Part IID RIEMANN SURFACES (2007–2008): Example Sheet 4

(a.g.kovalev@dpmms.cam.ac.uk)

1. Suppose that a holomorphic function f satisfies a polynomial equation

$$f^n(z) + a_{n-1}(z)f^{n-1}(z) + \ldots + a_1(z)f(z) + a_0(z) = 0$$

on an open disc $D \subset \mathbb{C}$, where the coefficients $a_i(z)$ are holomorphic on $\mathbb{C}$. Show that every analytic continuation of (f, D) also satisfies this equation.

2. Prove that the power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \ldots,$$

converges if $|z| < 1$ and diverges if $|z| > 1$. Further, prove that if $\varphi = p/2^q$ $(p, q \in \mathbb{Z})$, and $0 < r < 1$ then $\lim_{r\to 1-} f(re^{i\pi\varphi}) = \infty$. Deduce that the unit circle is the natural boundary for the function element $(f, \{|z| < 1\})$.

3. (i) Prove Schwartz lemma: if $f : \Delta \to \Delta$ is holomorphic and $f(0) = 0$ then either $|f(z)| < |z|$, for every $z \in \Delta - \{0\}$, or $f(z) = e^{i\theta}z$, for some real θ. Here $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. [Hint: consider the function $g(z) = f(z)/z$ and apply the *maximum modulus principle* to $g(z)$ on the closed discs $\{|z| \le 1 - \epsilon\}$, for any small $\epsilon > 0$.]

(ii) Deduce from Schwartz lemma that any biholomorphic map of Δ onto itself is a Möbius transformation (restricted to Δ). You may assume without proof a result (from IB Geometry examples) that a Möbius transformation maps Δ onto itself if and only if it is of the form $z \mapsto \dfrac{az + \bar{c}}{cz + \bar{a}}$, with $|a|^2 - |c|^2 = 1$.
[Hint: reduce the problem to the case when a biholomorphic map of Δ onto itself has a fixed point $z = 0$.]

(iii) The group $SU(1, 1)$ is defined as the group of complex 2×2 matrices preserving the standard Hermitian form of signature $(1, 1)$ on $\mathbb{C}^2$, i.e.

$$SU(1,1) = \left\{A \in GL(2,\mathbb{C}) \;:\; \det A = 1 \text{ and } A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\overline{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}.$$

Show that the group $\operatorname{Aut}\Delta$ of biholomorphic automorphisms of the open unit disc Δ is isomorphic to a 'projective special unitary group' $PSU(1, 1) = SU(1, 1)/\pm 1$.
(Compare with Q6 of example sheet 1.)

4. Let $\mathcal{F}$ be the complete analytic function $\sqrt{1 + \sqrt{z}}$. Show that the Riemann surface $S(\mathcal{F})$ contains exactly two germs $[f, z]$ with $z = 1$ and exactly four germs $[f, z]$ for each z such that $0 < |z - 1| < \frac{1}{2}$. [Hint: consider the possible values $f(z)$ for the function elements of $\mathcal{F}$.]

Let $0 < \epsilon < 1/2$ and consider the holomorphic map $\pi : [f, z] \in S(\mathcal{F}) \to z \in \mathbb{C}$. Verify that the path $\gamma(t) = 1 - \epsilon/2 + \epsilon t$, $0 \le t \le 1$, does not have a lift to $S(\mathcal{F})$ from $[g(1 - h(z)), 1 - \epsilon/2]$, where g, h are holomorphic functions near $1 - \epsilon/2$ and $1 - h(1 - \epsilon/2)$, respectively, satisfying $g(z)^2 = z$, $h(z)^2 = z$, $h(1) = 1$.

5. A group Γ acts **properly discontinuously** on a topological space X if and only if every $x \in X$ has a neighbourhood U, so that the sets $\gamma(U)$, for all $\gamma \in \Gamma$, are disjoint. Assuming the results of Q6(ii) of Example sheet 1, prove that any subgroup of biholomorphic automorphisms of $\mathbb{C}$ acting properly discontinuously is one of the following groups of translations,

$$\text{(i) } \{0\}, \qquad \text{(ii) } \mathbb{Z}\omega, \quad \omega \in \mathbb{C}^*, \qquad \text{or (iii) } \mathbb{Z}\lambda + \mathbb{Z}\mu, \quad \lambda\mu \in \mathbb{C}, \lambda\bar{\mu} \notin \mathbb{R}.$$

Deduce that the only Riemann surfaces whose universal cover is biholomorphic to $\mathbb{C}$ are $\mathbb{C}$ itself, $\mathbb{C}^*$, and the elliptic curves.

6. Show, using the uniformization theorem, that any holomorphic map from $\mathbb{C}$ to a compact connected Riemann surface of genus greater than 1 is constant.

Questions 7–10 are more challenging than others and some parts certainly go beyond limits of the examination. Nevertheless I hope that you will enjoy thinking about some of them.

7. (The j-invariant.) (a) The cross-ratio of four distinct points is defined by $\lambda = (z, z_1; z_2, z_3) = (z_0 - z_1)(z_2 - z_3)/((z_1 - z_2)(z_3 - z_0))$. Extend this definition to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, by taking the limit if some $z_k = \infty$, and verify that λ can take any complex value except 0, 1 and ∞. Show also that the only values of the cross-ratio obtainable from the same four points taken in some order are λ, $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(\lambda - 1)$, and $(\lambda - 1)/\lambda$.

(b) Let $\varphi(\lambda) = 4(\lambda^2 - \lambda + 1)^3/(27\lambda^2(\lambda - 1)^2)$. Show that two unordered quadruples are related by a Möbius transformation if (and only if) their cross-ratios λ, λ' satisfy $\varphi(\lambda) = \varphi(\lambda')$.

(c) In the lectures we saw that an elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is determined by the values of Weierstrass function $e_1 = \wp(1/2)$, $e_2 = \wp(\tau/2)$, $e_2 = \wp(1/2 + \tau/2)$. For $\text{Im}(\tau) > 0$, define $\lambda(\tau) = (e_1, e_2; e_3, \infty) = (e_1 - e_2)/(e_3 - e_2)$ and $J(\tau) = \varphi(\lambda(\tau))$. Show that $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is biholomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau')$ if and only if $J(\tau) = J(\tau')$ (Thus $J(\tau)$ parameterises uniquely the equivalence classes of biholomorphic elliptic curves.)

8. (Analytic continuation by reflections.) Let f be a function which is holomorphic on the upper half-plane $\mathbb{H}$ and continuous on $\mathbb{H} \cup I$, where $I \subset \mathbb{R}$ is an open interval. Suppose that $f(z) \in \mathbb{R}$ whenever $z \in I$. Prove that $f(z) = \overline{f(\bar{z})}$, for $\text{Im}(z) < 0$, defines an analytic continuation of f to $\mathbb{C} \setminus (\mathbb{R} \setminus I)$.
[Hint: it is convenient to use Morera's theorem from IB Complex Analysis. At some stage, consider a sequence of contours $\gamma_n(t)$, such that the γ_n's converge *uniformly with first derivatives* to a contour $\gamma(t)$ containing a subinterval of $I \subset \mathbb{R}$.]

Define, using Möbius transformations, the reflection in a circle in $\mathbb{R}^2$, generalising the reflections in straight lines. Now state carefully a general form of the principle of analytic continuation by reflections in lines or circles.

9. Consider the interior of hyperbolic triangle $T = \{z \in \mathbb{H} : 0 < \text{Re}(z) < 1, |z - 1/2| > 1/2\}$ in the upper half-plane $\mathbb{H}$. Let μ be a conformal equivalence map from T onto the upper half-plane and such that $\lim_{z\to 0} \mu(z) = 0$, $\lim_{z\to 1} \mu(z) = 1$, $\lim_{z\to\infty} \mu(z) = \infty$. (We assume the existence of such μ without proof here; it is a consequence of the Riemann mapping theorem. In fact, it is possible to give, with some further work, an 'explicit' construction of μ.) Assume further that μ extends continuously to the sides of the triangle T.

Show the following.

(a) μ has a well-defined analytic continuation, by reflections in the sides of T. By repeating the

reflections in the boundary arcs sufficiently many times, one obtains an analytic continuation of μ defined at any point of $\mathbb{H}$.
(b) The resulting holomorphic function on $\mathbb{H}$ (still denoted by μ) does not take values 0 and 1.
(c) μ admits no further analytic continuation outside $\mathbb{H}$.
(d) μ realizes $\mathbb{H}$ as the universal covering space of $\mathbb{C} \setminus \{0, 1\}$.

10. (Four views on the elliptic curves.) Let E be a compact connected Riemann surface. Show that the following are equivalent.

(1) E is the quotient $\mathbb{C}/\Lambda$ of the complex plane by a lattice.

(2) E is biholomorphic to a non-singular curve in $\mathbb{P}^2$ defined as the zero locus of a homogeneous cubic polynomial in the generalized Weierstrass normal form $XZ^2 - 4Y^3 - AX^2Y - BX^3$, for some complex constants A, B, $A^3 + 27B^2 \neq 0$.

(3) E is a compact Riemann surface of genus 1.

(4) there is a $2:1$ covering $E \to \mathbb{P}^1$ branched over four points.

You will need to recall appropriate results from several topics of the course (and some previous example(s)). Remember that we assume without proof that every compact Riemann surface carries non-constant meromorphic functions. You may also assume without proof that any abelian discrete subgroup of $\mathrm{Aut}(\Delta) = SU(1,1)/\pm 1$ is cyclic (this, and some topology, will be useful when showing that (3) implies (1)).

Suggestion for (3)⇒(2): use the group law on E (as defined in the lectures, by application of the Riemann–Roch) to find two meromorphic functions, f and h say, such that $(f) = P_1 + P_2 - 2P_0$ and $(h) = Q_1 + Q_2 + Q_3 - 3P_0$. ($P_i, Q_j$ are distinct points in E.) Now recall how the differential equation for $\wp$ was deduced and adapt the method to find a cubic polynomial expression $P(f, h) = h^2 + a_1 f^3 + \dots$ ($a_1 \in \mathbb{C}$) which does *not* have a pole at P_0.

Supervisors can obtain an annotated version of this example sheet from DPMMS.