Part III: Differential geometry (Michaelmas 2003, 2004, 2014, 2018)

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3 Riemannian geometry

3.1 Riemannian metrics and the Levi–Civita connection

Let M be a smooth manifold.

Definition. A bilinear symmetric positive-definite form

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

defined for every $p \in M$ and smoothly depending on p is called a **Riemannian metric** on M.

Positive-definite means that $g_p(v, v) > 0$ for every $v \neq 0$, $v \in T_pM$. Smoothly depending on p means that for every pair X_p,Y_p of C^∞-smooth vector fields on M the expression $g_p(X_p, Y_p)$ defines a C^∞-smooth function of $p \in M$.

Alternatively, consider a coordinate neighbourhood on M containing p and let x^i, $i = 1, \ldots, \dim M$ be the local coordinates. Then any two tangent vectors $u, v \in T_pM$ may be written as $u = u^i(\frac{\partial}{\partial x^i})_p$, $v = v^i(\frac{\partial}{\partial x^i})_p$ and $g_p(u, v) = g_{ij}(p)u^iv^j$, where the functions $g_{ij}(p) = g(\big(\frac{\partial}{\partial x^i}\big)_p, \big(\frac{\partial}{\partial x^j}\big)_p)$ express the coefficients of the metric g in local coordinates. One often uses the following notation for a metric in local coordinates

$$g = g_{ij}dx^idx^j.$$

The bilinear form (metric) g will be smooth if and only if the local coefficients $g_{ij} = g_{ij}(x)$ are smooth functions of local coordinates x^i on each coordinate neighbourhood.

Example 3.1. Recall (from Chapter 1) that any smooth regularly parameterized surface S in $\mathbb{R}^3$,

$$\mathbf{r} : (u, v) \in U \subset \mathbb{R}^2 \to r(u, v) \in \mathbb{R}^3.$$

is a 2-dimensional manifold (more precisely, we assume here that S satisfies all the defining conditions of an embedded submanifold). The first fundamental form[1] $Edu^2 + 2Fdudv + Gdv^2$ is a Riemannian metric on S.

The following formulae are proved in multivariate calculus.

- A curve on S may be given as $\gamma(t) = r(u(t), v(t))$, $a \leq t \leq b$. The *length* of γ is then computed as $\int_a^b |\dot\gamma(t)|dt = \int_a^b \sqrt{E\dot u^2 + 2F\dot u\dot v + G\dot v^2}\, dt$.
- The *area* of S is $\iint_U \sqrt{EG - F^2}\, du\, dv$.

[1] $E = (\mathbf{r}_u, \mathbf{r}_u)$, $F = (\mathbf{r}_u, \mathbf{r}_v)$, $G = (\mathbf{r}_v, \mathbf{r}_v)$ using the Euclidean inner product

Theorem 3.2. *Any smooth manifold M can be given a Riemannian metric.*

Proof. Indeed, M may be embedded in $\mathbb{R}^m$ by Whitney theorem (cf. Q9 Example Sheet 1). Then the restriction (more precisely, a pull-back) of the Euclidean metric of $\mathbb{R}^m$ to M defines a Riemannian metric on M. □

Remark. A metric, being a bilinear form on the tangent spaces, can be pulled back via a smooth map, f say, in just the same way as a differential form. But a pull-back f^*g of a metric g will be a well-defined metric only if f has an *injective* differential.

Remark. As a Riemannian metric on M is an inner product on the vector bundle TM, Theorem 3.2 is also a consequence of Q2 of Example Sheet 3.

Definition. A **connection on a manifold** M is a connection on its tangent bundle TM.

Recall that a choice of local coordinates x on M determines a choice of local trivialization of TM (using the basis vector fields $\frac{\partial}{\partial x^i}$ on coordinate patches). The transition function φ for two trivializations of TM is given by the Jacobi matrices of the corresponding change of coordinates $(\varphi^i_{i'}) = (\frac{\partial x^i}{\partial x^{i'}})$.

Let Γ^i_{jk} be the coefficients (Christoffel symbols) of a connection on M in local coordinates x^i. For any other choice $x^{i'}$ of local coordinates the transition law on the overlap becomes (cf. Chapter 2, eqn. (2.12a))

$$\Gamma^i_{jk} = \Gamma^{i'}_{j'k'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k} \tag{3.3}$$

One can see from the above formula that if Γ^i_{jk} are the coefficients of a connection on M then Γ^i_{kj} also are the coefficients of some well-defined connection on M (in general, this would be a different connection).

The difference $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ is called the **torsion** of a connection (Γ^i_{jk}). The transformation law for T^i_{jk} is $T^i_{jk} = T^{i'}_{j'k'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k}$, thus the torsion of a connection is a well-defined antisymmetric bilinear map sending a pair of vector fields X, Y to a vector field $T(X,Y) = T^i_{jk} X^j Y^k$ on M.

Definition. A connection on M is **symmetric** if its torsion vanishes, i.e. if $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Notation: given a connection (covariant derivative) $D : \Omega^0_M(TM) \to \Omega^1_M(TM)$ and a smooth vector field X on M, we write D_X for the composition of D and contraction of 1-forms (in $\Omega^1_M(TM)$) with X. Thus $D_X : \Omega^0_M(TM) \to \Omega^0_M(TM)$ is a linear differential operator acting on vector fields on M. In local coordinates, it is expressed as $(D_X Y)^i = X^j \partial_j Y^i + \Gamma^i_{jk} Y^j X^k$.

It is not difficult to see, by comparing with the definition on page 31, that a family of operators D_X, depending on a vector field X, defines a covariant derivative precisely if $D_X Y$ is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in Y and satisfies the Leibniz rule

$$D_X(hY) = (Xh)Y + h D_X Y, \tag{3.4}$$

for each $h \in C^\infty(M)$ and a vector field Y (recall Xh equals the contraction of dh with X).

Here is a way to define a symmetric connection independent of the local coordinates.

Proposition 3.5. *A connection D is symmetric if and only if $D_XY - D_YX = [X, Y]$.*

The proof is an (easy) straightforward computation.

Theorem 3.6. *On any Riemannian manifold (M, g) there exists a unique connection D such that*
(1) $d(g(X, Y))(Z) = g(D_ZX, Y) + g(X, D_ZY)$ for any vector fields X, Y, Z on M; and
(2) the connection D is symmetric.
D is called the **Levi–Civita connection** *of the metric g.*

The condition (1) in the above theorem is sometimes written more neatly as

$$dg(X, Y) = g(DX, Y) + g(X, DY).$$

Proof. Uniqueness. The conditions (1) and (2) determine the coefficients of Levi–Civita in local coordinates as follows. A 'coordinate vector field' $\frac{\partial}{\partial x^i}$ with constant coefficients has covariant derivative $D\frac{\partial}{\partial x^i} = \Gamma^p_{ik}\frac{\partial}{\partial x^p}dx^k$. The condition (1) with $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$ becomes

$$\frac{\partial}{\partial x^k} g_{ij} = \Gamma^p_{ik} g_{pj} + \Gamma^p_{jk} g_{ip}. \tag{3.7a}$$

Cycling i, j, k in the above formula, one can write two more relations

$$\frac{\partial}{\partial x^j} g_{ki} = \Gamma^p_{kj} g_{pi} + \Gamma^p_{ij} g_{kp}, \tag{3.7b}$$

$$\frac{\partial}{\partial x^i} g_{jk} = \Gamma^p_{ji} g_{pk} + \Gamma^p_{ki} g_{jp}. \tag{3.7c}$$

Let (g^{iq}) denote the inverse matrix to (g_{iq}), so $\Gamma^p_{jk} g_{qp} g^{iq} = \Gamma^i_{jk}$. Adding the first two equations of (3.7) and subtracting the third, dividing by 2, and multiplying both sides of the resulting equation by (g^{iq}), one obtains the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{iq}\left(\frac{\partial g_{qj}}{\partial x^k} + \frac{\partial g_{kq}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^q}\right) \tag{3.8}$$

(also taking account of the symmetry condition (2)). Thus if the Levi–Civita connection exists then its coefficients in local coordinates are expressed in terms of the metric by (3.8).

Exericise. By adapting the above method to arbitrary vector fields X, Y, Z on M, using the symmetry condition (2) in the form $D_XY - D_YX = [X, Y]$, show that the Levi–Civita connection is uniquely determined by the identity

$$g(D_XY, Z) = \frac{1}{2}\big(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(Y, [X, Z]) - g(Z, [Y, X]) + g(X, [Z, Y])\big). \tag{3.9}$$

Verify that D defined by (3.9) satisfies the conditions (1) and (2) in Theorem 3.6 (this might be argued by essentially following your calculation of (3.9) backwards).

Existence. *Proof 1.* One way of proving the existence is to check that the Γ^i_{jk} computed by the formula (3.8) are indeed the coefficients of a well-defined connection on M. This can be done by verifying that the Γ^i_{jk}'s transform in the right way, i.e. as in (3.3), under a change of local coordinates. The transformation law for g_{ij} is $g_{i'j'} = \dfrac{\partial x^i}{\partial x^{i'}} g_{ij} \dfrac{\partial x^j}{\partial x^{j'}}$, by the usual linear algebra. Differentiating this latter formula and using the respective formula for the induced inner product on the dual spaces, i.e. on the cotangent spaces to M, we can verify that the coefficients given by (3.8) indeed transform according to (3.3) and so the Levi–Civita connection of the metric g on M is well-defined.

Proof 2. Alternatively, assuming the exercise above, we shall be done if we show that $D_x Y$ defined by the formula (3.9) is $C^\infty(M)$-linear in X and satisfies the Leibniz rule in Y. For the first property, we note that $[fX, Z] = fXZ - Z(fX) = f[X,Z] - (Zf)X$, for every $f \in C^\infty(M)$. Then $2g(D_{fX}Y, Z)$ becomes

$$\begin{aligned} &fXg(Y,Z) + Yg(Z,fX) - Zg(fX,Y) - g(Y,[fX,Z]) - g(Z,[Y,fX]) + g(fX,[Z,Y]) \\ &= fXg(Y,Z) + (Yf)g(Z,X) + fYg(Z,X) - (Zf)g(X,Y) - fZg(X,Y) \\ &\qquad -g(Y, f[X,Z] - (Zf)X) - g(Z, (Yf)X + f[Y,X]) + fg(X,[Z,Y]), \end{aligned}$$

using the Leibniz rule for vector fields. It follows that $g(D_{fX}Y, Z) = g(fD_XY, Z)$, thus D is $C^\infty(M)$-linear in X.

For the Leibniz rule we calculate, with h a smooth function,

$$\begin{aligned} 2g(D_X(hY), Z) &= X(hg(Y,Z)) + hYg(Z,X) \\ &\qquad - Z(hg(X,Y)) - hg(Y,[X,Z]) - g(Z,[hY,X]) + g(X,[Z,hY]) \\ &= (Xh)g(Y,Z) + hXg(Y,Z) + hYg(Z,X) - (Zh)g(X,Y) - hZg(X,Y) \\ &\quad -hg(Y,[X,Z]) - hg(Z,[Y,X]) + (Xh)g(Z,Y) + hg(X,[Z,Y]) + (Zh)g(X,Y) \\ &= 2(Xh)g(Y,Z) + 2hg(D_XY,Z) \end{aligned}$$

which gives (3.4) as required. Since D_XY is clearly $\mathbb{R}$-linear in Y we have proved that D is a connection on M. □

3.2 Geodesics on a Riemannian manifold

Let $E \to M$ be a vector bundle endowed with a connection (Γ^i_{jk}). A parameterized smooth curve on the base M may be written in local coordinates by $(x^i(t)$. A lift of this curve to E is locally expressed as $(x^i(t), a^j(t))$ using local trivialization of the bundle E to define coordinates a^j along the fibres. A tangent vector $(\dot{x}(t), \dot{a}(t)) \in T_{(x^i(t),a^j(t))}E$ to a lifted curve will be *horizontal* (recall from the chapter 2, eqn. (2.10b)) at every t precisely when $a(t)$ satisfies a linear ODE

$$\dot{a}^i + \Gamma^i_{jk}(x) a^j \dot{x}^k = 0, \tag{3.10}$$

where $i, j = 1, \ldots, \operatorname{rank} E$, $k = 1, \ldots, \dim B$.

Now if $E = TM$ then there is also a *canonical* lift of any smooth curve $\gamma(t)$ on the base, as $\dot{\gamma}(t) \in T_{\gamma(t)}M$.

Definition. A curve $\gamma(t)$ on a Riemannian manifold M is called a **geodesic** if $\dot\gamma(t)$ at every t is horizontal with respect to the Levi–Civita connection.

Thus we are looking at a special case of (3.10) when $a = \dot x$. The condition for a path in M to be a geodesic may be written explicitly in local coordinates as

$$\ddot x^i + \Gamma^i_{jk}(x)\dot x^j \dot x^k = 0, \tag{3.11}$$

a non-linear second-order ordinary differential equation for a path $x(t) = (x^i(t))$ (here $i, j, k = 1, \dots, \dim M$). By the basic existence and uniqueness theorem from the theory of ordinary differential equations, it follows that for any choice of the initial conditions $x(0) = p$, $\dot x(0) = a$ there is a unique solution path $x(t)$ defined for $|t| < \varepsilon$ for some positive ε. Thus for any $p \in M$ and $a \in T_pM$ there is a uniquely determined (at least for any small $|t|$) geodesic with this initial data (i.e. 'coming out of p in the direction a'). Denote this geodesic by $\gamma_p(t,a)$ (or $\gamma(t,a)$ if this is not likely to cause confusion).

Proposition 3.12. *If $\gamma(t)$ is a geodesic on (M,g) then $|\dot\gamma(t)|_g = \text{const}$.*

Proof. We shall first make a rigorous sense of the equation

$$D_{\dot\gamma}\dot\gamma = 0 \tag{3.13}$$

and show that (3.13) is satisfied at each $\gamma(t)$ if and only if γ is a geodesic curve. The problem with (3.13) at the moment is that $\dot\gamma$ is not a vector field defined on any open set in M, but only along a curve γ. We define an extension, still denoted by $\dot\gamma$, on a coordinate neighbourhood U of $\gamma(0)$ as follows. It may be assumed, without loss, that $\dot\gamma(0) = (\dot x^i(0))$ has $\dot x^1(0) \neq 0$. We may further assume, taking a smaller U if necessary, that $\gamma \cap U$, is a graph of a smooth function $x^1 \mapsto (x^2(x^1), \dots, x^n(x^1))$. In particular, $\dot x^1(t) \neq 0$ for any small $|t|$ and also any hyperplane $x^1 = x^1_0$, such that $|x^1_0 - x^1(\gamma(0))|$ is small, meets the curve $\gamma \cap U$ in exactly one point. Denote by π the projection along hyperplanes $x^1 = const$ onto $\gamma \cap U$. Define, for every $p \in U$, $\dot\gamma(p) = \dot\gamma(\pi(p))$ and then $\dot\gamma$ is a smooth vector field on U, such that $(\dot\gamma)_p = \dot\gamma(t)$ whenever $p = \gamma(t)$.

Now let Γ^i_{jk} be the coefficients of the Levi–Civita in the coordinates on U. So $D_Z Y = (Z^l \partial_l Y^i + \Gamma^i_{jk} Y^j Z^k)\partial_i$ for any vector fields $Z = Z^l \partial_l$, $Y = Y^i \partial_i$ on U. Let $Y = Z = \dot\gamma$. Then at any point $p = \gamma(t)$ we have $Z^l \partial_l Y^i = \dot x^l \dfrac{\partial \dot x^i}{\partial x^l} = \ddot x^i$ by the chain rule. It follows that the equation (3.11) is equivalent to (3.13) if the latter if restricted to the points of the curve γ. It can also be seen, by inspection of the above construction, that $D_{\dot\gamma}\dot\gamma$ at the points of γ is independent of the choice of extension of $\dot\gamma(t)$ to a vector field on U.

We have $\dot\gamma(\dot\gamma, \dot\gamma)_g = (D_{\dot\gamma}\dot\gamma, \dot\gamma)_g + (\dot\gamma, D_{\dot\gamma}\dot\gamma)_g$ on U from the defining properties of the Levi–Civita. Hence $\dot\gamma\big(|\dot\gamma|^2_g\big) = 0$ at each $\gamma(t) \in U$, by (3.13). From the construction of the extension $\dot\gamma$ on U, we find that the directional partial derivative $\dot\gamma\big(|\dot\gamma|^2_g\big)$ at the points $\gamma(t)$ is expressed as $\dot x^l \dfrac{\partial}{\partial x^l}|\dot\gamma|^2_g = \dfrac{d}{dt}|\dot\gamma(t)|^2_g$ by the chain rule again, whence $|\dot\gamma|_g = const$ as we had to prove. □

Examples. 1. On $\mathbb{R}^n$ with the Euclidean metric $\sum(dx^i)^2$ we have $\Gamma^i_{ik} = 0$, so the Levi–Civita is just the exterior derivative $D = d$. The geodesics $\ddot{x}^i = 0$ are straight lines $\gamma_p(t,a) = p + at$ parameterized with constant velocity.

2. Consider the sphere S^n with the round metric (i.e. the restriction of the Euclidean metric to $S^n \subset \mathbb{R}^{n+1}$). Then $p \in S^n$ and $a \in T_pS^n$ may be regarded as the vectors in $\mathbb{R}^{n+1}$. Suppose $a \neq 0$, then the orthogonal reflection L in the 2-dimensional subspace $\mathcal{P} = \mathrm{span}\{p, p+a\}$ is an *isometry* of S^n. Now L preserves the metric and p and a, the data which determines the geodesic $\gamma_p(\cdot, a)$. As $\gamma_p(\cdot, a)$ is moreover uniquely determined it must be contained in the fixed point set of L. But the fixed point set is a curve, the great circle $\mathcal{P} \cap S^n$. We find that great circles, parameterized with velocity of constant length—and only these—are the geodesics on S^n.

Observe that for any geodesic $\gamma_p(t,a)$ and any real constant λ the path $\gamma_p(\lambda t, a)$ is also a geodesic and $\gamma_p(\lambda t, a) = \gamma_p(t, \lambda a)$.

By application of a general result in the theory of ordinary differential equations, a geodesic $\gamma_p(t,a)$ must depend smoothly on its initial conditions p, a. Furthermore, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ independent of a and such that if $|a| < \varepsilon_1$ then $\gamma_p(t,a)$ exists for all $-2\varepsilon_2 < t < 2\varepsilon_2$. It follows that $\gamma_p(1,a)$ is defined whenever $|a| < \varepsilon = \varepsilon_1\varepsilon_2$.

Definition. The **exponential map** at a point p of a Riemannian manifold (M, g) is

$$\exp_p : a \in \mathit{Ball}_{\,\varepsilon}(0) \subseteq T_pM \to \gamma(1; p, a) \in M.$$

Proposition 3.14. $(d\exp_p)_0 = \mathrm{id}(T_pM)$

Proof. We use the canonical identification $a \in T_pM \to \frac{d}{dt}(ta)|_{t=0}$ to define $(d\exp_p)_0$ as a linear map on T_pM (rather than on $T_0(T_pM)$).

Let $|a| < \varepsilon$, so $\gamma_p(t,a) = \gamma_p(1,ta)$ is defined for $0 \le t \le 1$. Then we have

$$\begin{aligned}(d\exp_p)_0 a &= \tfrac{d}{dt}\exp_p(ta)|_{t=0}\\ &= \tfrac{d}{dt}\gamma_p(1,ta)|_{t=0}\\ &= \tfrac{d}{dt}\gamma_p(t,a)|_{t=0}\\ &= \dot\gamma(0,a) = a.\end{aligned}$$ □

Corollary 3.15. *The exponential map* $\exp_m$ *defines a* diffeomorphism *from a neighbourhood of zero in* T_mM *to a neighbourhood of* m *in* M*.*

Proof. Apply the Inverse Mapping Theorem (page 11 of these notes). □

Corollary 3.15 means that the exponential map defines near every point p of a Riemannian manifold a system of local coordinates—called **normal** (or **geodesic**) **coordinates** at p. It is not difficult to see that the geodesics $\gamma_p(t,a)$ are given in these coordinates by rays emanating from the origin.

It also makes sense to speak of geodesic polar coordinates at $p \in M$ defined by the polar coordinates on T_pM via a diffeomorphism

$$f : (r, \mathbf{v}) \in \,]0, \varepsilon[\, \times S^{n-1} \;\to\; \exp_p(r\,\mathbf{v}) \in M. \tag{3.16}$$

Here $]0, \varepsilon[\times S^{n-1}$ is regarded as a subset in $T_pM \cong \mathbb{R}^n$ via the inner product $g(p)$. If $0 < r < \varepsilon$ then the image $\Sigma_r = f(\{r\} \times S^{n-1} \subset T_pM)$ of the metric sphere of radius r is well-defined on M and is called a **geodesic sphere** about p. (So Σ_r is an embedded submanifold of M.) The following remarkable result asserts that 'the geodesic spheres are orthogonal to their radii'.

Gauss Lemma. *The geodesic $\gamma_p(t, a)$ is orthogonal to Σ_r. Thus the metric g in geodesic polar coordinates has local expression $g = dr^2 + h(r, v)$, where for any $0 < r < \varepsilon$, $h(r, v)$ is the metric on Σ_r induced by restriction of g.*

Proof. Let X be an arbitrary smooth vector field on the unit sphere $S^{n-1} \subset T_pM$. Use polar coordinates to make sense of X as a vector field (independent of r) on the punctured unit ball $B \setminus \{0\} \subset T_pM$. Define a vector field $\tilde{X}(r, \mathbf{v}) = rX(\mathbf{v})$ on $B \setminus \{0\}$. The map $\exp_p$ induces a vector field $Y(f(r, \mathbf{v})) = (d\exp_p)_{r\mathbf{v}}\tilde{X}(r, \mathbf{v})$ on the punctured geodesic ball $B' \setminus \{p\} = \exp_p(B \setminus \{0\})$ in M.

We shall be done if we show that Y is everywhere orthogonal to the radial vector field $\frac{\partial}{\partial r}$. Note that, by construction, any geodesic from p is given in normal coordinates by $\gamma_p(t, a) = at$, so $\dot{\gamma}_p(t, a)/|a| = \frac{\partial}{\partial r}$. Here $|a|$ means the norm in the inner product g_p on the vector space T_pM. By application of Corollary 3.15, the family $\dot{\gamma}_p(t, a)$, where $|a| = 1$ and $0 < |t| < \varepsilon$, defines a smooth vector field on $B' \setminus \{p\}$. Recall from (3.13) that $D_{\dot{\gamma}}\dot{\gamma} = 0$ for any geodesic γ, where D denotes the Levi–Civita covariant derivative. Also $\frac{d}{dt}g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = \frac{d}{dt}g(\dot{\gamma}, \dot{\gamma}) = 0$ by Proposition 3.12, so $g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 1$. It remains to show that $g(Y, \dot{\gamma}) = 0$.

Using the diffeomorphism f in (3.16) to go to polar geodesic coordinates, we obtain

$$D_{\dot{\gamma}}Y - D_Y\dot{\gamma} = (df)\big(D_{\frac{\partial}{\partial r}}\tilde{X} - D_{\tilde{X}}\tfrac{\partial}{\partial r}\big) = (df)\frac{d}{dr}\tilde{X} = (df)(\tilde{X}/r) = Y/r,$$

with the help of Proposition 3.5. Therefore, we find

$$\frac{d}{dr}g(Y, \dot{\gamma}) = g(D_{\dot{\gamma}}Y, \dot{\gamma}) + g(Y, D_{\dot{\gamma}}\dot{\gamma}) = g(D_{\dot{\gamma}}Y, \dot{\gamma}) = g(D_Y\dot{\gamma} + \frac{1}{r}Y, \dot{\gamma}) = \frac{1}{r}g(Y, \dot{\gamma}).$$

as $2g(D\dot{\gamma}, \dot{\gamma}) = d\,g(\dot{\gamma}, \dot{\gamma}) = 0$ by Proposition 3.13. Thus $\frac{d}{dr}G = G/r$, where $G = g(Y, \dot{\gamma})$. Hence G is linear in r and $\frac{d}{dr}G$ independent of r. But $\lim_{r\to 0}\frac{d}{dr}G = \lim_{r\to 0} g(X, \frac{\partial}{\partial r}) = 0$, as $(d\exp_p)_0$ is an isometry by Proposition 3.14, and so $g(Y, \dot{\gamma}) = 0$ and the result follows. $\square$

3.3 Curvature of a Riemannian manifold

Let g be a metric on a manifold M. The (full) Riemann curvature $R = R(g)$ of g is, by definition, the curvature of the Levi–Civita connection of g. Thus $R \in \Omega^2_M(\mathrm{End}(TM))$, locally a matrix of differential 2-forms $R = \frac{1}{2}(R^i_{j,kl}dx^l \wedge dx^k)$, $i,j,k,l = 1 \dots n = \dim M$. The coefficients $(R^i_{j,kl})$ form the Riemann curvature tensor of (M,g). Given two vector fields X, Y, one can form an endomorphism field $R(X,Y) \in \Gamma(\mathrm{End}(TM))$; its matrix in local coordinates is $R(X,Y)^i_j = R^i_{j,kl}X^kY^l$ (as usual $X = X^k\partial_k$, $Y = Y^l\partial_l$). Denote $R_{kl} = R(\partial_k, \partial_l) \in \mathrm{End}(T_pM)$ (here p is any point in the coordinate neighbourhood).

Recall that in local coordinates a connection (covariant derivative) may be written as $d + A$, with $A = \Gamma^i_{jk}dx^k = A_k dx^k$. We write $D_k = D_{\frac{\partial}{\partial x^k}} = \frac{\partial}{\partial x^k} + A_k$. The definition of the curvature form of a connection (Chapter 2, p. 31) yields an expression in local coordinates

$$R^i_{j,kl} = \left(D_l D_k \tfrac{\partial}{\partial x^j} - D_k D_l \tfrac{\partial}{\partial x^j}\right)^i, \quad \text{or} \quad R_{kl} = -[D_k, D_l], \tag{3.17}$$

considering the coefficient at $dx^l \wedge dx^k$. Now $D_X = X^k D_k$ and so we have $-[D_X, D_Y] = -[X^kD_k, X^lD_l] = -X^k(\partial_k Y^l)D_l - X^kY^lD_kD_l + Y^k(\partial_k X^l)D_l + X^kY^lD_lD_k = X^kY^lR_{kl} - [X,Y]^lD_l$. We have thus proved

Lemma 3.18. $R(X,Y) = D_{[X,Y]} - [D_X, D_Y]$.

One also can combine (3.17) with (3.8) and thus obtain an explicit local expression for $R^i_{j,kl}$ in terms of the coefficients of the metric g and their first and second derivatives.

It is convenient to consider $R_{ij,kl} = g_{iq}R^q_{j,kl}$, which defines a map on 4-tuples of vector fields $(X,Y,Z,T) \mapsto g(R(X,Y)Z,T)$.

Proposition 3.19.

(i) $R_{ij,lk} = -R_{ij,kl} = R_{ji,kl}$;

(ii) $R^i_{j,kl} + R^i_{k,lj} + R^i_{l,jk} = 0$ *(the first Bianchi identity[2]);*

(iii) $R_{ij,kl} = R_{kl,ij}$.

Proof. (i) The first equality is clear. For the second equality, one has, from the definition of the Levi–Civita connection, $\frac{\partial g_{kl}}{\partial x^i} = g(D_i\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) + g(\frac{\partial}{\partial x^k}, D_i\frac{\partial}{\partial x^l})$, and further

$$\frac{\partial^2 g_{kl}}{\partial x^j \partial x^i} = g(D_jD_i\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) + g(D_i\frac{\partial}{\partial x^k}, D_j\frac{\partial}{\partial x^l}) + g(D_j\frac{\partial}{\partial x^k}, D_i\frac{\partial}{\partial x^l}) + g(\frac{\partial}{\partial x^k}, D_jD_i\frac{\partial}{\partial x^l}).$$

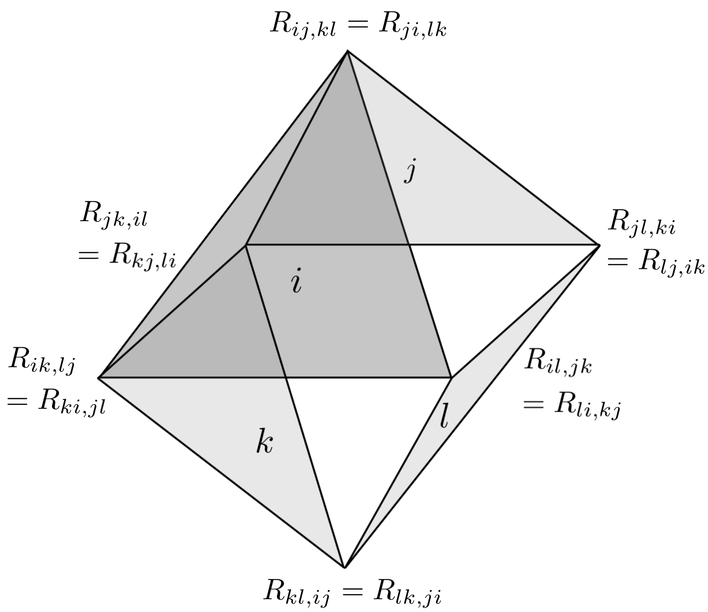
The right-hand side of the above expression is symmetric in i,j as $\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} = \frac{\partial^2 g_{kl}}{\partial x^j \partial x^i}$. The anti-symmetric part of the right-hand side (which has to be zero) equals $R_{ij,kl} + R_{ji,kl}$.

(ii) Firstly, $(D_k\frac{\partial}{\partial x^j})^i = \Gamma^i_{jk} = (D_j\frac{\partial}{\partial x^k})^i$, by the symmetric property of the Levi–Civita. The claim now follows by straightforward computation using (3.17).

We note for use in the proof of (iii) that multiplying (ii) by g_{iq} gives $R_{ij,kl} + R_{ik,lj} + R_{il,jk} = 0$.

(iii) We organize the argument using the vertices and faces of an octahedron (see the next page).

[2]also known as the algebraic Bianchi identity, not to be confused with the differential Bianchi identity in Chapter 2.



Assign to each vertex a simultaneous application of the two identities in (i). Then, for each *shaded face*, we may arrange the three coefficients of R to have the same first index (indicated by a letter in the middle of the face) so that the Bianchi identity (ii) can be applied. Adding the instances of (ii) for the two upper shaded faces and subtracting those for the two lower shaded faces, we obtain the required identity (iii) as all the terms in the vertices of the equatorial square cancel.[3]

$\square$

Remark. Notice that the proof of (ii) shows the first Bianchi identity is valid for *every symmetric connection* on M.

Corollary 3.20. *The Riemann curvature tensor* $(R_{ij,kl})_p$ *defines, at any point* $p \in M$ *a symmetric bilinear form on the fibres of* $\Lambda^2 T_pM$.

There are natural ways to extract "simpler" quantities (i.e. with less components) from the Riemann curvature tensor.

Definition. The **Ricci curvature** of a metric g at a point $p \in M$, $\mathrm{Ric}_p = \mathrm{Ric}(g)_p$, is the trace of the endomorphism $v \to R_p(x, v)y$ of T_pM depending on a pair of tangent vectors $x, y \in T_pM$.

Thus in local coordinates $\mathrm{Ric}(p)$ is expressed as a matrix $\mathrm{Ric} = (\mathrm{Ric}_{ij})$, $\mathrm{Ric}_{ij} = \sum_q R^q_{i,jq}$. That is, the Ricci curvature at p is a bilinear form on T_pM. A consequence of Proposition 3.19(iii) is that this bilinear form is *symmetric*, $\mathrm{Ric}_{ij} = \mathrm{Ric}_{ji}$.

Definition. The **scalar curvature** of a metric g at a point $p \in M$, $s = \mathrm{scal}(g)_p$ is a smooth function on M obtained by taking the trace of the bilinear form Ric_{ij} with respect to the metric g.

If local coordinates are chosen so that $g_{ij}(p) = \delta_{ij}$ at a point, then the latter definition means that $s(p) = \sum_i \mathrm{Ric}_{ii}(p) = \sum_{i,j} R_{ij,ji}(p)$. For a general g_{ij}, the formula may be written as $s = \sum_i g^{ij}\,\mathrm{Ric}_{ij}$, where g^{ij} is the induced inner product on the cotangent space with respect to the dual basis, algebraically (g^{ij}) is the inverse matrix of (g_{ij}).

[3] I learned this argument from the lectures of M.M. Postnikov.

3.3.1 Some examples

(1) It makes sense to consider the condition

$$\mathrm{Ric} = \lambda g \tag{3.21}$$

for some constant $\lambda \in \mathbb{R}$, as both the metric and its Ricci curvature are symmetric bilinear forms on the tangent spaces to M. When the condition (3.21) is satisfied, the Riemannian manifold (M, g) is called *Einstein manifold*. In particular, if (3.21) holds with $\lambda = 0$ then M is said to be *Ricci-flat*.

(2) Recall that if Σ is a surface in $\mathbb{R}^3$ (smooth, regularly parameterized by (u, v) in an open set in $\mathbb{R}^2$) then there is a metric induced on Σ, expressed as the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$. The second fundamental form $Ldu^2 + 2Mdudv + Ndv^2$ is defined by taking the inner products $L = (\mathbf{r}_{uu}, \mathbf{n})$, $M = (\mathbf{r}_{uv}, \mathbf{n})$, $N = (\mathbf{r}_{vv}, \mathbf{n})$ with the unit normal vector to Σ, $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / |\mathbf{r}_u \times \mathbf{r}_v|$ (the subscripts u and v denote respective partial derivatives). The quantity

$$K = \frac{LN - M^2}{EG - F^2}$$

is called the *gaussian curvature* of Σ. A celebrated *theorema egregium*, proved by Gauss, asserts that K is determined by the coefficients of first fundamental form, i.e. by the metric on Σ (and so K is independent of the choice of an isometric embedding of Σ in $\mathbb{R}^3$).

Taking up a general view on Σ as a 2-dimensional Riemannian manifold, one can check that $2(EG - F^2)^{-1} R_{12,21} = s$, the scalar curvature of Σ. From the results of the next section, we shall see (among other things) that the scalar curvature of a surface Σ is twice its gaussian curvature $s = 2K$.

3.4 Riemannian submanifolds

When a manifold M^n is an embedded submanifold of a Riemannian manifold, say V^{n+r}, the Riemannian metric g_V on V induces, by restriction, a Riemannian metric g_M on M. What is the relation between the Levi–Civita connection $\tilde{D}$ of g_V and the Levi–Civita connection D of g_M?

To see this relation, it is convenient to consider the vector bundle $E = \iota^*(TV)$ over M, where $\iota : M \hookrightarrow V$ is the embedding map. (Informally, E is just the restriction of TV to M if the latter is regarded as a subset of V.)

In the next proposition, we write x^k for local coordinates on M, y^γ for local coordinates on V, and $\alpha, \beta, \gamma = 1, \dots, n + r$.

Proposition 3.22. *Any connection $\tilde{\nabla}$ on V induces in a canonical way a connection on E with the coefficients $\Gamma^\alpha_{\beta k} = \frac{\partial y^\gamma}{\partial x^k} \Gamma^\alpha_{\beta\gamma}$, where $\Gamma^\alpha_{\beta\gamma}$ are the coefficients of $\tilde{\nabla}$ and $y = y(x)$ is the local expression of the embedding ι.*

We shall still denote by $\tilde{\nabla}$ the connection on E defined by the above proposition. For $p \in E$, consider the tangent space T_pE as a subspace of T_pV and then the corresponding horizontal subspace of T_pE is just the intersection $S_p \cap T_pE$, where $S_p \subset T_pV$ is the horizontal subspace for the connection on V.

There is also an interpretation in terms of the covariant derivatives (needed for the proof of Gauss–Weingarten formulae below). Any local vector field X on M (respectively local section σ of E) can be extended smoothly to a local vector field $\tilde{X}$ (respectively $\tilde{\sigma}$) on V. Then $(\tilde{\nabla}_{\tilde{X}}\tilde{\sigma})|_M = \tilde{\nabla}_X\sigma$, where in the left-hand side we use the connection on E. In particular, the right-hand side is independent of the choices of extensions $\tilde{X}$ and $\tilde{\sigma}$.

Thus the connection $\tilde{\nabla}$ on E makes natural sense from all three points of view. Note that we did not require any metric to define this induced connection.

Each fibre E_x of E contains T_xM as a subspace. Using now the metric on M we obtain a direct sum decomposition

$$E_x = T_xM \oplus (T_xM)^{\perp}. \tag{3.23}$$

The disjoint union of the orthogonal complements $\sqcup_{x\in M}(T_xM)^{\perp}$ forms a vector bundle of rank r over M called the **normal bundle of M in V**, denoted $N_{M/V}$. Exercise: verify that $N_{M/V}$ is indeed a well-defined vector bundle (recall Theorems 1.8 and 2.4).

For any two vector fields X, Y on M, we can decompose the covariant derivative $(\tilde{\nabla}_XY)_x = (\nabla_XY)_x + (h(X,Y))_x$, according to (3.23), where $h(X,Y)$ is some section of $N_{M/V}$. It turns out that ∇ is a well-defined covariant derivative (connection) on M and h is a bilinear map $T_xM \times T_xM \to (T_xM)^{\perp}$ (depending smoothly on x). Furthermore, in the case when $\tilde{\nabla} = \tilde{D}$ is the Levi–Civita connection on V we obtain.

Theorem 3.24 (Gauss formula)**.** *For any vector fields X, Y on M,*

$$\tilde{D}_XY = D_XY + II(X,Y),$$

where D is the Levi–Civita connection of the induced metric on M, and II is a symmetric bilinear map called the <u>second fundamental form</u> *of M in V.*

Theorem 3.25 (Weingarten formula)**.** *For any vector field X on M and section ξ of the normal bundle $N_{M/V}$,*

$$\tilde{D}_X\xi = -\mathcal{S}_\xi X + \nabla'_X\xi,$$

where for any ξ, $\mathcal{S}_\xi$ is a endomorphism of the vector bundle TM called the <u>shape operator</u> *and ∇' is a connection on $N_{M/V}$. Furthermore, the shape operator is symmetric with respect to the induced Riemannian metric M,*

$$g_M(\mathcal{S}_\xi X, Y) = g_M(X, \mathcal{S}_\xi Y) = g_V(II(X,Y), \xi),$$

for any vector field Y on M.

By direct application of the above, we can compute the Riemann curvature $R = (R_{ij,kl})$ of M in terms of the curvature of the ambient manifold and the second fundamental form.

Theorem 3.26 (Gauss)**.**

$$R(X,Y,Z,T) = \tilde{R}(X,Y,Z,T) + g_V(II(X,Z), II(Y,T)) - g_V(II(X,T), II(Y,Z)).$$

Corollary 3.27. *The curvature of a submanifold M of a* flat *manifold is determined by the second fundamental form of M.*

When M is a surface in the Euclidean $\mathbb{R}^3$, this is equivalent to *theorema egregium* discussed in the previous section.

3.5 Laplace–Beltrami operator

Throughout this section M is a connected oriented Riemannian manifold of dimension n. Let g denote a metric on M and let the orientation be given by a nowhere-zero n-form ε.

Starting from the vector fields $\frac{\partial}{\partial x^1}, \ldots \frac{\partial}{\partial x^n}$ at a point x in a coordinate neighbourhood U, we can apply Gram–Schmidt process with x as a parameter. Thus we obtain a new system of (smooth) vector fields $e_1, \ldots, e_n$ which give an orthonormal basis of tangent vectors on a perhaps smaller neighbourhood of p (still denote this neighbourhood by U). Let $\omega_1, \ldots, \omega_n$ on U be the dual 1-forms to $e_1, \ldots, e_n$, in the sense that

$$\omega_j(e_i) = \delta_{ij} \qquad \text{at any point in } U.$$

Then ω_j give at every point p of U a basis of T^*_pM, the dual basis to e_j.

The metric on M induces, for every $p = 0, \ldots, n$ an inner product on the bundle $\Lambda^p T^*M$ by making $\{\omega_{i_1}(x) \wedge \ldots \wedge \omega_{i_p}(x) : 1 \le i_1 < \ldots < i_p \le n\}$ an orthonormal basis of $\Lambda^p T^*_x M$.

If ω'_j is another system of local 1-forms, on another coordinate neighbourhood U' say, and ω'_j are orthonormal at every point in U' then

$$\omega'_1 \wedge \ldots \wedge \omega'_n = \det(\Phi)\, \omega_1 \wedge \ldots \wedge \omega_n \quad \text{on } U' \cap U,$$

for some *orthogonal* matrix Φ (depending on $x \in U' \cap U$). Assuming, as we can on an oriented M, that all the coordinate neighbourhoods are chosen so that the Jacobians $\det(\Phi)$ are positive on the overlaps, we find that $\omega_1 \wedge \ldots \wedge \omega_n$ is a well-defined nowhere-zero n-form ω_g on all of M. We can further ensure that $\omega_g = a\varepsilon$ for some positive function $a \in C^\infty(M)$. Then ω_g is called the **volume form** of M.

In (positively oriented) local coordinates, $\omega_g = \sqrt{\det g_{ij}} dx^1 \wedge \ldots \wedge dx^n$.

Definition. The **Hodge star** on M is a linear operator on the differential forms

$$* : \Lambda^p T^*_x M \to \Lambda^{n-p} T^*_x M,$$

such that for any two p-forms $\alpha, \beta \in \Lambda^p T^*_x M$ one has $\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g \, \omega_g(x)$, where ω_g is the volume form on M.

It follows that if ω_i is an orthonormal basis of a cotangent space $T^*_x M$ then necessarily $*(\omega_1 \wedge \ldots \wedge \omega_p) = \omega_{p+1} \wedge \ldots \wedge \omega_n$. In particular, $*1 = \omega_g$ and $*\omega_g = 1$. By permutations of indices and by linearity, the Hodge star is then uniquely determined for any differential form on M. Further, it follows that $** = (-1)^{p(n-p)}$ on the p-forms.

Using the Hodge star we construct a differential operator

$$\delta : \Omega^p(M) \to \Omega^{p-1}(M)$$

putting $\delta = (-1)^{n(p+1)+1} * d *$ if $p \ne 0$ and $\delta = 0$ on $\Omega^0(M) = C^\infty(M)$.

Definition. The **Laplace–Beltrami operator**, or **Laplacian**, on M is a linear differential operator $\Delta : \Omega^p(M) \to \Omega^p(M)$ given by

$$\Delta = \delta d + d\delta.$$

Straightforward computation shows that when M is the Euclidean $\mathbb{R}^n$ the definition gives $\Delta f = -\dfrac{\partial^2 f}{(\partial x^1)^2} - \ldots - \dfrac{\partial^2 f}{(\partial x^n)^2}$ for any smooth function f. For a general metric $g = (g_{ij})$, the local expression becomes $\Delta_g f = -\dfrac{1}{\sqrt{\det g}} \dfrac{\partial}{\partial x^j}\left(\sqrt{\det g}\, g^{ij}\, \dfrac{\partial f}{\partial x^i}\right)$.

Proposition 3.28. *The operator δ is the adjoint*[4] *of d in the sense that*

$$\int_M \langle d\alpha, \beta\rangle_g\, \omega_g = \int_M \langle \alpha, \delta\beta\rangle_g\, \omega_g,$$

for every compactly supported $\alpha \in \Omega^{p-1}(M), \beta \in \Omega^p(M)$.

Using the inner product on the spaces $\Lambda^p T^*_p M$, $p \in M$, we can define an inner product on $\Omega^p(M)$, called the L^2 *inner product*, by putting $\langle \alpha, \beta\rangle_{L^2} = \int_M \langle \alpha, \beta\rangle_g\, \omega_g$. The inner product makes each $\Omega^p(M)$ into a normed space, with L^2-norm defined by $\|\alpha\| = (\langle \alpha, \alpha\rangle_{L^2})^{1/2}$. In particular, $\alpha = 0$ if and only if $\|\alpha\| = 0$.

Thus Proposition 3.28 says that $\langle d\alpha, \beta\rangle_{L^2} = \langle \alpha, \delta\beta\rangle_{L^2}$ and, consequently, $\langle \Delta\alpha, \beta\rangle_{L^2} = \langle \alpha, \Delta\beta\rangle_{L^2}$. It follows immediately that the Laplace–Beltrami operator is self-adjoint.

A differential form $\alpha \in \Omega^p(M)$ is called **harmonic** if $\Delta\alpha = 0$.

Corollary 3.29. *Every harmonic differential form on a compact manifold is closed and co-closed: $\Delta\alpha = 0$ if and only if both $d\alpha = 0$ and $\delta\alpha = 0$.*

Proof. Integration by parts, $0 = \langle \delta d\alpha + d\delta\alpha, \alpha\rangle_{L^2} = \langle \delta\alpha, \delta\alpha\rangle_{L^2} + \langle d\alpha, d\alpha\rangle_{L^2}$. □

It is also easily checked that $*\Delta = \Delta *$ on any $\Omega^p(M)$. Therefore the Hodge star of any harmonic form is again harmonic.

Hodge Decomposition Theorem. *Let M be a compact oriented Riemannian manifold. For every $0 \le p \le \dim M$, the space $\mathcal{H}^p$ of harmonic p-forms is finite-dimensional. Furthermore, there are L^2-orthogonal direct sum decompositions*

$$\begin{aligned}\Omega^p(M) &= \Delta\Omega^p(M) \oplus \mathcal{H}^p \\ &= d\delta\Omega^p(M) \oplus \delta d\Omega^p(M) \oplus \mathcal{H}^p \\ &= d\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M) \oplus \mathcal{H}^p\end{aligned}$$

(where we formally put $\Omega^{-1}(M) = \{0\}$).

Remark: the compactness condition on M cannot be removed.[5]

[4] It is more correct to say that δ is the 'formal adjoint' of d for reasons that have to do with the Analysis.
[5] The reason is that certain results in Analysis fail on non-compact sets, but this is another story.

Short summary of the proof. We need to introduce the concept of a **weak solution** of

$$\Delta\omega = \alpha. \tag{3.30}$$

A weak solution of (3.30) is by definition, a linear functional $l : \Omega^p(M) \to \mathbb{R}$ which is
(i) bounded, $|l(\beta)| \le C\|\beta\|$, for some $C > 0$ independent of β, and
(ii) satisfies $l(\Delta\varphi) = \langle \alpha, \varphi\rangle_{L^2}$.

Any solution ω of (3.30) defines a weak solution by putting $l_\omega(\beta) = \langle \omega, \beta\rangle_{L^2}$.

The proof of Hodge Decomposition Theorem requires some results from Functional Analysis.

Regularity Theorem. *Any weak solution l of* (3.30) *is of the form $l(\beta) = \langle \omega, \beta\rangle_{L^2}$, for some $\omega \in \Omega^p(M)$ (and hence defines a solution of* (3.30)*).*

Compactness Theorem. *Assume that a sequence $\alpha_n \in \Omega^p(M)$ satisfies $\|\alpha_n\| < C$ and $\|\Delta\alpha_n\| < C$, for some C independent of n. Then α_n contains a Cauchy subsequence.*

We shall assume the above two theorems (and the Hahn–Banach theorem below) without proof.

Compactness Theorem implies at once that $\mathcal{H}^p$ must be finite-dimensional (for, otherwise, there would exist an infinite orthonormal sequence of forms). As $\mathcal{H}^p$ is finite-dimensional, we can write an L^2-orthogonal decomposition $\Omega^p(M) = \mathcal{H}^p \oplus (\mathcal{H}^p)^\perp$.

It is easy to see that $\Delta\Omega^p(M) \subseteq (\mathcal{H}^p)^\perp$ (use Proposition 3.28). For the reverse inclusion, suppose that $\alpha \in (\mathcal{H}^p)^\perp$. We want to show that the equation (3.30) has a solution. Assuming the Regularity Theorem, we shall be done if we obtain a *weak* solution $l : \Omega^p(M) \to \mathbb{R}$ of (3.30).

Define l first on a subspace $\Delta\Omega^p(M)$, by putting $l(\Delta\eta) = \langle \eta, \alpha\rangle_{L^2}$. It is not hard to check that l is well-defined. Further, (ii) is automatically satisfied (on this subspace); we claim that (i) holds too. To verify the latter claim, we show that l is *bounded below* on $\Delta\Omega^p(M)$ using, once again, the Compactness Theorem.

In order to extend l to all of $\Omega^p(M)$, we appeal to

Hahn–Banach Theorem. *Suppose that L is a normed vector space, and L_0 a subspace of L, and $l : L_0 \to \mathbb{R}$ a linear functional satisfying $l(x_0) < \|x_0\|$, for all $x_0 \in L_0$. Then l extends to a linear functional on L with $l(x) < \|x\|$ for all $x \in L$.*

Thus we obtain a weak solution of (3.30) and deduce that $\Omega^p(M) = \Delta\Omega^p(M) \oplus \mathcal{H}^p$ as desired. The two other versions of the L^2-orthogonal decomposition of $\Omega^p(M)$ follow readily by application of Proposition 3.28. □

Corollary 3.31. *Every de Rham cohomology class $a \in H^r(M)$ of a compact oriented Riemannian manifold M is represented by a unique harmonic differential r-form $\alpha \in \Omega^r(M)$, $[\alpha] = a$. Thus $\mathcal{H}^r \cong H^r(M)$.*

Proof. Uniqueness. If α_1, α_2 are harmonic p-forms and $\alpha_1 - \alpha_2 = d\beta$ then $\|d\beta\|^2 = \langle d\beta, \alpha_1 - \alpha_2\rangle_{L^2} = \langle \beta, \delta(\alpha_1 - \alpha_2)\rangle_{L^2} = 0$.

Existence. If α is such that $d\delta\alpha = 0$ then $\|\delta\alpha\| = 0$. Hence any closed p-form must be in $d\Omega^{p-1}(M) \oplus \mathcal{H}^p$. □

Corollary 3.31 is a surprising result: an *analytical* object (harmonic forms) turns out to be equivalent to a *topological* object (de Rham cohomology) via some *differential geometry*. Here is a way to see 'why such a result can be true'.

A de Rham cohomology class, $a \in H^r(M)$ say, can be represented by many differential forms; consider the (infinite-dimensional) affine space

$$\begin{aligned} B_a &= \{\xi \in \Omega^r(M) \mid d\xi = 0,\ [\xi] = a \in H^r(M)\} \\ &= \{\xi \in \Omega^r(M) | \xi = \alpha + d\beta, \text{ for some } \beta \in \Omega^{r-1}(M)\}. \end{aligned}$$

When does a closed form α have the *smallest* L^2-norm amongst all the closed forms in a given de Rham cohomology class B_a?

Such a form must be a critical point of the function $F(\alpha + d\beta) = \|\alpha + d\beta)\|^2$ on B_a, so the partial derivatives of F in any direction should vanish. That is, we must have

$$0 = \frac{d}{dt}\bigg|_{t=0} \langle \alpha + t\, d\beta, \alpha + t\, d\beta \rangle_{L^2} = 2\langle \alpha, d\beta \rangle_{L^2}.$$

Integrating by parts, we find that $\int_M \langle \delta\alpha, \beta \rangle_g = 0$ must hold for every $\beta \in \Omega^{r-1}(M)$. This forces $\delta\alpha = 0$, and so the extremal points of F are precisely the harmonic forms α.

Page references for Chapter 3

all to Gallot–Hulin–Lafontaine except * to Warner

Riemannian metrics — pp.52–53,

Levi–Civita connection — pp.69–71,

curvature of a Riemannian manifold — pp.107–108,111–112,155–156

geodesics, Gauss Lemma — pp.80–85,89–90,

Riemannian submanifolds — pp.217–220,

*the Laplacian and the Hodge Decomposition Theorem — [W], pp. 140–141,220–226.