n is $(p, 13n^{3/4})$-jumbled, since

$$13n^{3/4}k\geqslant\min\left\{320n^{3/2}, \binom{k}{2}\right\}$$

for all k. Moreover the graphs in this example show that G_n is not (p, n^{β})-jumbled for any $\beta<3/4$, so this result is best possible. However, in this example the subgraphs with large error are fairly localised, and most subgraphs have smaller error. In fact we shall now show (Theorem 1.4) that if $\omega(n)\to\infty$ is a function of n then the given conditions on G_n imply that G_n contains a subgraph G_n^* of order $n(1+o(1))$ which is $(p, \omega n^{1/2})$-jumbled, so that the errors in G_n^* are much smaller.

Before proving Theorem 1.4 we need an analogue of Lemma 1.2 for multipartite subgraphs.

Lemma 1.3. *Let $r\geqslant 2$ be an integer and let p, η, m, n and G be as in the statement of Lemma 1.2. Let H be an induced r-partite subgraph of G whose vertex classes have orders $k_1, k_2, \ldots, k_r$. Then*

$$\left|e(H)-p\sum_{1\leqslant i<j\leqslant r}k_ik_j\right|\leqslant 360m\eta^{-2}(1-\eta)^{-2}.$$

Proof. Choose an integer l with $r/3\leqslant l\leqslant r/2$. There are $\binom{r}{l}$ partitions of the vertex classes into two groups, one with l classes and the other with $r-l$ classes. Each edge of H joins classes in different groups for $2\binom{r-2}{l-1}$ of these partitions. Consider some fixed partition, and let H_1 be the subgraph formed by the classes in the first group. Put $h=|H_1|$ and $H_2=H-H_1$. By Lemma 1.2

$$e(H_1)=p\binom{h}{2}+B(2A)$$

and

$$e(H_2)=p\binom{k-h}{2}+B(2A)$$

where $k=|H|$ and $A=40\,m\eta^2(1-\eta)^2$. Hence $e(H_1, H_2)=ph(k-h)+B(4A)$. Summing over all partitions gives

$$2\binom{r-2}{l-1}e(H)=2\binom{r-2}{l-1}p\sum_{i<j}k_i k_j+\binom{r}{l}B(4A)$$

so

$$e(H)=p\sum_{i<j}k_i k_j+\frac{r(r-1)}{2(r-l)\,l}B(4A)$$
$$=p\sum_{i<j}k_i k_j+B(9A). \quad \square$$

Theorem 1.4. *Let p, η, α, n, ω be positive real numbers with $0<p, \eta<1$ such that ηn is an integer with $2\leqslant\eta n\leqslant n-2$. Let G be a graph of order n in which for every induced subgraph H of order ηn,*

$$\left|e(H)-p\binom{\eta n}{2}\right|\leqslant\eta n\alpha$$

holds. Then G contains a subgraph G^* of order at least

$$\left(1-\frac{880}{\eta(1-\eta)^2\omega}\right)n$$

which is $(p, \omega\alpha)$-jumbled.

Proof. We first construct G_0 by repeatedly removing "dense" subgraphs H_1, H_2, ..., H_r such that $e(H_i)-p\binom{k_i}{2}>k_i\omega\alpha$, where $|H_i|=k_i$ and $H_j\subset G-\bigcup_{i<j}H_i$. Stop when it is no longer possible to choose another H_*, and let $G_0=G-\bigcup_{i=1}^{r}H_i$.

Let $H=\bigcup_{i=1}^{r} H_i$ and $k=|H|=\sum_{i=1}^{r} k_i$. By Lemma 1.2 $e(H)\leqslant p\binom{k}{2}+2A$, where $A=40\eta^{-1}(1-\eta)^{-2}n\alpha$, and by Lemma 1.3

$$e(H)-\sum_{i=1}^{r} e(H_i)\geqslant p\sum k_i k_j-9A,$$

so

$$\sum_{i=1}^{r} e(H_i)\leqslant \sum_{i=1}^{r} p\binom{k_i}{2}+11A,$$

giving $\sum_{i=1}^{r} k_i\omega\alpha\leqslant 11A$ and $k\leqslant 11A/\omega\alpha$. Now construct G^* by removing from G_0 'sparse' subgraphs $F_1, \ldots, F_s$ such that $e(F_i)-p\binom{f_i}{2}<-f_i\omega\alpha$, where $f_i=|F_i|$. By a similar argument, $|G_0-G^*|<11A/\omega\alpha$. Thus $|G-G^*|<22A/\omega\alpha$ as asserted. □

There are a couple of ways in which one might wish to weaken the conditions of Theorem 1.4 but they fail to give the desired conclusion. The first is to require $\left|e(H)-p\binom{\eta n}{2}\right|\leqslant\eta n\alpha$ holds not for all H of order ηn but for almost all. However, this is a requirement satisfied by a complete bipartite graph $K_{n/2,\, n/2}$ (assuming $p=\frac{1}{2}$) and so it cannot be strong enough. The second way to weaken the conditions is to ask only that the root mean square value of $\left|e(H)-p\binom{\eta n}{2}\right|$ be small. This too is inadequate, since the r.m.s. value depends only on the number of pairs of edges and this in turn depends only on the degree sequence. Indeed if G has order n and has $E=p\binom{n}{2}$ edges, with degree sequence $(p(n-1)+\varepsilon_i)_{i=1}^{n}$, then looking at all subgraphs H of order k one finds

$$\binom{n}{k}^{-1}\sum_{H}\left(e(H)-p\binom{k}{2}\right)^2$$
$$=\frac{1}{2}p(n-k)\binom{k}{2}\binom{n-2}{2}^{-1}\left\{(1-p)(n-k-1)+\frac{k-2}{E}\sum_{i=1}^{n}\varepsilon_i^2\right\}.$$

Once again this condition fails to discriminate against bipartite graphs.

Finally, suppose we take a typical jumbled graph, say a $(\frac{1}{2}, n^{1/2})$-jumbled graph G of order n. One might ask, in the spirit of Theorem 1.4, whether G contains a subgraph G^* of order almost n in which the errors in subgraphs of order k are small, provided k is small (say $k<n^{\frac{1}{2}}$; of course in G we have no information about such small subgraphs). The answer is no. Consider say the graph of example (6) when $k=2$ and n is a square. The $\frac{1}{2}\binom{n}{2}$ edges are covered by several complete subgraphs of order $\sqrt{n}$, each edge being in the same number of these. Thus if $|G^*|=(1-\varepsilon)n$, one of these complete subgraphs meets G^* in at least $(1-2\varepsilon)\sqrt{n}$ vertices, for otherwise a short calculation shows $G-G^*$ would have more than $\binom{\varepsilon n}{2}$ edges. We can therefore say no more about small subgraphs in G^* than we can in G.

§2. Properties of jumbled graphs

We now examine consequences of the definition of jumbled graphs. These graphs have many properties which are well known to hold for random graphs. The following two lemmas will be fundamental in our study.

Lemma 2.1. *Let G be a (p,α)-jumbled graph of order n, and let $0<\varepsilon<1$. Then at least $(1-\varepsilon)n$ of the vertex degrees of G lie in the range $p(n-1)\pm 10\alpha\varepsilon^{-1}$.*

Proof. Let S be a subgraph of order s, and let the sum of the degrees (in G) of the vertices of S be sd. Then $sd=2e(S)+e(S,G-S)$. But

$$e(S)+e(S,G-S)=e(G)-e(G-S)$$
$$=p\binom{n}{2}-p\binom{n-s}{2}+B(\alpha n)+B(\alpha(n-s))$$

and

$$e(S)=p\binom{s}{2}+B(\alpha s),$$

so

$$sd=2p\binom{s}{2}+ps(n-s)+B(\alpha n)+B(\alpha(n-s))+B(\alpha s)$$
$$=ps(n-1)+B(2\alpha n).$$

Thus $d=p(n-1)+B(2\alpha ns^{-1})$. Taking S to be the $\lfloor\varepsilon n/2\rfloor$ vertices of smallest degree, we see that the average of these degrees is at least $p(n-1)-10\alpha\varepsilon^{-1}$ (since the lemma is vacuous if $\varepsilon n<10$, and otherwise $\lfloor\varepsilon n/2\rfloor\geqslant 2\varepsilon n/5$). The proof is completed by then taking S to be the $\lfloor\varepsilon n/2\rfloor$ vertices of largest degree. □

Lemma 2.2. *Let G be a (p, α)-jumbled graph of order n, and let $0<\varepsilon<1$. Let H be an induced subgraph of G of order k. Then at least $n-\varepsilon k$ of the vertices of G have between $pk-21\alpha\varepsilon^{-1}$ and $pk+21\alpha\varepsilon^{-1}$ neighbours in H.*

Proof. By Lemma 2.1 at least $(1-\varepsilon/2)k$ of the vertices of H have degree $p(k-1)+B(20\alpha\varepsilon^{-1})=pk+B(21\alpha\varepsilon^{-1})$ in H. If $G-H\neq\emptyset$, let S be a non-empty subgraph of $G-H$ with $|S|=s$. Define d by $sd=e(S, H)$. Then

$$\begin{aligned} sd &= e(S\cup H)-e(S)-e(H) \\ &= p\binom{s+k}{2}-p\binom{s}{2}-p\binom{k}{2}+B(\alpha(s+k))+B(\alpha s)+B(\alpha k) \\ &= psk+B(2\alpha(s+k)), \end{aligned}$$

so

$$d=pk+B(2\alpha(1+ks^{-1})).$$

Choose S to be the $\lfloor \varepsilon k/4\rfloor$ vertices of $G-H$ which send the least number of edges to H. Since we may as well assume $k\varepsilon\geqslant 21$ we see $\lfloor\varepsilon k/4\rfloor\geqslant\varepsilon k/4-1\geqslant\varepsilon k/5$ so $B(2\alpha(1+ks^{-1}))=B(2\alpha(1+5\varepsilon^{-1}))=B(12\alpha\varepsilon^{-1})$. Thus all but $\varepsilon k/4$ vertices of $G-H$ have at least $pk-12\alpha\varepsilon^{-1}$ neighbours in H, and likewise all but $\varepsilon k/4$ vertices have at most $pk+12\alpha\varepsilon^{-1}$ neighbours. □

For a set U of vertices of a graph G, denote by $N(U)$ the set of vertices of $G-U$ which are joined to every vertex in U, and denote by $\overline{N(U)}$ those vertices of $G-U$ joined to none of the vertices of U. If U_1 and U_2 are disjoint sets of vertices, we denote $|N(U_1)\cap\overline{N(U_2)}|$ by $v(U_1, U_2)$.

Under the conditions of Theorem 1.1 it is possible to show easily that almost all k-tuples of vertices have about $p^k n$ common neighbours. This is done by enumerating the k-tuples, letting d_i be the number of neighbours of the ith k-tuple, and estimating $\sum d_i$ and $\sum\binom{d_i}{2}$. This works for k up to $\log_b n$, where $b=1/p$. The following theorem does the same for any jumbled graph, though the argument works only as far as $k=\frac{1}{2}\log_b n$.

Theorem 2.3. *Let G be a (p, α)-jumbled graph of order n, let $k, l\geqslant 0$ be integers and let $0<\varepsilon<1$. Then*

$$|v(U_1, U_2)-p^k q^l n|<21(k+l)^2\alpha\varepsilon^{-1}$$

for at least $(1-\varepsilon)\binom{n}{k}\binom{n-k}{l}$ *choices of sets* U_1 *and* U_2, *where* $|U_1|=k$, $|U_2|=l$ *and* $q=1-p$.

Proof. Let $\delta=\varepsilon(k+l)^{-1}$. First choose $u_1^1\in G$ with degree $pn+B(21\alpha\delta^{-1})$; by Lemma 2.2 there are at least $(1-\delta)n$ choices for u_1^1. Then choose $u_1^2\in G-U_1$ with $p^2n+2B(21\alpha\delta^{-1})$ neighbours in common with u_1^1. By Lemma 2.2 there are at least $(1-\delta)(n-1)$ choices for u_1^2. Repeating the procedure in $G-\{u_1^1, u_1^2\}$ shows there are at least $(1-\delta)^k n(n-1)\dots(n-k+1)$ choices of sequences u_1^1, $u_1^2, \dots, u_1^k$ with $p^kn+kB(21\alpha\delta^{-1})$ common neighbours. Let $U_1=\{u_1^1, \dots, u_1^k\}$. Likewise we may choose a sequence $u_2^1, \dots, u_2^l$ such that if $U_2=\{u_2^1, \dots, u_2^l\}$ then $v(U_1, U_2)=p^kq^ln+(k+l)B(21\alpha\delta^{-1})$, and this may be done in at least $(1-\delta)^l(n-k)\dots(n-k-l+1)$ ways. Since U_1 and U_2 may arise from $k!l!$ different choices of sequences, we get at least $(1-\delta)^{k+l}\binom{n}{k}\binom{n-k}{l}\geqslant(1-\varepsilon)\binom{n}{k}\binom{n-k}{l}$ different choices of U_1 and U_2 such that

$$|v(U_1, U_2)-p^kq^ln|<21(k+l)^2\alpha\varepsilon^{-1}. \quad \square$$

Note that even under the conditions of Theorem 1.1 we cannot show $|N(U)|\sim p^kn$ for *all* k-subsets U, even for $k=3$. In the graph of example (9), for instance, $|N(U)|\sim 2^{-d}n$ where d is the dimension of the subspace spanned by U; d may take any value between $\lceil\log_2|U|\rceil$ and $|U|$. However, for the Paley graphs (example (6)) Bollobás and Thomason [6] have shown $|N(U)|\sim p^kn$ for all $k\ll\frac{1}{2}\log_2 n$.

Given a set $U\subset V$, let $\Gamma(U)$ denote the set of vertices of $G-U$ joined to at least one vertex of U. Then $\Gamma(U)=G-U-\overline{N(U)}$. A graph is called an *expander* graph if, loosely speaking, $\Gamma(U)$ is as large as possible for every set U. Under our definition of a jumbled graph we can speak usefully of $\Gamma(U)$ for *all* U only if $|U|\gg\alpha$. However, under the conditions of Theorem 1.1 we get information about $\Gamma(U)$ for smaller U, as is discussed in [25].

The diameter

The next few graph properties we shall look at, such as diameter and connectivity, are only interesting when the graph has no isolated vertices or vertices of low degree. For this reason we shall consider (p, α)-jumbled graphs of order n and minimum degree pn. To apply these results to jumbled graphs in general, note by Lemma 2.1 a jumbled graph has most vertex degrees near pn and so contains a large subgraph of large minimum degree, to which the following theorems will apply.

It is easy to see that the diameter of a graph in $\mathcal{G}(n, p)$ is almost surely at most 2 if $p^2n-2\log n \to \infty$. For a jumbled graph we can do almost as well.

Theorem 2.4. *Let G be a (p, α)-jumbled graph. Let $u, w \in G$ be vertices with degree at least d. If $pd > 4\alpha$ there is a $u-w$ path of length at most 3 in G. In particular, if $\delta(G) > 4\alpha p^{-1}$ then G has diameter at most 3.*

Proof. Choose $U \subset \Gamma(u)$, $W \subset \Gamma(w)$ with $|U| = |W| = d$. If $U \cap W \neq \emptyset$, we are home; otherwise the number of $U-W$ edges is

$$p|U||W| + B(|U|\alpha) + B(|W|\alpha) + B((|U|+|W|)\alpha)$$
$$\geqslant pd^2 - 4d\alpha > 0,$$

so there is a $u-w$ path of length at most 3. □

Connectivity

For random graphs it is well known that the connectivity is almost surely the same as the minimum degree (see Bollobás and Thomason [8] for a proof that this holds over the entire range $0 \leqslant p \leqslant 1$). For jumbled graphs we have the following.

Theorem 2.5. *Let G be a (p, α)-jumbled graph of order n. Then*

$$\kappa(G) > \delta(G) - 4\alpha p^{-1} + 1.$$

Proof. Let S be a vertex cut of G. As in the proof of Theorem 2.4, if $k > 4\alpha p^{-1}$ there is at least one edge between any two subgraphs of order k, so a smallest component of $G-S$ has order at most $4\alpha p^{-1}$. But such a component together with S contains at least $\delta(G)+1$ vertices. □

Note that the expanding properties referred to after Theorem 2.3 of a jumbled graph G satisfying the conditions of Theorem 1.1 allow us to rule out very small components of $G-S$ and so to show $\kappa(G) = \delta(G)$ for such graphs.

Hamilton cycles

A graph in $\mathcal{G}(n, p)$ is almost surely hamiltonian if $np - \log n - \log\log n \to \infty$. Likewise if p is not too small a jumbled graph is hamiltonian, and indeed has many hamilton cycles. We will find all these cycles by means of the following lemma.

Lemma 2.6. *Let G be a graph, and let P be a path in G of length $l \geqslant 0$. (A path of length 0 is just a vertex.) If G has no independent set of order $\kappa(G)-l+1$ then G has a hamilton cycle containing P.*

Proof. A theorem of Chvátal and Erdös [12] says if G has no independent set of order $\kappa(G)$ then G has a hamilton cycle between any two specified vertices. Applying this to the graph G', obtained from G by removing the middle $l-1$ vertices of P (or removing P if $l=0$) gives the result claimed. □

Theorem 2.7. *Let G be a (p, α)-jumbled graph, and let P be a path in G of length $l \geqslant 0$. If*

$$\delta(G) \geqslant 6\alpha p^{-1} + l,$$

G has a hamilton cycle containing P.

Proof. The largest independent set in G has order at most $2\alpha p^{-1}+1$, which by Theorem 2.5 is no larger than $\kappa(G)-l$. Apply Lemma 2.6. □

Corollary 2.8. *Let G be a (p, α)-jumbled graph of order n with minimum degree at least pn. If*

$$(p-k/n)^2 n \geqslant 6(\alpha+2k),$$

where k is a nonnegative integer, then G has a set of $k+1$ edge-disjoint hamilton cycles.

Proof. Theorem 2.7 gives the case $k=0$, and shows G has a hamilton cycle. Removing the edges of this gives a graph G' of order n which is $(p-2/n, \alpha+2)$-jumbled with minimum degree $pn-2$. We again apply Theorem 2.7 to find a hamilton cycle, and repeating k times gives the result claimed. □

Calkin [10] asked whether there is an exponentially large number of hamilton cycles in the Paley graph (example (6)). By Corollary 2.8 we can find a set of $n/100$ edge-disjoint cycles and Corollary 2.2 of [23] then assures us of at least $(n/100)^2$ hamilton cycles. But we can do much better.

Corollary 2.9. *Let G be a (p, α)-jumbled graph of order n, with minimum degree $pn \geqslant m = \lceil 6\alpha p^{-1} \rceil$. Then G has at least $\frac{1}{2}(pn)!/m!$ hamilton cycles.*

Proof. Choose a vertex x. There are at least $pn(pn-1)\dots(pn-l+1)$ paths of length l beginning at x. If $l=pn-m$, Theorem 2.7 shows each of these paths is in a hamilton cycle, and each cycle contains at most two such paths. □

Applying this to the Paley graphs gives us almost $(n/2)!$ hamilton cycles.

Induced subgraphs

We now examine small induced subgraphs of a jumbled graph. There are various reasons for this. For instance, Erdös [14] has conjectured the minimum number of monochromatic K_k's in an edge 2-colouring of K_n is $\binom{n}{k}2^{1-\binom{k}{2}}(1+o(1))$, and it is known that this bound is attained if the colouring is random. Theorem 2.10 shows it is enough for the colouring to be jumbled; this was shown by Giraud [17] in the case $k=4$ for graphs satisfying the conditions of Theorem 1.1. Theorem 2.10 also shows that a $(\frac{1}{2}, \alpha)$-jumbled graph of order n contains an induced copy of *every* graph of order r if $n \geqslant cr^4 2^{2r}$ and α is of order $n^{\frac{1}{2}}$. Such graphs were called *r-full* by Bollobás and Thomason [6], who in answer to a question of Rosenfeld showed that several of our examples of jumbled graphs are r-full.

The restriction to $p \leqslant \frac{1}{2}$ in the following theorem is for convenience only, and for graphs with $p > \frac{1}{2}$ the theorem can be applied to the complement.

Theorem 2.10. *Let F be a graph of order $r \geqslant 3$ with m edges, and let z be the order of its automorphism group. Let G be a (p, α)-jumbled graph of order n, where $p \leqslant \frac{1}{2}$. Suppose ε satisfies $0<\varepsilon<1$ and $\varepsilon^2 p^r n \geqslant 42\alpha r^2$. Then the number of induced subgraphs of G which are isomorphic to F lies between*

$$(1-\varepsilon)^r p^m q^{\binom{r}{2}-m} z^{-1} n^r$$

and

$$(1+\varepsilon)^r p^m q^{\binom{r}{2}-m} z^{-1} n^r,$$

where $q=1-p$.

Proof. Define $\delta>0$ by

$$(1+\delta)^{(r+1)/2}=1+\varepsilon.$$

Then

$$1-\varepsilon<(1-\delta)^{(r+1)/2},$$

and, since $\varepsilon<1$,

$$(1-\delta)^{r-1}>(2-2^{2/(r+1)})^{r-1}>\tfrac{1}{4}$$

holds. Moreover $\delta>\varepsilon/r$. We conclude therefore

$$\delta^2(1-\delta)^{r-1}p^{r-1}n\geq\varepsilon^2p^rn/4r^2p\geq 21\alpha,$$

and it is the outer inequality we will use.

Let $V(F)=\{w_1, \dots, w_r\}$. We estimate in how many ways we may choose a sequence $x_1, \dots, x_r$ of vertices of G spanning an induced subgraph isomorphic to F (so x_i corresponds to w_i etc.).

To begin with, we have n choices for x_1, and of these all but at most $21\alpha/\delta p$ are 'normal', that is, have degrees in the range $pn(1+B(\delta))$; this comes from Lemma 2.2 using the value $21\alpha/\delta pn$ for the ε there. Suppose now we have chosen $x_1, \dots, x_j$, $j<r$, and let X_i^j, $j<i\leq r$, be the set of vertices of $G-\{x_1, \dots, x_j\}$ such that $x\in X_i^j$ is joined to x_k, $k\leq j$, if and only if $w_iw_k\in F$. In other words, when we later choose x_i, we will have to choose it from X_i^j. Note that the X_i^j need not be distinct. We now have exactly $|X_{j+1}^j|$ choices for x_{j+1}. Lemma 2.2 once again shows that, for each $i>j+1$, all but at most $21\alpha/\delta p$ choices have $p|X_i^j|(1+B(\delta))$ neighbours in X_i^j; for these choices we will have

$$|X_i^{j+1}|=p|X_i^j|(1+B(\delta))$$

or

$$|X_i^{j+1}|=q|X_i^j|(1+B(\delta))$$

according as $w_{j+1}w_i$ is or is not an edge of F. We call x_{j+1} normal if it has $p|X_i^j|(1+B(\delta))$ neighbours in X_i^j for each i, so there are at most $(r-j-1)21\alpha/\delta p$ abnormal choices for x_{j+1}.

For $2\leq i\leq r$, let $f(i)$ be the number of neighbours of w_i among $\{w_1, w_2, \dots, w_{i-1}\}$. If $x_1, \dots, x_j$ are all normal choices, we have

$$|X_{j+1}^j|\geq(1-\delta)^jp^fq^{j-f}n,$$

where $f=f(j+1)$, so there are at least

$$(1-\delta)^jp^fq^{j-f}n-(r-j-1)\,21\alpha/\delta p$$
$$\geq(1-\delta)^{j+1}p^fq^{j-f}n$$

normal choices for x_{j+1}; this inequality holds since (for $j+1<r$)

$$\delta(1-\delta)^j p^f q^{j-f} n \frac{\delta p}{(r-j-1)21\alpha}$$

$$\geqslant \frac{\delta^2(1-\delta)^{r-1} p^{j+1} n}{21(r-j-1)\alpha}$$

$$\geqslant \frac{\delta^2(1-\delta)^{r-1} p^{r-1} n}{21\alpha} \geqslant 1,$$

and if $j+1=r$ all choices are normal. Hence there are at least

$$(1-\delta)^{\binom{r+1}{2}} p^m q^{\binom{r}{2}-m} n^r$$

normal choices for the sequence $x_1, \ldots, x_r$. Likewise we have

$$|X_{j+1}^j| \leqslant (1+\delta)^j p^f q^{j-f} n,$$

so we obtain at once that there are at most

$$N=(1+\delta)^{\binom{r}{2}} p^m q^{\binom{r}{2}-m} n^r$$

normal choices for $x_1, \ldots, x_r$. It can be seen that each edge $w_j w_i$ of F contributes a factor $(1+\delta)p$ to N, since

$$|X_i^j| \leqslant (1+\delta)\, p\, |X_i^{j-1}|,$$

and likewise each non-edge contributes a factor $(1+\delta)q$. However, if x_j is an abnormal choice we may say only $|X_i^j| \leqslant |X_i^{j-1}|$, and so the factors $(1+\delta)p$ or $(1+\delta)q$ are lost for any edge or non-edge incident with a vertex chosen abnormally. So the number of choices of $x_1, \ldots, x_r$ where some fixed subsequence $x_{i_1}, \ldots, x_{i_k}$ is chosen abnormally is at most

$$N\left[(1+\delta)^{(r-k)k+\binom{k}{2}} p^t q^{(r-k)k+\binom{k}{2}-t}\right]^{-1} n^{-k} \left(\frac{21\alpha r}{\delta p}\right)^k,$$

where t is the number of edges of F incident with at least one of $w_{i_1}, \ldots, w_{i_k}$.

Hence the number of choices of $x_1, \ldots, x_r$ with k abnormal choices is at most

$$\binom{r}{k}[(1+\delta)p]^{-(r-k)k-\binom{k}{2}}n^{-k}\left(\frac{21\alpha r}{\delta p}\right)^k N$$

$$\leqslant \binom{r}{k}\left[\frac{21\alpha r}{\delta pn[(1+\delta)p]^{(r-1)/2}}\right]^k N$$

$$\leqslant \binom{r}{k}\delta^k\left[\frac{21\alpha r^3}{\varepsilon^2 np^{(r+1)/2}}\right]^k N \leqslant \binom{r}{k}\delta^k N,$$

so the total number of choices for $x_1, \ldots, x_r$ does not exceed $(1+\delta)^r N$.

We have now shown that the number of ways to construct the sequence $x_1, \ldots, x_r$ lies between

$$(1-\delta)^{\binom{r+1}{2}} p^m q^{\binom{r}{2}-m} n^r$$

and

$$(1+\delta)^{\binom{r+1}{2}} p^m q^{\binom{r}{2}-m} n^r.$$

The proof is completed by noting that

$$(1+\delta)^{\binom{r+1}{2}} = (1+\varepsilon)^r,$$

$$(1-\delta)^{\binom{r+1}{2}} \geqslant (1-\varepsilon)^r,$$

and that each subgraph isomorphic to F corresponds to exactly z sequences $x_1, \ldots, x_r$. □

This theorem shows that $(\frac{1}{2}, \alpha)$-jumbled graphs with α of order $n^{\frac{1}{2}}$ are r-full for r up to about $\frac{1}{2}\log_2 n$. It is natural to ask whether the estimates of Theorem 2.10 could be extended to larger values of r by using a slightly less blunt method. Alas, this is not so. Let us estimate the number of K_r's in the graph of example (11) with $k=2s$ and $r=s+t$. Let W be a subset of the vertex set. Regarding the vertex set as elements of a vector space, let U be the subspace generated by W. Then the set of vertices joined to every element of W is precisely $U^{\perp}$, the subspace orthogonal to U under the dot product. Now $\dim U^{\perp}=k-\dim U$ and if W spans a complete subgraph then $U=U^{\perp}$ so $\dim U\leqslant s$. If we construct K_r's by first choosing s independent mutually orthogonal vectors and then choosing t more in the subspace spanned by the first s, we have at least

$$\frac{1}{r!}(2^{k-1}-1)(2^{k-2}-2)(2^{k-3}-4)\dots(2^{s+1}-2^{s-1})(2^s-s-1)\dots(2^s-r)$$

$$\sim\frac{\pi}{r!}n^s2^{-\binom{s}{2}}n^t2^{-st}=\frac{\pi}{r!}n^r2^{-\binom{r}{2}+\binom{t}{2}}$$

K_r's, where $\pi=\prod_{i>0}(1-4^{-i})$, which is $2^{\binom{t}{2}}\pi$ times the expected number suggested by Theorem 2.10.

Cliques and the chromatic number

Theorem 2.10 shows that the clique number of a jumbled graph must be at least $\frac{1}{2}\log_b n$ (if α is of order $(pn)^{\frac{1}{2}}$). The graph of example (12) shows the clique number need not exceed $\log_b n$, since a clique corresponds to a set of linearly independent vectors. This contrasts with the case of a random graph, where the clique number is known very precisely (see Matula [20] and Bollobás and Erdös [5]), and is around $2\log_b n$. More striking is the possibility of large cliques. We mentioned that a clique has order at most $2\alpha(1-p)^{-1}+1$. Several of our examples have such large cliques. In the Paley graph (example (6)) if n is a square, the elements in the subfield of order $n^{\frac{1}{2}}$ form a clique.

The graph of example (11), looked at in the previous section, is particularly interesting from the point of view of maximal cliques, since every maximal clique is the same size. Indeed if W is the set of vertices of a clique then W is contained in a subspace of dimension $k/2$ which spans a clique. Thus the greedy algorithm always produces a clique of the maximum order. The Paley graph has fewer large cliques, and many small maximal ones. In the case $n=p^2$ and $p\equiv 3 \pmod 4$, every edge is contained in a clique of order $n^{\frac{1}{2}}$. Another large clique is given by the vertices z^i, $1\leqslant i\leqslant(p+1)/2$, together with 0; here $z=g^{2(p-1)}$ and g is a primitive root for the prime p. To see that these vertices form a clique, it is enough to show that $1-z^i$ is a square. Putting $z^i=y$, we see

$$(1-y)^{(p^2-1)/2}=(1-y)^{(p-1)(p+1)/2}$$

$$=\left[\frac{(1-y)^p}{(1-y)}\right]^{(p+1)/2}$$

$$=\left[\frac{1-y^{p}}{1-y}\right]^{(p+1)/2}=\left[\frac{1-y^{-1}}{1-y}\right]^{(p+1)/2}$$

$$=(-y)^{(p+1)/2}=1,$$

the fourth equality holding since $y^{p+1}=1$.

Bounds on the chromatic number follow at once from bounds on the clique (or independence) number. Thus for a $(\frac{1}{2}, n^{\frac{1}{2}})$-jumbled graph of order n, the chromatic number is at most $2n/\log_2 n$, need not be less than $n/\log_2 n$, and must be at least $\frac{1}{4}n^{\frac{1}{2}}$. The greedy algorithm may use as many as $n/2$ colours, as for instance on the graph of example (4) if the vertices are ordered $x_1, y_1, \ldots, x_{n/2}, y_{n/2}$.

Contractions and subdivisions

Given a graph G, we define the *contraction-clique number* $ccl(G)$ to be the largest integer k such that G has a subcontraction to K_k. Bollobás, Catlin and Erdös [4] showed that for $G \in \mathcal{G}(n, p)$, p constant,

$$ccl(G)=n(\log(1-p)/\log n)^{\frac{1}{2}}(1+o(1))$$

almost surely, and so proved that almost all graphs satisfy Hadwiger's conjecture. Thomason [24] proved that graphs of order n with $p\binom{n}{2}$ edges satisfy

$$ccl(G)>pn(\log_2(pn))^{-\frac{1}{2}}/6;$$

in fact, using the technique of [24] and Lemma 2.3 one can show that, for (p, α)-jumbled graphs with p constant and $\alpha=O(n^{1-\varepsilon})$,

$$ccl(G)\geqslant n(\log(1-p)/\log n)^{\frac{1}{2}}(1+o(1)).$$

Note that the graph of example (5) shows we may have $ccl(G)\geqslant pn$.

Finally, we consider the *topological clique number* of G, denoted $tcl(G)$, which is the largest k for which G contains a subdivision of K_k. By estimating $tcl(G)$ for $G \in \mathcal{G}(n, p)$ Erdös and Fajtlowicz [15] showed that almost every graph is a counterexample to Hajós' conjecture. Bollobás and Catlin [3] showed

$$tcl(G)=2(n/(1-p))^{\frac{1}{2}}(1+o(1))$$

for almost every $G \in \mathcal{G}(n, p)$, p constant. It is interesting that we are able to give good bounds on $tcl(G)$ for a jumbled graph.

Lemma 2.11. *Let G be a (p, α)-jumbled graph of order n. Then*

$$tcl(G) < 2(\alpha + n^{\frac{1}{2}})(1-p)^{-1}.$$

Proof. Let W be the set of branch vertices of a subdivision of K_k contained in G. Then $|W| = k$. There are at most $p\binom{k}{2} + \alpha k$ edges of G in W, so G contains at least $(1-p)\binom{k}{2} - \alpha k$ disjoint paths (of length at least 2) joining the vertices of W. Thus

$$n - k \geqslant (1-p)\binom{k}{2} - \alpha k,$$

or

$$k\left[k - \frac{2\alpha}{1-p} + \frac{1+p}{1-p}\right] - \frac{2n}{1-p} \leqslant 0.$$

But if

$$k \geqslant 2(\alpha + n^{\frac{1}{2}})(1-p)^{-1}$$

this inequality fails to hold. □

The lower bound we give could be sharpened with more work or if the conditions of Theorem 1.1 were assumed. But still the upper and lower bounds we give are of the same order if p is constant and α is of order $n^{1/2}$.

Theorem 2.12. *Let G be a (p, α)-jumbled graph of order n, and suppose $\varepsilon > 0$ satisfies $\varepsilon p^2 n > 40\alpha$. Then*

$$tcl(G) \geqslant \lfloor (1-\varepsilon)(pn)^{\frac{1}{2}} \rfloor.$$

Proof. We may assume $\varepsilon < 1$; let $k = \lfloor (1-\varepsilon)(pn)^{1/2} \rfloor$. By Lemma 2.1 there are $n/3 \geqslant k$ vertices of degree at least $pn - 20\alpha$. Let W be a set of such vertices with $|W| = k$. We will construct one by one a set of $\binom{k}{2}$ paths of length at most 3 joining the vertices of W such that W is thereby the set of branch vertices of a subdivided K_k. To do this, let $u, w \in W$ and let G^* be the graph obtained from G by removing $W - \{u, w\}$ along with the other vertices of the paths constructed so far. Then we have removed at most

$$k - 2 + 2\left[\binom{k}{2} - 1\right] < k^2$$

vertices, so in G^* u and w have degree at least d, where

$$d \geqslant pn - 20\alpha - k^2 > \varepsilon pn/2 > 4\alpha p^{-1}.$$

By Theorem 2.4 there is a $u-w$ path of length at most 3 in G^*, and this is the path we require. □

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