Part III: Riemannian Geometry (Lent 2017)

Example Sheet 2

1. Give an example of a *non-compact* complete Riemannian manifold with Ricci curvature (strictly) positive-definite at each point.

2. Let G be a Lie group whose Lie algebra $\mathfrak{g}$ has trivial centre. Suppose that G admits a bi-invariant (i.e. left- and right-invariant) Riemannian metric. Show that G and its universal cover are compact. Deduce that $SL(2,\mathbb{R})$ admits no bi-invariant metric.

3. (i) Show that the Hodge star on $\Lambda^2(\mathbb{R}^4)^*$ determines an orthogonal decomposition $\Lambda^2(\mathbb{R}^4)^* = \Lambda^+ \oplus \Lambda^-$ into the ± 1 eigenspaces and $\dim \Lambda^+ = \dim \Lambda^- = 3$. Deduce that on every oriented 4-dimensional Riemannian manifold M there is a decomposition of 2-forms $\Omega^2(M) = \Omega^+ \oplus \Omega^-$, so that $\alpha \wedge \alpha = \pm|\alpha|_g^2\, \omega_g$, for every $\alpha \in \Omega^\pm$, where ω_g is the volume form. (2-forms in the subspaces $\Omega^\pm$ are called, respectively, the *self-* and *anti-self-dual* forms on M.)

   (ii) Now assume that M is a *compact* 4-dimensional oriented Riemannian manifold. Show that the expression $\int_M \alpha \wedge \beta$, for closed $\alpha, \beta \in \Omega^2(M)$, induces a well-defined symmetric bilinear form on the de Rham cohomology $H^2_{\mathrm{dR}}(M)$. Let $(b^+(M), b^-(M))$ denote the signature of this bilinear form. Show that $b^\pm(M) = \dim \mathcal{H}^\pm$, where $\mathcal{H}^\pm$ denotes the space of harmonic (anti-)self-dual forms on M.

4. (i) Derive explicit formulas for $*$, δ, and Laplace–Beltrami operator in Euclidean space. In particular, show that if

$$\alpha = \sum_{i_1 < \ldots < i_p} \alpha_I dx_{i_1} \wedge \ldots \wedge dx_{i_p} \qquad (I = i_1, \ldots, i_p),$$

   then

$$\Delta\alpha = -\sum_{i_1 < \ldots < i_p} \Big(\sum_{i=1}^{n} \frac{\partial^2 \alpha_I}{\partial {x_i}^2} \Big)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

   (ii) For $u, v \in C^\infty(M)$, show that $\Delta(uv) = (\Delta u)v - 2\langle du, dv\rangle_g + u\Delta v$ (M is an oriented Riemannian manifold).

5. Calculate explicitly the expression of the Laplacian for functions:

   (a) on the hyperbolic plane $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, where the metric is $g(x,y) = \dfrac{dx^2 + dy^2}{y^2}$;

   (b) on the unit sphere $S^n \subset \mathbb{R}^{n+1}$, in local coordinates given by stereographic projections. (The metric on S^n is the standard 'round' metric induced by the embedding.)

   *Express the Laplacian on the Euclidean $\mathbb{R}^{n+1} \setminus \{0\}$ in terms of the Laplacian on the unit sphere S^n (recall that the Euclidean metric can be expressed as $g = dr^2 + r^2 dS^2$, where $r = |x|$, $x \in \mathbb{R}^{n+1}$, and dS^2 is the 'round' metric on S^n). Deduce a formula for the Laplacian on spherically-symmetric functions $f(r)$.

6. Let α and β be n-forms on a compact oriented manifold M^n such that $\int_M \alpha = \int_M \beta$. Prove that α and β differ by an exact form. (Stokes' theorem may be assumed.)

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http://www.dpmms.cam.ac.uk/∼agk22/riemannian2.pdf

7. Show that the partial differential equation $\Delta f = \varphi$ for a function $f \in C^\infty(M)$ on a compact oriented Riemannian manifold (M, g), with a given $\varphi \in C^\infty(M)$, has a solution if and only if $\int_M \varphi\, \omega_g = 0$. ($\omega_g$ denotes the volume form.) Is the solution unique? *Discuss the solvability of $\Delta(\Delta f) = \varphi$ when $f, \varphi \in C^\infty(M)$ and more generally when f, φ are p-forms.

8. Let M be a compact oriented Riemannian manifold and F a diffeomorphism of M which preserves the volume form on M. We say that a form $\alpha \in \Omega^p(M)$ is *invariant* under F if $\alpha \circ F = \alpha$ and we say that the Laplacian Δ is *invariant* under F if $\Delta\alpha \circ F = \Delta(\alpha \circ F)$, for all $\alpha \in \Omega^p(M)$. Suppose that Δ and α are invariant under F and α is L^2-orthogonal to each harmonic form on M. Prove that there is an invariant solution η of $\Delta\eta = \alpha$.

9. (Holonomy transformations.) Show that the parallel transport defined by the Levi–Civita connection over any closed loop based at $x \in M$ defines an orthogonal linear transformation of T_xM which is in $SO(T_xM)$ when M is oriented.

   An *orthogonal almost complex structure* on a manifold (M, g) is an endomorphism J of its tangent bundle TM such that $J^2 = -1$ and $g(JX, JY) = g(X, Y)$, for all $X, Y \in \mathrm{Vect}(M)$. If M admits such J, show that M is orientable and even-dimensional. Show that $\omega = g(J\cdot, \cdot)$ defines a 2-form on M with $\omega^n \neq 0$ at each point ($\dim M = 2n$).

   Show further that the following statements are equivalent:
   (a) $\nabla J = 0$,
   (b) $\nabla \omega = 0$,
   (c) the parallel transport defined by ∇ along closed loops is represented by elements of $U(n) \subset SO(2n)$ (after some natural identifications).

   Here ∇ denotes the (induced) Levi-Civita connection on respective vector bundles. (Each of (a),(b),(c) is in fact equivalent to M being a *Kähler complex manifold* with Kähler form ω and J corresponding to multiplication by i in local complex coordinates.)

10. (i) For any two bilinear forms h, k on tangent spaces to M, define a $(0,4)$-tensor $(h\cdot k)(X, Y, Z, T) = h(X, Z)k(Y, T) + h(Y, T)k(X, Z) - h(X, T)k(Y, Z) - h(Y, Z)k(X, T)$, where $X, Y, Z, T \in T_xM$. Show that the curvature tensor $R = (R_{ij,kl})$ of a Riemannian n-dimensional manifold (M, g), $n \geq 4$, has an $SO(n)$-invariant, orthogonal decomposition $R = \dfrac{s}{2n(n-1)}\, g \cdot g + \dfrac{1}{n-2}(\mathrm{Ric} - \dfrac{s}{n} g) \cdot g + W$, where W satisfies $W(X, Y, Z, T) + W(Z, X, Y, T) + W(Y, Z, X, T) = 0$ (1st Bianchi identity) and $\sum_{i=1}^n W(X, e_i, Y, e_i) = 0$ for all $X, Y, Z, T \in T_xM$ and where e_i is an orthonormal basis of T_xM. (W is called the *Weyl tensor* of (M, g)).

    (ii) Suppose that $\dim M = 4$ and M is oriented. We consider R as a symmetric bilinear form on the fibres of $\Lambda^2 T^*M$. Let a bilinear form, $B : \Lambda^+ \times \Lambda^- \to \mathbb{R}$ be the restriction of R defined using the decomposition into self- and anti-self-dual forms as in Question 3. Show that B is equivalent to the trace-free part Ric_0 of the Ricci curvature (with respect to the metric g), that is, $g^{ik}(\mathrm{Ric}_0)_{kj} = g^{ik}\,\mathrm{Ric}_{kj} - \frac{1}{4} s\, \delta^i_j$ in local coordinates, where (g^{ij}) denotes the inverse matrix of $g = (g_{ij})$ (the summation convention is assumed).

    Show further that R, with respect to the decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, has the form

$$\begin{pmatrix} W^+ + \frac{s}{12} I & B \\ B^T & W^- + \frac{s}{12} I \end{pmatrix},$$

    where $W^\pm : \Lambda^\pm \times \Lambda^\pm \to \mathbb{R}$ are symmetric bilinear forms with $\mathrm{tr}\, W^- + \mathrm{tr}\, W^+ = 0$ and $W = W^+ \oplus W^-$ is an $SO(4)$-invariant orthogonal decomposition of the Weyl tensor.

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