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HAMILTONIAN CYCLES AND UNIQUELY EDGE COLOURABLE GRAPHS

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0. Introduction

A theorem of Smith (see Tutte [8]) states that in any cubic graph the number of hamiltonian cycles containing a given edge is even. If the graph is cubic and bipartite, a theorem of Kotzig (see Bosák [2]) tells us that the total number of hamiltonian cycles in the graph is even too. These two theorems are in fact consequences of a more general result, which we prove in Section 1 below. We also look at sets of edge-disjoint hamiltonian cycles in multigraphs (loops are allowed). Let $m \geq 2$ and for two edges x and y of a multigraph G (with at least three vertices) let $P(x, y)$ be the set of all collections of m edge-disjoint hamiltonian cycles in G. The main result of Section 2 states that $|P(x, y)|$ is even.

These results were discovered whilst investigating uniquely edge colourable graphs. We denote by $\chi'(G)$ the edge chromatic number of a graph G. (We adopt the terminology of [1].) If G has no isolated vertices, and if all edge colourings of G induce the same partition of the edges into independent sets, we say that G is *uniquely k-edge colourable* (where $k = \chi'(G)$); this is sometimes abbreviated to *uniquely edge colourable.* Let α and β be two of the colours used to colour a uniquely k-edge colourable graph, and let $C_{\alpha\beta}$ be the subgraph induced by the edges of colour α and the edges of colour β. We may swap the colours α and β in any component of $C_{\alpha\beta}$ and get another edge colouring of G; hence $C_{\alpha\beta}$ is connected, and is a path or an (even) cycle. If G is k-regular then $C_{\alpha\beta}$ is a hamiltonian cycle, since there is an edge of colour α (and one of colour β) at each vertex.

Obviously any uniquely 2-edge colourable graph is a path or an even cycle; it is clear also that the star $K_{1,k}$ is uniquely k-edge colourable ($K_{1,k}$ has vertex set $\{u\} \cup \{v_1, \ldots, v_k\}$ and edge set $\{uv_1, \ldots, uv_k\}$). Suppose now that G is uniquely 3-edge colourable. If G contains a triangle we may contract the triangle to a single vertex and get another uniquely 3-edge colourable multigraph; conversely we may replace any vertex of degree 3 by a triangle to get a larger uniquely 3-edge colourable graph. This fact led Greenwell and Kronk [4] to conjecture that every uniquely 3-edge colourable graph other than $K_{1,3}$ contains a triangle; they

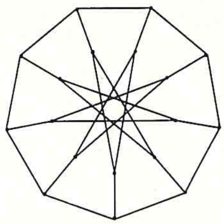

Fig. 1. Tutte's counterexample.

also conjectured that every cubic graph with exactly three hamiltonian circuits is uniquely edge colourable. A counterexample to the first conjecture was found by Tutte [9]; see Fig. 1.

A conjecture of Cantoni (see [9]) states that every cubic planar graph with exactly three hamiltonian cycles contains a triangle. This leads naturally to the conjecture stated by Fiorini [3], that every uniquely 3-edge colourable planar graph other than $K_{1,3}$ contains a triangle.

For $\chi'(G) \geqslant 4$ the stars are the only uniquely edge colourable graphs; we prove this in Section 3. It was first stated by Wilson [10] as a conjecture.

1. Hamiltonian cycles

Throughout this section we shall be concerned with hamiltonian paths in a multigraph $G=(V, E)$ which begin with a certain sequence of edges. (Paths and cycles are always considered as sequences or sets of edges, rather than as sequences of vertices.) We select a path $s=e_1, \ldots, e_m$ in G, where the endvertices of the edge e_i are v_i and v_{i+1}, $1 \leqslant i \leqslant m$. The path s is called a *stick*. The definitions to follow, and the statement of Theorem 1.1, depend on our choice of s; we obtain corollaries to Theorem 1.1 by making suitable specific choices of s.

Let $|V|=n$, and for a vertex $v \in V$ let $d(v)$ be the degree of v in G. Further let $\varepsilon(v)$ be the number of edges between v and the set of vertices $\{v_1, \ldots, v_m\}$, that is, all the vertices of the stick except the last. Let $h=e_1, \ldots, e_{n-1}$ be a hamiltonian path beginning with the stick s, where the edge e_i has endvertices v_i and v_{i+1}, $1 \leqslant i \leqslant n-1$. Let e_n be another edge with endvertices v_n and v_k, $k \geqslant m+1$, where $e_n \neq e_{n-1}$. Then the set $ϙ=\{e_1, \ldots, e_n\}$ is called a *lollipop*.[1] It contains two hamiltonian paths beginning with the stick s, namely $h=e_1, \ldots, e_n$ and $h'=e_1, \ldots, e_{k-1}, e_n, e_{n-1}, \ldots, e_{k+1}$. Note that if e_n is a loop then $h=h'$; we regard ϙ as then containing two copies of h.

We now define the *lollipop graph* ϙ(G, s) to be a multigraph whose vertex set is the set of hamiltonian paths of G beginning with the stick s. ϙ(G, s) has an edge e for each lollipop ϙ of G, the endvertices of e being the vertices h and h' of ϙ(G, s). Again, note that if $h=h'$ then e will be a loop of ϙ(G, s).

[1] The letter ϙ (koppa) is an episemon, originally coming between π and ρ in the Greek alphabet.

Suppose h is a hamiltonian path in G beginning with the stick s and ending in a vertex v_n. Then the degree of h in ♀(G, s) is exactly the number of copies of h contained in the lollipops, namely $d(v_n) - \varepsilon(v_n) - 1$; this holds even if there are loops in G at v_n.

Theorem 1.1. *The number of hamiltonian paths in G beginning with the stick s and ending in a vertex of the set* $W = \{w \in V\colon d(w) - \varepsilon(w)$ *is even*$\}$ *is even.*

Proof. These paths are exactly the vertices of odd degree in ♀(G, s).

Corollary 1.2. *Let G be a multigraph, let* $u, v \in V$, *and suppose that* $d(w)$ *is odd for each vertex* $w \in V - \{u, v\} \neq \emptyset$. *Then the number of hamiltonian paths in G from u to v is even.*

Proof. We may assume that u and v are adjacent vertices (if they are not we may add an edge between them); let e be an edge between u and v. We choose the stick s to be the edge e with $u = v_1$ and $v = v_2$; if $w \in V$ then $\varepsilon(w)$ is the number of edges from u to w. Consequently a hamiltonian path h beginning with s and ending in w gives rise to exactly $\varepsilon(w)$ hamiltonian paths from u to v. But by Theorem 1.1 the number of such paths ending in the set $W = \{w \in V\colon \varepsilon(w) \text{ is odd}\}$ is even.

Note that the case of Corollary 1.2 in which G is cubic and u is adjacent to v is precisely Smith's theorem.

Corollary 1.3. *Let G be a multigraph with n vertices,* $n \geq 4$. *Let* $u, v, w \in V$ *and suppose that* $d(x)$ *is odd if* $x \in V - \{u, v, w\}$. *Suppose that every path of length* $n-2$ *from v to w passes through the vertex u. Then the number of paths of length* $n-2$ *from u to v which do not contain w is even.*

We prove Corollary 1.3 in the following equivalent form.

Corollary 1.4. *Let G be a multigraph with n vertices,* $n \geq 4$. *Let* $u, v, w \in V$, *with* $uw, wv \in E$, *and let* $d(x)$ *be odd if* $x \in V - \{u, v, w\}$. *Suppose that every* $(n-1)$*-cycle in G passes through the vertex u. Then the number of hamiltonian cycles containing both the edges uw and wv is even.*

Proof. We take our stick to be $s = e_1, e_2$ where $e_1 = uw$, $e_2 = wv$, $v_1 = u$, $v_2 = w$ and $v_3 = v$. Let h be a hamiltonian path starting with s and ending in a vertex v_n. Then v_n cannot be joined to w since there is no $(n-1)$-cycle in G which doesn't pass through the vertex u. Thus v_n is joined to u by $\varepsilon(v_n)$ edges and so h gives rise to $\varepsilon(v_n)$ hamiltonian cycles containing the edges e_1 and e_2. By Theorem 1.1, the number of such paths ending in the set $W = \{x \in V\colon \varepsilon(x) \text{ is odd}\}$ is even, and the result then follows.

In the particular case when G is cubic and bipartite, let $w \in V$, and let w have neighbours u_1, u_2 and u_3. By Corollary 1.4 the number of hamiltonian cycles containing the edges u_1w and wu_2 is even; similarly for u_1w and wu_3 and for u_2w and wu_3. Thus the total number of hamiltonian cycles in G is even, and we obtain Kotzig's theorem.

If we restrict ourselves to cubic graphs we can obtain the following stronger result.

Corollary 1.5. *Let G be a cubic graph, and let H be the number of hamiltonian cycles in G. For any vertex $v \in V$, let $g(v)$ be the number of $(n-1)$-cycles not containing v, and for any two incident edges e and f let $h(e, f)$ be the number of hamiltonian cycles containing both e and f. Then*

$$g(v) \equiv h(e, f) \equiv H (\text{mod } 2).$$

Proof. Let $s = e_1, e_2$ be a stick in G. Let a be the number of hamiltonian paths beginning with s and ending in a vertex adjacent to v_1 but not v_2. Let b be the number of hamiltonian paths beginning with s and ending in a vertex adjacent to v_2 but not v_1. Let c be the number of hamiltonian paths beginning with s and ending in a vertex adjacent to both v_1 and v_2. Then $h(e_1, e_2) = a + c$, and since G is cubic, $g(v_0) = b + c$. By Theorem 1.1, $a + b$ is even, and so $h(e_1, e_2) \equiv g(v_0)$ (mod 2). Let now f_1, f_2 and f_3 be the edges incident with a vertex w. The number of hamiltonian cycles not containing the edge f_1 is $h(f_2, f_3)$, so by Smith's theorem $H \equiv h(f_2, f_3)$ (mod 2), and the proof is complete.

Corollary 1.6. *Let G be a graph in which every vertex has even degree. Let u be a vertex of G, and let e be an edge incident to u. Then the number of hamiltonian paths in G which begin at u, contain e, and end in a vertex not adjacent to u, is even.*

Given a multigraph G and a hamiltonian path h beginning with a stick s we can always construct the lollipops which contain h and thus find the vertices adjacent to h in the lollipop graph ♀(G, s); thus we have an algorithm for constructing the component of ♀(G, s) which contains h. This is particularly simple in the case when G is cubic, since then the components of ♀(G, s) are paths and cycles. This algorithm is illustrated in Fig. 2, where given one hamiltonian cycle containing the two dark edges we may find another, since there is no 9-cycle which doesn't contain the vertex x. (This algorithm, applied to cubic planar graphs, was discovered independently by Price [6].)

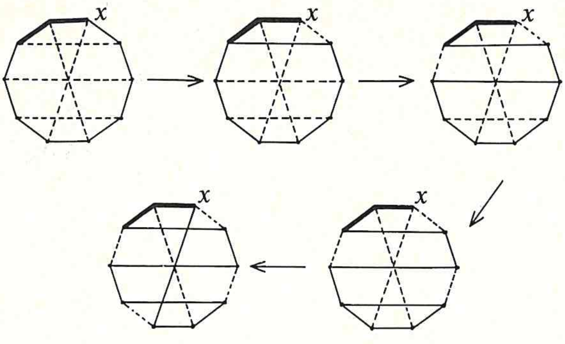

Fig. 2. An algorithm illustrated.

2. Hamiltonian decompositions

Given a multigraph $G=(V, E)$, a partition of E into edge-disjoint hamiltonian cycles is called a *hamiltonian decomposition* of G. A pair $\{h, \bar{h}\}$ of edge-disjoint hamiltonian cycles is called a *hamiltonian pair*. Let now G be 4-regular, that is, $d(v)=4$ for each $v \in V$, and let P be the set of all hamiltonian pairs. Since G is 4-regular a hamiltonian pair is a hamiltonian decomposition of G. For $x, y \in E$, let $P(x, y)$ be the set of hamiltonian pairs in which x and y lie in the same cycle, and let $Q(x, y)$ be the set of hamiltonian pairs in which x and y lie in different cycles; thus $Q(x, y)=P-P(x, y)$. Note that if x, y_1, y_2 and y_3 are the edges incident to a vertex $v \in V$, then $P=\bigcup_{i=1}^{3} P(x, y_i)$ and so $|P|=\sum_{i=1}^{3}|P(x, y_i)|$; in particular if each $|P(x, y_i)|$ is even then so is $|P|$.

I would like to express here my thanks to Mr. Richard Pinch, of Trinity College, Cambridge, whose computing work helped guide me towards the next theorem.

Theorem 2.1. *Let G be a 4-regular multigraph with at least three vertices, and let x and y be any two edges of G. Then the number of hamiltonian pairs in which x and y lie in the same cycle is even.*

Proof. Suppose that the theorem is false, and let G be a counter-example with fewest vertices. Then $|P|>0$, so G is connected and has no loops. Since the only loopless 4-regular multigraph on 3 vertices is the fat triangle (Fig. 3) it follows that $|V| \geqslant 4$.

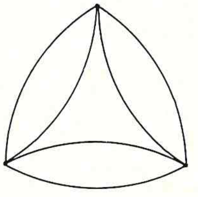

Fig. 3. The fat triangle.

Let z_1 and z_2 be edges with a common endvertex v; say v is joined to vertices u_1 and u_2 by z_1 and z_2 respectively and to vertices $\bar{u}_1$ and $\bar{u}_2$ by edges $\bar{z}_1$ and $\bar{z}_2$ respectively. The multigraph G' is constructed from G by removing v, z_1, z_2, $\bar{z}_1$ and $\bar{z}_2$, and by then adding the edge z between u_1 and u_2 and the edge $\bar{z}$ between $\bar{u}_1$ and $\bar{u}_2$. Given $\{h, \bar{h}\} \in P(z_1, z_2)$ with $z_1, z_2 \in h$, say, then $\bar{z}_1, \bar{z}_2 \in \bar{h}$, and there is a corresponding hamiltonian pair $\{h', \bar{h}'\}$ in G' with $z \in h'$ and $\bar{z} \in \bar{h}'$. Similarly it is clear that to each pair $\{k', \bar{k}'\} \in Q(z, \bar{z})$ there corresponds a pair $\{k, \bar{k}\} \in P(z_1, z_2)$, and so $|P(z_1, z_2)| = |Q(z, \bar{z})|$. But since G' is not a counterexample to the theorem it follows by the remarks made earlier that G' contains evenly many hamiltonian pairs, and so $|Q(z, \bar{z})|$ is even. Hence in G, $|P(z_1, z_2)|$ is even for any two incident edges z_1 and z_2, and in particular $|P|$ is even.

Let now x and y be any two edges of G, and let $x, y_1, y_2, \ldots, y_{r-1}, y_r = y$ be a sequence of edges forming a path whose end edges are x and y. Now for any edge z, the identity

$$Q(x, y) = P(x, z)\, \Delta\, P(z, y)$$

holds (where the triangle denotes symmetric difference) since z is in either the cycle containing x or that containing y. Hence we have for $1 \leqslant i \leqslant r-1$,

$$\begin{aligned} |P(x, y_{i+1})| &= |P| - |Q(x, y_{i+1})| \equiv |Q(x, y_{i+1})| \\ &= |P(x, y_i)\, \Delta\, P(y_i, y_{i+1})| \equiv |P(x, y_i)| + |P(y_i, y_{i+1})| \\ &\equiv |P(x, y_i)| \pmod 2, \end{aligned}$$

since y_i and y_{i+1} have a common endvertex. Thus

$$|P(x, y)| = |P(x, y_r)| \equiv |P(x, y_{r-1})| \equiv \cdots \equiv |P(x, y_1)| \equiv 0 \pmod 2,$$

contradicting our choice of G as a counterexample.

Theorem 2.1 answers a question of Sloane [7], who asked whether the existence of a hamiltonian pair in a graph G implied the existence of another such pair. Sloane showed that if G contains a hamiltonian pair then it contains a third hamiltonian cycle; Sloane's result was improved somewhat by Ninčák [5] who showed that G must contain at least six hamiltonian cycles. Corollary 2.2 includes a further improvement on the estimate of the number of hamiltonian cycles in G.

Corollary 2.2. *Let G be a $2m$-regular multigraph with at least three vertices, where $m \geqslant 1$. If G has a hamiltonian decomposition, then*

(i) *each edge of G is in at least $3m-2$ hamiltonian cycles,*

(ii) *G contains at least $m(3m-2)$ hamiltonian cycles, and*

(iii) *G has at least $(3m-2)(3m-5) \cdots 7.4 \geqslant 3^{m-1}(m-1)!$ hamiltonian decompositions.*

In particular if G has a unique hamiltonian decomposition then G is a cycle.

Proof. We prove statements (i), (ii) and (iii) by induction on m; they are obvious if $m=1$. Suppose $m=2$. By Theorem 2.1 the number $|P|$ of hamiltonian decompositions of G is even. Suppose $e \in E$ and $\{h_1, \bar{h}_1\}$, $\{h_2, \bar{h}_2\} \in P$ with $e \in h_i$, $i=1, 2$. Then there is an edge $f \in h_1\text{-}h_2$, so $\{h_1, \bar{h}_1\} \in P(e, f)$, and since $|P(e, f)|$ is even it follows that there is a third hamiltonian pair in G. Thus $|P| \geq 4$, G has at least 8 hamiltonian cycles and each edge is in at least 4 hamiltonian cycles.

Now suppose $k>2$ and the statements are true for all values of $m \leq k-1$. Let $e \in E$ and let $\{h_1, \ldots, h_k\}$ be the given hamiltonian decomposition, with $e \in h_1$, say. Let G_i be the 4-regular subgraph induced by $h_1 \cup h_i$, $2 \leq i \leq k$. G_i has a hamiltonian decomposition, and there are at least three further hamiltonian decompositions $\{h_{il}, \bar{h}_{il}\}$, $1 \leq l \leq 3$, where $e \in h_{il}$. Now if $i \neq j$ then $h_{il} \cap h_{jl'} \subset h_1$ and so $h_{il} \neq h_{jl'}$. Let $H=\{h_1\} \cup \{h_{il}: 2 \leq i \leq k,\ 1 \leq l \leq 3\}$; then $|H|=3k-2$ and so statement (i) is proved. Since each hamiltonian cycle contains $n=|V|$ edges it follows that G contains at least $kn \cdot (3k-2)/n$ hamiltonian cycles, and so statement (ii) is proved. Further, if $h \in H$ let $G_h=(V, E\text{-}h)$. Then G_h is $2(k-1)$-regular and has a hamiltonian decomposition, namely $\{h_2, \ldots, h_k\}$ if $h=h_1$ and $\{h_2, \ldots, h_{i-1}, \bar{h}_{il}, h_{i+1}, \ldots, h_k\}$ if $h=h_{il}$, $2 \leq i \leq k$, $1 \leq l \leq 3$. Thus G_h has at least $(3k-5) \cdots 7 \cdot 4$ hamiltonian decompositions, and so G has at least $(3k-2)\ (3k-5) \cdots 7 \cdot 4$, proving statement (iii).

An examination of a few arbitrarily chosen 4-regular graphs with fewer than 20 vertices suggested that the number of hamiltonian pairs in a 4-regular graph with n vertices increases rapidly with n. However, for every $n \geq 10$ there is a graph on n vertices with exactly 32 hamiltonian pairs. Consider first the 4-regular graph T_n, $n \geq 5$, with vertex set $\{0, 1, \ldots, n-1\}$ and with the vertex j joined to the vertices $j \pm 1$ and $j \pm 2$ (addition mod n). T_{12} is illustrated in Fig. 4.

For $0 \leq k \leq n-1$, the sequence of vertices $0, 1, \ldots, k-1$, $k+1$, k, $k+2$, $k+3, \ldots, n-1$ gives rise to a hamiltonian cycle, and the remaining edges also form a hamiltonian cycle; thus T_n has at least n hamiltonian pairs. If n is odd the cycle $0, 1, 2, \ldots, n-1$ also yields a hamiltonian pair. Suppose now that $\{h, \bar{h}\}$ is a hamiltonian pair. It is easily shown that if neither h nor $\bar{h}$ is given by $0, 1, \ldots, n-1$ then h, say, must contain a path of the form $j, j+2, j+1, j+3$, say the path 0, 2, 1, 3. Since 3, 2, 4 is a path in $\bar{h}$ the edge $(3, 4)$ must be in h, so $(3, 5) \in \bar{h}$, so $(4, 5) \in h$ etc., and we see that $\{h, \bar{h}\}$ is one of the pairs described above, and that

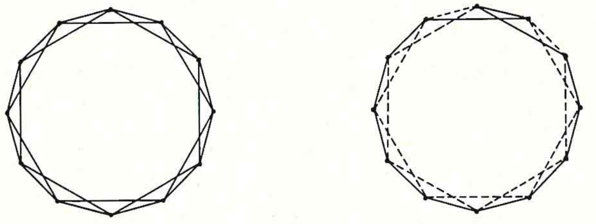

Fig. 4. The graph T_{12} and a typical decomposition.

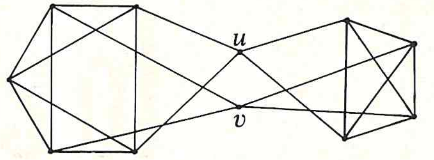

Fig. 5. A graph with 11 vertices and 32 hamiltonian pairs.

T_n has exactly $2\{\frac{1}{2}n\}$ hamiltonian pairs, $\{r\}$ denoting the least integer greater than or equal to the real number r.

Now let $n \geq 10$, and let $n_1 + n_2 = n$, with $n_i \geq 5$, $i = 1, 2$. Let G_i, $i = 1, 2$, be formed from T_{n_i} by removing the vertex 0 and its incident edges and adding vertices u_i and v_i; u_i is joined to 1 and $n_i - 1$ in T_{n_i} and v_i is joined to 2 and $n_i - 2$. Form G by identifying u_1 with u_2 and v_1 with v_2 (see Fig. 5). Then the number of hamiltonian pairs in G is $2p_1p_2$, where p_i is the number of pairs in T_{n_i} in which the edges $(0, 1)$ and $(0, n-1)$ are in different cycles. But by the above remarks $p_i = 4$ and so G has exactly 32 hamiltonian pairs.

3. Uniquely edge colourable graphs

Let G be a graph with $\chi'(G) = 4$, and suppose that G is edge coloured with the colours b, g, r and y. We denote by $u(b)$, say, a vertex u of degree 3 none of whose incident edges are coloured b, and by $v(g, r)$, say, a vertex v of degree 2 whose incident edges are coloured neither g nor r; that is, they are coloured b and y.

If G is uniquely edge colourable, then the subgraph induced by the edges of two given colours is connected, and so is a path or a cycle. We call these colour paths and colour cycles.

Lemma 3.1. *Suppose that $K_{1,4}$ is not the only uniquely 4-edge colourable graph. Then there is a uniquely 4-edge colourable graph G satisfying one of the following two properties:*

(i) *G is 4-regular, or*

(ii) *There are two vertices $u, v \in V$ such that $d(w) = 4$ for each $w \in V - \{u, v\}$; furthermore u and v both have degree 2 and their incident edges are coloured with the same two colours.*

Proof. Let H be a uniquely 4-edge colourable graph. We saw earlier that the subgraph induced by the edges of any two given colours is connected. In particular if H is a tree this means that H has no path of length three: thus $H = K_{1,4}$. Suppose now $H \neq K_{1,4}$. If v is a vertex of degree 1, then the removal of v and its incident edge gives a graph H' which is also uniquely 4-edge colourable; since then H is not a tree we may assume that each vertex of H has degree at least 2. We set about adding edges and vertices to H to obtain uniquely edge colourable graphs with fewer vertices of degree less than 4. If at some stage our graph were to have two vertices of degree 3, u and v say, then either $u = u(b)$ and $v = v(b)$ or $u = u(b)$ and $v = v(g)$. In the first case we add the b-coloured edge uv, and in the second we add the vertex w with a b-coloured edge uw and a g-coloured edge vw. This shows that we may assume H has at most one vertex of degree 3; since H cannot have just one vertex of odd degree, it has none at all.

Let now H have q vertices of degree 2, all other vertices having degree 4. If $q = 0$ then H is regular and we may take $G = H$, so we assume $q \geqslant 1$. Let H have p colour paths; then $p \leqslant \binom{4}{2} = 6$. Furthermore each vertex of degree 2 is an endvertex of exactly 4 colour paths (for instance, $u(b, g)$ is an endvertex of the b–r, b–y, g–r and g–y colour paths), and so $2p = 4q$; that is, $p = 2q$. Since $q \geqslant 1$ we have $p \geqslant 2$, and since each path has two ends we must then have $q \geqslant 2$; thus $q = 3$ or $q = 2$.

Suppose that $q = 3$ (and so $p = 6$) and that u, v, w are the vertices of degree 2. If $u = u(b, g)$ and $v = v(b, g)$, say, then neither u nor v is an endvertex of the b–g colour path, which is impossible since the b–g colour path has two ends. Thus we may assume that $u = u(b, g)$ and $v = v(g, r)$. Then we may add a g-coloured edge uv. We now have two vertices of degree 3 and by the remarks above this reduces to the case $q = 2$.

In the final case $q = 2$ let u and v be the vertices of degree 2, and let $u = u(b, g)$. Then the colour paths are coloured b–r, b–y, g–r and g–y, and so either $v = v(b, g)$ or $v = v(r, y)$, since v is the other endvertex of each of these paths. If $v = v(b, g)$ we may take $G = H$. If $v = v(r, y)$ we may identify u and v to get a 4-regular uniquely edge colourable graph.

Theorem 3.2. *The only uniquely k-edge colourable graph for $k \geqslant 4$ is the star, $K_{1,k}$.*

Proof. If G is uniquely k-edge colourable and G' is the subgraph induced by the edges of k' of the colours, $k' \leqslant k$, then G' is uniquely k'-edge colourable, so we need prove Theorem 3.2 only in the case $k = 4$.

Suppose then that $G \neq K_{1,4}$ is a uniquely 4-edge colourable graph. We may assume that G satisfies property (i) or property (ii) of Lemma 3.1. If G satisfies property (i) then any colour cycle of G is a hamiltonian cycle which is contained in a hamiltonian pair, hence G has at least 3 hamiltonian pairs. But given any hamiltonian pair we may colour one cycle b–g and the other r–y to get an edge colouring of G: this means that G has exactly 3 hamiltonian pairs. But this is impossible by Theorem 2.1 and so G must satisfy property (ii).

Suppose then G has property (ii), and so has two vertices $u(b, g)$ and $v(b, g)$, say. Then the $(b–g)$-coloured subgraph of G is an $(n-2)$-cycle C_1 (recall that G has n vertices) and the $(r–y)$-coloured subgraph is a hamiltonian cycle C_2. Let the neighbours of u and v be u_1, u_2 and v_1, v_2 respectively. Construct the multigraph G' from G by removing u and v and their incident edges and adding the edges

$x = u_1u_2$ and $y = v_1v_2$. Then C_1 and C_2 give rise to a hamiltonian pair $\{C'_1, C'_2\}$ in G' such that $\{x, y\} \subseteq C'_2$. By Theorem 2.1 there is another hamiltonian pair $\{D'_1, D'_2\}$ in G' such that $\{x, y\} \subseteq D'_2$. Hence there is an $(n-2)$-cycle D_1 in G and an edge-disjoint hamiltonian cycle D_2 such that $\{C_1, C_2\} \neq \{D_1, D_2\}$. By colouring D_1 with b and g and colouring D_2 with r and y we get a new edge colouring of G. This contradiction completes the proof of the theorem.

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