Part III: Differential geometry (Michaelmas 2004)

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The line and surface integrals studied in vector calculus require a concept of a curve and a surface in Euclidean 3-space. These latter objects are introduced via a *parameterization*: a smooth map of, respectively, an interval on the real line $\mathbb{R}$ or a domain in the plane $\mathbb{R}^2$ into $\mathbb{R}^3$. In fact, one often requires a so-called *regular parameterization*. For a curve $\mathbf{r}(t)$, this means non-vanishing of the 'velocity vector' at any point, $\dot{\mathbf{r}}(t) \neq 0$. On a surface a point depends on two parameters $\mathbf{r} = \mathbf{r}(u, v)$ and regular parameterization means that the two vectors of partial derivatives in these parameters are linearly independent at every point $\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v) \neq 0$.

Differential Geometry develops a more general concept of a smooth n-dimensional *differentiable manifold*.[1] and a systematic way to do differential and integral calculus (and more) on manifolds. Curves and surfaces are examples of manifolds of dimension $d = 1$ and $d = 2$ respectively. However, in general a manifold need not be given (or considered) as lying in some ambient Euclidean space.

1.1 Manifolds: definitions and first examples

The basic idea of smooth manifolds, of dimension d say, is to introduce a class of spaces which are 'locally modelled' (in some precise sense) on a d-dimensional Euclidean space $\mathbb{R}^d$.

A good way to start is to have a notion of open subsets (sometimes one says 'to have a topology') on a given set of points (but see Remark on page 2).

Definition.
1. A **topological space** is a set, M say, with a specified class of **open subsets**, or **neighbourhoods**, such that
   (i) $\emptyset$ and M are open;
   (ii) the intersection of any two open sets is open;
   (iii) the union of any number of open sets is open.
2. A topological space M is called **Hausdorff** if any two points of M possess non-intersecting neighbourhoods.
3. A topological space M is called **second countable** if one can find a countable collection $\mathcal{B}$ of open subsets of M so that any open $U \subset M$ can be written as a union of sets from $\mathcal{B}$.

The last two parts of the above definition will be needed to avoid some pathological examples (see below).

[1] More precisely, it is often useful to also consider appropriate 'structures' on manifolds (e.g. Riemannian metrics), as we shall see in due course.

The knowledge of open subsets enables one to speak of *continuous maps*: a map between topological spaces is continuous if the inverse image of any open set is open. Exercise: check that for maps between open subsets (in the usual sense) of Euclidean spaces this definition is equivalent to other definitions of a continuous map.

A *homeomorphism* is a bijective continuous map with continuous inverse. More explicitly, to say that 'a bijective mapping φ of U onto V is a homeomorphism' means that '$D \subset U$ is open if and only if $\varphi(D) \subset V$ is open'.

Let M be a topological space. A homeomorphism $\varphi : U \to V$ of an open set $U \subseteq M$ onto an open set $V \subseteq \mathbb{R}^d$ will be called a **local coordinate chart** (or just 'a chart') and U is then a **coordinate neighbourhood** (or 'a coordinate patch') in M.

Definition. A C^∞ **differentiable structure**, or smooth structure, on M is a collection of coordinate charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^d$ (same d for all α's) such that

(i) $M = \cup_{\alpha \in A} U_\alpha$;

(ii) any two charts are 'compatible': for every α, β the change of local coordinates $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth (C^∞) map on its domain of definition, i.e. on $\varphi_\alpha(U_\beta \cap U_\alpha) \subseteq \mathbb{R}^d$.

(iii) the collection of charts φ_α is maximal with respect to the property (ii): if a chart φ of M is compatible with all φ_α then φ is included in the collection.

A bijective smooth map with a smooth inverse is called a *diffeomorphism*. Notice that clause (ii) in the above definition implies that any change of local coordinates is a diffeomorphism between open sets $\varphi_\alpha(U_\beta \cap U_\alpha)$ and $\varphi_\beta(U_\beta \cap U_\alpha)$ of $\mathbb{R}^d$.

In practice, one only needs to worry about the first two conditions in the above definition. Given a collection of compatible charts covering M, i.e. satisfying (i) and (ii), there is a unique way to extend it to a maximal collection to satisfy (iii). I leave this last claim without proof but refer to Warner, p. 6.

Definition . A topological space equipped with a C^∞ differential structure is called a **smooth manifold**. Then d is called the dimension of M, $d = \dim M$.

Sometimes in the practical examples one starts with a differential structure on a set of points M (with charts being bijective maps onto open sets in $\mathbb{R}^d$) and then *defines* the open sets in M to be precisely those making the charts into homeomorphisms. More explicitly, one then says that $D \subset M$ is open if and only if for every chart $\varphi : U \to V \subseteq \mathbb{R}^d$, $\varphi(D \cap U)$ is open in $\mathbb{R}^d$. (For this to be well-defined, every finite intersection of coordinate neighbourhoods must have an open image in $\mathbb{R}^d$ under some chart.) We shall refer to this as the **topology induced by a C^∞ structure**.

Remarks . 1. Some variations of the definition of the differentiable structure are possible. Much of the material in these lectures could be adapted to C^k rather than C^∞ differentiable manifolds, for any integer $k > 0$. [2]

On the other hand, replacing $\mathbb{R}^d$ with the complex coordinate space $\mathbb{C}^n$ and smooth maps with holomorphic (complex analytic) maps between domains in $\mathbb{C}$, leads of an important special class of complex manifolds—but that is another story.

[2] if $k = 0$ then the definition of differentiable structure has no content

By a manifold I will always mean in these lectures a smooth (real) manifold, unless explicitly stated otherwise.

2. Here is an example of what can happen if one omits a Hausdorff property. Consider the following 'line with a double point'

$$M = (-\infty, 0) \cup \{0', 0''\} \cup (0, \infty),$$

with two charts being the obvious 'identity' maps (the induced topology is assumed) $\varphi_1 : U_1 = (-\infty, 0) \cup \{0'\} \cup (0, \infty) \to \mathbb{R}$, $\varphi_2 : U_2 = (-\infty, 0) \cup \{0''\} \cup (0, \infty) \to \mathbb{R}$, so $\varphi_1(0') = \varphi_1(0'') = 0$. It is not difficult to check that M satisfies all the conditions of a smooth manifold, except for the Hausdorff property ($0'$ and $0''$ cannot be separated).

Omitting the 2nd countable property would allow, e.g. an uncountable collection (disjoint union) of lines

$$\sqcup_{0<\alpha<1} \mathbb{R}_\alpha,$$

each line $\mathbb{R}_\alpha$ equipped with the usual topology and charts being the identity maps $\mathbb{R}_\alpha \to \mathbb{R}$. [3]

Examples. The Euclidean $\mathbb{R}^d$ is made into a manifold using the identity chart. The complex coordinate space $\mathbb{C}^n$ becomes a $2n$-dimensional manifold via the chart $\mathbb{C}^n \to \mathbb{R}^{2n}$ replacing every complex coordinate z_j by a pair of real coordinates $\operatorname{Re} z_j, \operatorname{Im} z_j$.

The sphere $S^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i^2 = 1\}$ is made into a smooth manifold of dimension n, by means of the two stereographic projections onto $\mathbb{R}^n \cong \{x \in \mathbb{R}^{n+1} : x_0 = 0\}$, from the North and the South poles $(\pm 1, 0, \ldots, 0)$. The corresponding change of coordinates is given by $(x_1, \ldots, x_n) \mapsto (x_1/|x|^2, \ldots, x_n/|x|^2)$.

The real projective space $\mathbb{R}P^n$ is the set of all lines in $\mathbb{R}^{n+1}$ passing through 0. Elements of $\mathbb{R}P^n$ are denoted by $x_0 : x_1 : \ldots : x_n$, where not all x_i are zero. Charts can be given by $\varphi_i(x_0 : x_1 : \ldots : x_n) = (x_0/x_i, \ldots \hat{i} \ldots, x_n/x_i) \in \mathbb{R}^n$ with changes of coordinates given by

$$\varphi_j \circ \varphi_i^{-1} : (y_1, \ldots, y_n) \mapsto y_1 : \ldots : (1 \text{ in } i\text{th place}) : \ldots : y_n \in \mathbb{R}P^n \mapsto \left(\frac{y_1}{y_j}, \ldots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j} \ldots \frac{y_n}{y_j}\right),$$

smooth functions on their domains of definition (i.e. for $y_j \neq 0$). Thus $\mathbb{R}P^n$ is a smooth n-dimensional manifold.

Definition. Let M, N be smooth manifolds. A continuous map $f : M \to N$ is called **smooth** (C^∞) if for each $p \in M$, for some (hence for every) charts φ and ψ, of M and N respectively, with p in the domain of φ and $f(p)$ in the domain of ψ, the composition $\psi \circ f \circ \varphi^{-1}$ (which is a map between open sets in $\mathbb{R}^n$, $\mathbb{R}^k$, where $n = \dim M$, $k = \dim N$) is smooth on its domain of definition.

Exercise: write out the domain of definition for $\psi \circ f \circ \varphi^{-1}$.

Two manifolds M and N are called **diffeomorphic** is there exists a smooth bijective map $M \to N$ having smooth inverse. Informally, diffeomorphic manifolds can be thought of as 'the same'.

[3] For a more interesting example on a 'non 2nd countable manifold' see Example Sheet Q1.12.

1.2 Matrix Lie groups

Consider the general linear group $GL(n,\mathbb{R})$ consisting of all the $n \times n$ real matrices A satisfying $\det A \neq 0$. The function $A \mapsto \det A$ is continuous and $GL(n,\mathbb{R})$ is the inverse image of the open set $\mathbb{R} \setminus \{0\}$, so it is an open subset in the n^2-dimensional linear space of all the $n \times n$ real matrices. Thus $GL(n,\mathbb{R})$ is a manifold of dimension n^2. Note that the result of multiplication or taking the inverse depends smoothly on the matrix entries.

Similarly, $GL(n,\mathbb{C})$ is a manifold of dimension $2n^2$ (over $\mathbb{R}$).

Definition. A group G is called a **Lie group** if it is a smooth manifold and the map $(\sigma,\tau) \in G \times G \to \sigma\tau^{-1} \in G$ is smooth.

Let A be an $n \times n$ complex matrix. The norm given by $|A| = n \max_{ij} |a_{ij}|$ has a useful property that $|AB| \leq |A||B|$ for any A, B. The exponential map on the matrices is defined by

$$\exp(A) = I + A + A^2/2! + \ldots + A^n/n! + \ldots.$$

The series converge absolutely and uniformly on any set $\{|A| \leq \mu\}$, by the Weierstrass M-test. It follows that e.g. $\exp(A^{\mathrm{t}}) = (\exp(A))^{\mathrm{t}}$ and $\exp(C^{-1}AC) = C^{-1}\exp(A)C$, for any invertible matrix C. Furthermore, the term-by-term differentiated series also converge uniformly and so $\exp(A)$ is C^∞-smooth in A. (This means smooth as a function of $2n^2$ real variables, the entries of A.)

The logarithmic series

$$\log(I + A) = A - A^2/2 + \ldots + (-1)^{n+1}A^n/n + \ldots$$

converge absolutely for $|A| < 1$ and uniformly on any closed subset $\{|A| \leq \varepsilon\}$, for $\varepsilon < 1$, and $\log(A)$ is smooth in A.

One has

$$\exp(\log(A)) = A, \qquad \text{when } |A - I| < 1. \tag{1.1}$$

This is true in the formal sense of composing the two series in the left-hand side. The formal computations are valid in this case as the double-indexed series in the left-hand side is *absolutely convergent*.

For the other composition, one has

$$\log(\exp(A)) = A \qquad \text{when } |A| < \log 2. \tag{1.2}$$

again by considering a composition of power series with a similar reasoning.

Remark. Handling the power series of complex matrices in (1.1) and (1.2) is quite similar to handling 1×1 matrices, i.e. complex numbers. Warning: not all the usual properties carry over wholesale, as the multiplication of matrices is not commutative. E.g., in general, $\exp(A)\exp(B) \neq \exp(A + B)$. However, the identity $\exp(A)\exp(-A) = I$ does hold (and this is used in the proof of Proposition 1.3 below).

Proposition 1.3. *The orthogonal group $O(n) = \{A \in GL(n, \mathbb{R}) : AA^t = I\}$ has a smooth structure making it into a manifold of dimension $n(n-1)/2$.*

The charts take values in the $\frac{n(n-1)}{2}$-dimensional linear space of skew-symmetric $n \times n$ real matrices. E.g.

$$\varphi : A \in \{A \in O(n) : |A - I| \text{ is small } \} \mapsto B = \log(A) \in \{B : B^t = -B\}$$

is a chart in a neighbourhood of $I \in O(n)$. The desired smooth structure is generated by a family of charts of the form $\varphi_C(A) = \log(C^{-1}A)$, where $C \in O(n)$.

The method of proof of Proposition 1.3 is not specific to the orthogonal matrices and works for many other subgroups of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ (Example Sheet 1, Question 4).

1.3 Tangent space to a manifold

If $x(t)$ is a smooth regular curve in $\mathbb{R}^n$ then the velocity vector $\dot{x}(0)$ is a tangent vector to this curve at $t = 0$. In a change of coordinates $x'_i = x'_i(x)$ the coordinates of this vector are transformed according to the familiar chain rule, applied to $x'(x(t))$. One consequence is that the statement "two curves pass through the same point with the same tangent vector" is independent of the choice of coordinates, that is to say a tangent vector (understood as velocity vector) is a *geometric* object.

The above observation is *local* (depends only on what happens in a neighbourhood of a point of interest), therefore the definition may be extended to an arbitrary smooth manifold.

Definition. A tangent vector to a manifold M at a point $p \in M$ is a map a assigning to each chart (U, φ) with $p \in U$ an element in the coordinate space $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ in such a way that if (U', φ') is another chart then

$$a'_i = \left(\frac{\partial x'_i}{\partial x_j}\right)_p a_j, \tag{1.4}$$

where x_i, x'_i are the local coordinates on U, U' respectively. [4] All the tangent vectors at a given point p form the **tangent space** denoted T_pM.

Remarks. It is easy to check that T_pM is naturally a *vector space.*
The transformation law (1.4) is the *defining property* of a tangent vector.

Notation. A choice local coordinates x_i on a neighbourhood $U \subseteq M$ defines a linear isomorphism $T_pM \to \mathbb{R}^n$. A basis of T_pM corresponding to the standard basis of $\mathbb{R}^n$ via this isomorphism is a usually denoted by $(\frac{\partial}{\partial x_i})_p$. The expression of a tangent vector in local coordinates $a(U, \varphi) = (a_1, \dots, a_n)$ then becomes $a_i(\frac{\partial}{\partial x_i})_p$.

[4] Here and below a *convention* is used that if the same letter appears as an upper and lower index then the *summation* is performed over the range of this index. E.g. the summation in $j = 1, \dots, n$ in this instance.

As can be seen from (1.4), the standard basis vectors of T_pM given by the local coordinates x_i and x'_i are related by

$$\left(\frac{\partial}{\partial x_i}\right)_p = \left(\frac{\partial x'_j}{\partial x_i}\right)_p \left(\frac{\partial}{\partial x'_j}\right)_p. \tag{1.4$'$}$$

This is what one would expect in view of the chain rule from the calculus. The formula (1.4′) tells us that every tangent vector $a_i(\frac{\partial}{\partial x_i})_p$ gives a well-defined first-order 'derivation'

$$a_i(\frac{\partial}{\partial x_i})_p : f \in C^\infty(M) \to a_i \frac{\partial f}{\partial x_i}(p) \in \mathbb{R}. \tag{1.5}$$

In fact the following converse statement is true although I shall not prove it here[5]. Given $p \in M$, every linear map $a : C^\infty(M) \to \mathbb{R}$ satisfying Leibniz rule $a(fg) = a(f)g(p) + f(p)a(g)$ arises from a tangent vector as in (1.5).

Example 1.6. Let $\mathbf{r} = \mathbf{r}(u, v)$, $(u, v) \in U \subseteq \mathbb{R}^2$ be a smooth regular-parameterized surface in $\mathbb{R}^3$. Examples of tangent vectors are the partial derivatives $\mathbf{r}_u$, $\mathbf{r}_v$ —these correspond to just $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ in the above notation, as the parameterization by u, v is an instance of a coordinate chart.

Definition. A vector space with a multiplication $[\cdot, \cdot]$, bilinear in its arguments (thus satisfying the distributive law), is called a **Lie algebra** if the multiplication is anti-commutative $[a, b] = -[b, a]$ and satisfies the Jacobi identity $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$.

Theorem 1.7 (The Lie algebra of a Lie group)**.** *Let G be a Lie group of matrices and suppose that* log *defines a coordinate chart near the identity element of G. Identify the tangent space $\mathfrak{g} = T_IG$ at the identity element with a linear subspace of matrices, via the* log *chart, and then $\mathfrak{g}$ is a Lie algebra with $[B_1, B_2] = B_1B_2 - B_2B_1$.*

The space $\mathfrak{g}$ is called the Lie algebra of G.

Proof. It suffices to show that for every two matrices $B_1, B_2 \in \mathfrak{g}$, the $[B_1, B_2]$ is also an element of $\mathfrak{g}$. As $[B_1, B_2]$ is clearly anticommutative and the Jacobi identity holds for matrices, $\mathfrak{g}$ will then be a Lie algebra.

The expression

$$A(t) = \exp(B_1t) \exp(B_2t) \exp(-B_1t) \exp(-B_2t)$$

defines, for $|t| < \varepsilon$ with sufficiently small ε, a path $A(t)$ in G such that $A(0) = I$. Using for each factor the local formula $\exp(Bt) = I + Bt + \frac{1}{2}B^2t^2 + o(t^2)$, as $t \to 0$, [6] we obtain

$$A(t) = I + [B_1, B_2]t^2 + o(t^2), \qquad \text{as } t \to 0.$$

[5] A proof can be found in Warner 1.14-1.20. Note that his argument requires C^∞ manifolds (and does not extend to C^k).

[6] The notation $o(t^k)$ means a remainder term of order higher than k, i.e. $r(t)$ such that $\lim_{t\to 0} r(t)/t^k = 0$.

Hence

$$B(t) = \log A(t) = [B_1, B_2]t^2 + o(t^2) \qquad \text{and} \qquad \exp(B(t) = A(t)$$

hold for any *sufficiently small* $|t|$ and so $B(t) \in \mathfrak{g}$ (as $B(t)$ is in the image of the log chart). Hence $B(t)/t^2 \in \mathfrak{g}$ for every small $t \neq 0$ (as $\mathfrak{g}$ is a vector space). But then also $\lim_{t\to 0} B(t)/t^2 = \lim_{t\to 0}([B_1, B_2] + o(1)) = [B_1, B_2] \in \mathfrak{g}$ (as $\mathfrak{g}$ is a closed subset of matrices). □

Notice that the idea behind the above proof is that the Lie bracket $[B_1, B_2]$ on a Lie algebra $\mathfrak{g}$ is an 'infinitesimal version' of the commutator $g_1 g_2 g_1^{-1} g_2^{-1}$ in the corresponding Lie group G.

Definition. Let M be a smooth manifold. A disjoint union $TM = \sqcup_{p\in M} T_pM$ is called the **tangent bundle** of M.

Theorem 1.8 (The 'manifold of tangent vectors')**.** *The tangent bundle TM has a canonical differentiable structure making it into a smooth $2n$-dimensional manifold, where $n = \dim M$.*

The charts identify any $\sqcup_{p\in U} T_pM \subseteq TM$, for an coordinate neighbourhood $U \subseteq M$, with $U \times \mathbb{R}^n$. [7] Exercise: check that TM is Hausdorff and second countable (if M is so).

Definition. A (smooth) **vector field** on a manifold M is a map $X : M \to TM$, such that
(i) $X(p) \in T_pM$ for every $p \in M$, and
(ii) in every chart, X is expressed as $a_i(x)\frac{\partial}{\partial x_i}$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i.

Theorem 1.9. *Suppose that on a smooth manifold M of dimension n there exist n vector fields $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$, such that $X^{(1)}(p), X^{(2)}(p), \ldots, X^{(n)}(p)$ form a basis of T_pM at every point p of M. Then TM is isomorphic to $M \times \mathbb{R}^n$.*

Here 'isomorphic' means that TM and $M \times \mathbb{R}^n$ are diffeomorphic as smooth manifolds and for every $p \in M$, the diffeomorphism restricts to an isomorphism between the tangent space T_pM and vector space $\{p\} \times \mathbb{R}^n$. (Later we shall make a more systematic definition including this situation as a special case.)

Proof. Define $\pi : \vec{a} \in TM \to p \in M$ if $\vec{a} \in T_pM \subset TM$. On the other hand, for any $\vec{a} \in TM$, there is a unique way to write $\vec{a} = a_i X^{(i)}$, for some $a_i \in \mathbb{R}$. Now define

$$\Phi : \vec{a} \in TM \to (\pi(\vec{a}); a_1, \ldots, a_n) \in M \times \mathbb{R}^n.$$

It is clear from the construction and the hypotheses of the theorem that Φ is a bijection and Φ converts every tangent space into a copy of $\mathbb{R}^n$. It remains to show that Φ and Φ^{-1} are smooth.

[7] The topology on TM is induced from the smooth structure.

Using an arbitrary chart $\varphi : U \subseteq M \to \mathbb{R}^n$, and the corresponding chart $\varphi_T : \pi^{-1}(U) \subseteq TM \to \mathbb{R}^n \times \mathbb{R}^n$ one can locally express Φ as

$$(\varphi, \mathrm{id}_{\mathbb{R}^n}) \circ \Phi \circ \varphi_T^{-1} : (x, (b_j)) \in \varphi(U) \times \mathbb{R}^n \to (x, (a_i)) \in \varphi(U) \times \mathbb{R}^n,$$

where x_i are local coordinates on U, and $\vec{a} = b_j \frac{\partial}{\partial x_j}$. Writing $X^{(i)} = X_j^{(i)}(x)\frac{\partial}{\partial x_j}$, we obtain $b_j = a_i X_j^{(i)}(x)$, which shows that Φ^{-1} is smooth. The matrix $X(x) = (X_j^{(i)}(x))$ expresses a *change of basis* of T_pM, from $(\frac{\partial}{\partial x_j})_p$ to $X^{(j)}(p)$, and is smooth in x, so the inverse matrix $C(x)$ is smooth in x too. Therefore $a_i = b_j C_i^j(x)$ verifies that Φ is smooth. □

Remark. The hypothesis of Theorem 1.9 is rather restrictive. In general, a manifold need not admit any non-vanishing smooth (or even continuous) vector fields at all (as we shall see, this is the case for any even-dimensional sphere S^{2n}) and the tangent bundle TM will not be a product $M \times \mathbb{R}^n$.

Definition. The differential of a smooth map $F : M \to N$ at a point $p \in M$ is a linear map

$$(dF)_p : T_pM \to T_{F(p)}N$$

given in any charts by $(dF)_p : (\frac{\partial}{\partial x_i})_p \mapsto (\frac{\partial y_j}{\partial x_i})(p)\ (\frac{\partial}{\partial y_j})_{F(p)}$. Here x_i are local coordinates on M, y_j on N, defined respectively by charts φ around p and ψ around $F(p)$, and $y_j = F_j(x_1, \dots, x_n)$ for $j = 1, \dots, \dim N$ $(n = \dim M)$ is the expression $\psi \circ F \circ \varphi^{-1}$ for F in these local coordinates.

It follows, by direct calculation in local coordinates using (1.4′), that the differential is independent of the choice of charts and that the chain rule $d(F_2 \circ F_1)_p = (dF_2)_{F_1(p)} \circ (dF_1)_p$ holds.

Every (smooth) vector field, say X on M, defines a linear differential operator of first order $X : C^\infty(M) \to C^\infty(M)$, according to (1.5) (with p allowed to vary in M). Suppose that $F : M \to N$ is a diffeomorphism. Then for each vector field X on M, $(dF)X$ is a well-defined vector field on N. For any $f \in C^\infty(N)$, the chain rule in local coordinates $\frac{\partial y_j}{\partial x_i}(p)\ \frac{\partial}{\partial y_j}\big|_{F(p)} f = \frac{\partial}{\partial x_i}\big|_p (f \circ F)$ $\quad$ (x_i on M and y_j on N) yields a coordinate-free relation

$$\big(((dF)X)f\big) \circ F = X(f \circ F), \tag{1.10}$$

Let X, Y be vector fields regarded as differential operators on $C^\infty(M)$. Then $[X, Y] = XY - YX$ defines a vector field: its local expression is $(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j})\frac{\partial}{\partial x_i}$. Direct calculation shows that $[\cdot, \cdot]$ satisfies the Jacobi identity and so the space $V(M)$ of all (smooth) vector fields on a manifold is an infinite-dimensional *Lie algebra*.

Left-invariant vector fields

Let G be a Lie group, $e \in G$ the identity element, and denote $\mathfrak{g} = T_eG$. The group operations can be used to construct non-vanishing vector fields on G as follows.

For every $g \in G$, the left translation $L_g : h \in G \to gh \in G$ is a diffeomorphism of G. Let $\xi \in \mathfrak{g}$ be a non-zero element. Define

$$X_\xi : g \in G \to (dL_g)_e \xi \in T_g G \subset TG. \tag{1.11}$$

Then $X_\xi \neq 0$ at any point $g \in G$, for any $\xi \neq 0$, because the linear map $(dL_g)_e$ is invertible. Furthermore, X_ξ is a well-defined smooth vector field on G.

To verify the latter claim we consider the smooth map $L : (g, h) \in G \times G \to gh \in G$, using *local coordinate charts* $\varphi_{g_0} : U_{g_0} \to \mathbb{R}^m$, $\varphi_e : U_e \to \mathbb{R}^m$ defined near $g_0, e \in G$, respectively. Here $m = \dim G$. The local expression of L near (g_0, e) via these charts, $L_{\text{loc}} = \varphi_{g_0} \circ L \circ (\varphi_{g_0}^{-1}, \varphi_e^{-1})$, is a smooth map $L_{\text{loc}} : U_{g_0} \times U_e \to U_{g_0}$. Now the local expression for $(dL_g)_e$ is just the partial derivative map $D_2 L_{\text{loc}}$, linearizing L_{loc} in the second m-tuple of variables. It is clearly smooth in the first m variables, and so $(dL_g)_e \xi$ depends smoothly on g.

To sum up, we have proved

Proposition 1.12. *If $\xi_1, \dots, \xi_m$ is a basis of the vector space $\mathfrak{g}$ then $X_{\xi_1}(h), \dots, X_{\xi_m}(h)$ define $m = \dim G$ vector fields whose values at each $h \in G$ give a basis of $T_h G$.*

Hence, in view of Theorem 1.9, we obtain

Theorem 1.13. *The tangent bundle TG of any Lie group G is isomorphic to the product $G \times \mathbb{R}^{\dim G}$.*

The smoothness of X_ξ and (1.11) together imply that $(dL_g)_h X_\xi(h) = X_\xi(gh)$, i.e.

$$(dL_g) X_\xi = X_\xi \circ L_g. \tag{1.14}$$

and we shall call any vector field satisfying (1.14) **left-invariant**.

It follows from Proposition 1.12 that the vector space $l(G)$ of all the left-invariant vector fields on G form a finite-dimensional subspace (of dimension m) in the space $V(G)$ of all vector fields. In fact more is true.

Theorem 1.15. *The space of all the left-invariant vector fields on G is a finite-dimensional Lie algebra, hence a Lie subalgebra of $V(G)$.*

Proof. The theorem may be restated as saying that for every pair X_ξ, X_η of left-invariant vector fields, $[X_\xi, X_\eta]$ is again left-invariant. To this end, we calculate using (1.10)

$$\begin{aligned}
(dL_g)[X_\xi, X_\eta] f \circ L_g &= [X_\xi, X_\eta](f \circ L_g) = X_\xi X_\eta (f \circ L_g) - X_\eta X_\xi (f \circ L_g) \\
&= X_\xi\big((dL_g) X_\eta f \circ L_g\big) - X_\eta\big((dL_g) X_\xi f \circ L_g\big) \\
&= (dL_g) X_\xi ((dL_g) X_\eta f) \circ L_g - (dL_g) X_\eta ((dL_g) X_\xi f) \circ L_g \\
&= [(dL_g) X_\xi, (dL_g) X_\eta] f \circ L_g
\end{aligned}$$

and using the left-invariant property (1.14) of X_ξ and X_η for the next step

$$= [X_\xi \circ L_g, X_\eta \circ L_g] f \circ L_g = ([X_\xi, X_\eta] \circ L_g) f \circ L_g.$$

Thus $(dL_g)[X_\xi, X_\eta] f \circ L_g = ([X_\xi, X_\eta] \circ L_g) f \circ L_g$, for each $g \in G$, $f \in C^\infty(G)$, so the vector field $[X_\xi, X_\eta]$ satisfies (1.14) and is left-invariant. $\square$

It follows that $[X_\xi, X_\eta] = X_\zeta$ for some $\zeta \in \mathfrak{g}$, thus the Lie bracket on $l(G)$ induces one on $\mathfrak{g}$. We can identify this Lie bracket more explicitly for the matrix Lie groups.

Theorem 1.16. *If G is a matrix Lie group, then the map $\xi \in \mathfrak{g} \to X_\xi \in l(G)$ is an isomorphism of the Lie algebras, where the Lie bracket on $\mathfrak{g}$ is as defined in Theorem 1.7.*

We shall prove Theorem 1.16 in the next section.

1.4 Submanifolds

Suppose that M is a manifold, $N \subset M$, and N is itself a manifold, denote by $\iota : N \to M$ the inclusion map.

Definition. N is said to be an **embedded submanifold**[8] of M if

(i) the map ι is smooth;

(ii) the differential $(d\iota)_p$ at any point of $p \in N$ is an injective linear map;

(iii) ι is a homeomorphism onto its image, i.e. a $D \subseteq N$ is open in the topology of manifold N if and only if D is open in the topology induced on N from M (i.e. the open subsets in N are precisely the intersections with N of the open subsets in M).

Remark. Often the manifold N is not given as a subset of M but can be identified with a subset of M by means of an *injective* map $\psi : N \to M$. In this situation, the conditions in the above definition make sense for $\psi(N)$ (regarded as a manifold diffeomorphic to N). If these conditions hold for $\psi(N)$ then one says that the map ψ *embeds* N in M and writes $\psi : N \hookrightarrow M$.

Example. A basic example of embedded submanifold is a (parameterized) curve or surface in $\mathbb{R}^3$. Then the condition (i) means that the parameterization is smooth and (ii) means that the parameterization is regular (cf. introductory remarks on p.1).

Remark. A map ι satisfying conditions (i) and (ii) is called an *immersion* and respectively N is said to be an *immersed submanifold*. The condition (iii) eliminates e.g. the irrational twist flow $t \in \mathbb{R} \to [(t, \alpha t)] \in \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, on the torus.

A surface or curve in $\mathbb{R}^3$ (or more generally in $\mathbb{R}^n$) is often defined by an equation or a system of equations, i.e. as the zero locus of a smooth map on $\mathbb{R}^3$ (respectively on $\mathbb{R}^n$). E.g. $x^2 + y^2 - 1 = 0$ (a circle) or $(x^2 + y^2 + b^2 + z^2 - a^2)^2 - 4b^2(x^2 + y^2) = 0, \quad b > a > 0$ (a torus). However a smooth (even polynomial) map may in general have 'bad' points in its zero locus (cf. Example Sheet 1, Q.8). When does a system of equations on a manifold define a submanifold?

Definition. A value $q \in N$ of a smooth map f between manifolds M and N is called a **regular value** if for any $p \in M$ such that $f(p) = q$ the differential of f at p is surjective, $(df)_p(T_pM) = T_qN$.

Theorem 1.17. *Let $f : M \to N$ be a smooth map between manifolds and $q \in N$ a regular value of f. The inverse image of a regular value $P = f^{-1}(q) = \{p \in M : f(p) = q\}$ (if it is non-empty) is an embedded submanifold of M, of dimension* $\dim M - \dim N$.

[8] In these notes I will sometimes write 'submanifold' meaning 'embedded submanifold'.

We shall need the following result from advanced calculus.

Inverse Mapping Theorem. *Suppose that $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map defined on an open set U, $0 \in U$ and $f(0) = 0$. Then f has an smooth inverse g, defined on some neighbourhood of 0 with $g(0) = 0$, if and only if $(df)_0$ is an invertible linear map of $\mathbb{R}^n$.*

Note that the Inverse Mapping Theorem, as stated above, is a *local* result, valid only if one restricts attention to a suitably chosen neighbourhood of a point. The statement will in general no longer hold with neighbourhoods replaced by manifolds. (Consider e.g. the map of $\mathbb{R}$ to the unit circle $S^1 \subset \mathbb{C}$ given by $f(x) = e^{ix}$.)

Proof of Theorem 1.17. Firstly, P is Hausdorff and second countable because M is so.

Let p be an arbitrary point of P. We may assume without loss of generality that there are local coordinates x_i, y_j, $i = 1, \ldots, n$, $j = 1, \ldots, k = \dim N$, $n + k = \dim M$, defined in a neighbourhood of p in M and local coordinates defined in a neighbourhood of $f(p)$ in N such that $x_i(p) = y_j(p) = 0$ and f is expressed in these local coordinates as a map $f = (f_1, \ldots, f_k)$ on a neighbourhood of 0 in $\mathbb{R}^n$ with values in $\mathbb{R}^k$, $f(0) = 0$, $\det(\partial f_i/\partial y_j)(0, 0) \neq 0$.

Then

$$\det \begin{pmatrix} 1 & 0 \\ (\partial f_i/\partial x_i) & (\partial f_i/\partial y_j) \end{pmatrix} (0, 0) \neq 0$$

and so, by the Inverse Function Theorem, the x_i, f_j form a valid set of *new local coordinates* on a (perhaps smaller) neighbourhood of p in M. The local equation for the intersection of P with that neighbourhood takes in the new coordinates a simple form $f_j = 0$, $j = 1, \ldots, k$. Furthermore the first projection $(x_i, f_j) \mapsto (x_i)$ restricts to a *homeomorphism* from a neighbourhood of p in P onto a neighbourhood of zero in $\mathbb{R}^n$. Define this first projection to be a coordinate chart on P with x_i the local coordinates. Then the family of all such charts covers P and it remains to verify that any two charts defined in this way are in fact compatible.

So let x_i, f_j, x'_i, f'_j be two sets of local coordinates near p as above. Then, by the construction, for every p <u>in P</u>, we have that $x'_i = x'_i(x, f)$, $f'_j = f'_j(x, f)$ with $f'_j(x, 0) = 0$ *identically in* x. Therefore, $(\partial f'_i/\partial x_j)(0, 0) = 0$. Then

$$\det \begin{pmatrix} \partial x'_{i'}/\partial x_i & \partial x'_{i'}/\partial f_j \\ \partial f'_{j'}/\partial x_i & \partial f'_{j'}/\partial f_j \end{pmatrix} (0, 0) = \det \begin{pmatrix} \partial x'_{i'}/\partial x_i & \partial x'_{i'}/\partial f_j \\ 0 & \partial f'_{j'}/\partial f_j \end{pmatrix} (0, 0) \neq 0,$$

so we must have $\det(\partial x'_{i'}/\partial x_j)(0) \neq 0$ for the $n \times n$ Jacobian matrix, and the change from x_i's to $x'_{i'}$'s is a diffeomorphism near 0. $\square$

Remarks. 1. It is *not* true that every submanifold of M is obtainable as the inverse image of a regular value for some smooth map on M. One counterexample is $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$ (an exercise: check this!).

2. Sometimes in the literature one encounters a statement 'a subset P is (or is not) a submanifold of M'. Every subset P of a manifold M has a topology induced from M. It turns out that there is at most one smooth structure on the topological space P such that P is an embedded submanifold of M, but I shall not prove it here.

Theorem 1.18 (Whitney embedding theorem)**.** *Every smooth n-dimensional manifold can be embedded in $\mathbb{R}^{2n}$ (i.e. is diffeomorphic to a submanifold of $\mathbb{R}^{2n}$).*

We shall assume the Whitney embedding theorem without proof here (note however, Examples Sheet 1 Q.9). A proof of embedding in $\mathbb{R}^{2n+1}$ is e.g. in Guillemin and Pollack, Ch.1 §9.

It is worth to remark that the possibility of embedding any manifold in *some* $\mathbb{R}^N$, with N possibly very large, does not particularly simplify the study of manifolds in practical terms (but is relatively easier to prove). The essence of the Whitney embedding theorem is the *minimum possible dimension* of the ambient Euclidean space as way of measuring the 'topological complexity' of the manifold. The result is sharp, in that the dimension of the ambient Euclidean space could not in general be lowered (as can be checked by considering e.g. the Klein bottle).

We can now give, as promised, a proof of Theorem 1.16

Theorem 1.16. *Suppose that $G \subset GL(n,\mathbb{R})$ is a subgroup and an embedded submanifold of $GL(n,\mathbb{R})$, and smooth structure on G is defined by the log-charts. Then the map $\xi \in \mathfrak{g} \to X_\xi \in l(G)$ is an isomorphism of the Lie algebras, where the Lie bracket on $\mathfrak{g}$ is the Lie bracket of matrices, as in Theorem 1.7.*

Proof of Theorem 1.16. We want to show $[X_\xi, X_\eta] = X_{[\xi,\eta]}$ for a matrix Lie group where the LHS is the Lie bracket of left-invariant vector fields and the RHS is defined using Theorem 1.7. Note first that for $G = GL(n\mathbb{R})$, all the calculations can be done on an open subset of $\mathbb{R}^{n^2} = \mathrm{Matr}(n,\mathbb{R})$ with coordinates x^i_j, $i.j = 1.\dots,n$. The map L_g, and hence also dL_g, is the usual left multiplication by a fixed matrix $g = (x^i_j)$. Respectively, the left-invariant vector fields are $X_\xi(g) = x^i_k \xi^k_j \frac{\partial}{\partial x^i_j}$ and $[X_\xi, X_\eta] = X_{[\xi,\eta]}$ is an easy calculation. Thus the theorem holds for $GL(n,\mathbb{R})$.

Now consider a general case $G \subset GL(n,\mathbb{R})$. Denote, as before, the inclusion map by ι. Any left-invariant vector field X_ξ on G ($\xi \in \mathfrak{g}$) may be identified by means of $d\iota$ with a vector field defined on a subset $G \subset GL(n,\mathbb{R})$. Further, the left translation L_g on G is the restriction of the left translation on $GL(n,\mathbb{R})$ (if $g \in G$). We find that the vector fields $(d\iota)X_\xi$ ($\xi \in \mathfrak{g}$) correspond bijectively to the restrictions to $G \subset GL(n,\mathbb{R})$ of the left-invariant vector fields X_ξ on $GL(n,\mathbb{R})$, such that $\xi \in \mathfrak{g}$. Let $X_\xi, X_\eta \in l(GL(n,\mathbb{R}))$ with $\xi, \eta \in \mathfrak{g}$, where $\mathfrak{g}$ is understood as the image of log-chart for G near I. We have $[X_\xi, X_\eta] = X_{[\xi,\eta]}$, by the above calculations on $GL(n,\mathbb{R})$, and the Lie bracket of matrices $[\xi,\eta] \in \mathfrak{g}$ by Theorem 1.7. Therefore, $X_{[\xi,\eta]}$ restricts to give a well-defined left-invariant vector field on G, and $[X_\xi, X_\eta] = X_{[\xi,\eta]}$ holds for any $X_\xi, X_\eta \in l(G)$ as claimed. □

Remark. Notice that the differential $d\iota$ considered in the above proof identifies $\mathfrak{g}$ with a linear subspace of $n \times n$ matrices, by considering G as a hypersurface in $GL(n,\mathbb{R}) \subset \mathbb{R}^{n^2}$. In the example of $G = O(n)$, for any path $A(t)$ or orthogonal matrices with $A(0) = I$, we calculate $0 = \frac{d}{dt}\big|_{t=0} A(t)A^*(t) = \dot{A}(0)A^*(0) + A(0)\dot{A}^*(0) = \dot{A}(0) + \dot{A}^*(0)$, i.e. $\dot{A}(0)$ is skew-symmetric. Thus the log chart at the identity actually maps onto a neighbourhood of zero in the tangent space T_IG —which explains why the log-chart construction in §1.2 worked.

1.5 Exterior algebra of differential forms. De Rham cohomology

See the reference card on multilinear algebra

The differential forms

Consider a smooth manifold M of dimension n. The dual space to the tangent space T_pM, $p \in M$, is called the **cotangent space** to M at p, denoted T_p^*M. Suppose that a chart and hence the local coordinates are given on a neighbourhood of p. The dual basis to $(\frac{\partial}{\partial x_i})_p$ is traditionally denoted by $(dx_i)_p$, $i = 1, \ldots, n$. (Sometimes I may drop the subscript p from the notation.) Thus an arbitrary element of T_p^*M is expressed as $\sum_i a_i(dx_i)_p$ for some $a_i \in \mathbb{R}$.

Recall from (1.4′) that a change of local coordinates, say from from x_i to x'_i, induces a change of basis of tangent space T_pM from $\frac{\partial}{\partial x_i}$ to $\frac{\partial}{\partial x'_i}$. By linear algebra, there is a corresponding change of the dual basis of T_p^*M, from dx'_i to dx_i given by the *transposed* matrix. Thus the transformation law is

$$dx'_j = \frac{\partial x'_j}{\partial x_i} dx_i. \tag{1.19}$$

Like (1.4′) the eq.(1.19) is *a priori* merely a notation which resembles familiar results from the calculus. See however further justification in the remark on the next page.

A disjoint union of all the cotangent spaces $T^*M = \sqcup_{p \in M} T_p^*M$ of a given manifold M is called the **cotangent bundle** of M. The cotangent bundle can be given a smooth structure making it into a manifold of dimension $2\dim M$ by an argument very similar to one for the tangent bundle (but with a change of notation, replacing any occurrence of (1.4′) with (1.19)).

A smooth field of linear functionals is called a (smooth) **differential 1-form** (or just 1-form). More precisely, a differential 1-form is a map $\alpha : M \to T^*M$ such that $\alpha_p \in T_p^*M$ for every $p \in M$ and α is expressed in any local coordinates $x = (x_1, \ldots, x_n)$ by $\alpha = \sum_i a_i(x) dx_i$ where $a_i(x)$ are some smooth functions of x.

Remark. The 1-forms are of course the dual objects to the vector fields. In particular, $a_i(x(p))$ is obtained as the value of α_p on the tangent vector $(\frac{\partial}{\partial x_i})_p$. Consequently, α is smooth on M if and only if $\alpha(X)$ is a smooth function for every vector field X on M. Notice that the latter condition does not use local coordinates.

One can similarly consider a space $\Lambda^r T_p^*M$ of alternating multilinear functions on $T_pM \times \ldots \times T_pM$($r$ factors), for any $r = 0, 1, 2, \ldots n$, and proceed to define the r-th exterior power $\Lambda^r T^*M$ of the cotangent bundle of M and the (smooth) **differential r-forms** on M. Details are left as an exercise.

The space of all the smooth differential r-forms on M is denoted by $\Omega^r(M)$ and r is referred to as the degree of a differential form. If $r = 0$ then $\Lambda^0 T^*M = M \times \mathbb{R}$ and $\Omega^0(M) = C^\infty(M)$. The other extreme case $r = \dim M$ is more interesting.

Theorem 1.20 (Orientation of a manifold)**.** *Let M be an n-dimensional manifold. The following are equivalent:*

(a) there exists a nowhere vanishing smooth differential n-form on M;

(b) there exists a family of charts in the differentiable structure on M such that the respective coordinate domains cover M and the Jacobian matrices have positive determinants on every overlap of the coordinate domains;

(c) the bundle of n-forms $\Lambda^n T^*M$ is isomorphic to $M \times \mathbb{R}$.

Proof (gist). That (a)$\Leftrightarrow$(c) is proved similarly to the proof of Theorem 1.9.

Using linear algebra, we find that the transformation of the differential forms of top degree under a change of coordinates is given by

$$dx_1 \wedge \ldots \wedge dx_n = \det\big(\frac{\partial x_j}{\partial x'_i}\big) dx'_1 \wedge \ldots \wedge dx'_n.$$

Now (a)$\Rightarrow$(b) is easy to see.

To obtain, (b)$\Rightarrow$(a) we assume the following.

Theorem 1.21 (Partition of unity)**.** *For any open cover $M \subset \cup_{\alpha \in A} U_\alpha$, there exists a countable collection of functions $\rho_i \in C^\infty(M)$, $i = 1, 2, ...$, such that the following holds:*
(i) for any i, the closure of $\operatorname{supp}(\rho_i) = \{x \in M : \rho_i(x) \neq 0\}$ *is compact and contained in U_α for some $\alpha = \alpha_i$ (i.e. depending on i);*
(ii) the collection is locally finite: each $x \in M$ has a neighbourhood W_x such that $\rho_i(x) \neq 0$ on W_x for only finitely many i; and
(iii) $\rho_i \geq 0$ on M for all i and $\sum_i \rho_i(x) = 1$ for all $x \in M$.
The collection $\{\rho_i\}$ satisfying the above is called a partition of unity *subordinate to $\{U_\alpha\}$.*

Choose a partition of unity $\{\rho_i\}$ subordinate to the given family of coordinate neighbourhoods covering M. For each i, choose local coordinates $x_i^{(\alpha)}$ valid on the support of ρ_i. Define $\omega_\alpha = dx_1^{(\alpha)} \wedge \ldots \wedge dx_n^{(\alpha)}$ in these local coordinates, then $\rho_i \omega_{\alpha_i}$ is a well-defined (smooth) n-form on all of M (extended by zero outside the coordinate domain) and $\omega = \sum_i \rho_i \omega_{\alpha_i}$ is the required n-form. $\square$

A manifold M satisfying any of the conditions (a),(b),(c) of the above theorem is called **orientable**. A choice of the differential form in (a), or family of charts in (b), or diffeomorphism in (c) defines an orientation of M and a manifold endowed with an orientation is said to be oriented.

Exterior derivative

Recall that the differential of a smooth map $f : M \to N$ between manifolds is a linear map between respective tangent spaces. In the special case $N = \mathbb{R}$, the $(df)_p$ at each $p \in M$ is a linear functional on T_pM, i.e. an element of the dual space T_p^*M. In local coordinates x_i defined near p we have $df(x) = \frac{\partial f}{\partial x_i}(x) dx_i$, thus df is a well-defined differential 1-form, whose coefficients are those of the gradient of f.

Remark. Observe that any local coordinate x_i on an open domain $U \subset M$ is a smooth function on U. Then the formal symbols dx_i actually make sense as the *differentials* of these smooth functions (which justifies the previously introduced notation, cf.(1.19)).

Theorem 1.22 (exterior differentiation)**.** *There exists unique linear operator* $d : \Omega^k(M) \to \Omega^{k+1}(M)$, $k \geq 0$, *such that*

(i) *if* $f \in \Omega^0(M)$ *then* df *coincides with the differential of a smooth function* f*;*

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega}\omega \wedge d\eta$ *for any two differential forms* ω,η*;*

(iii) $dd\omega = 0$ *for every differential form* ω.

Proof (gist). On an open set $U \subseteq \mathbb{R}^n$, or in the local coordinates on a coordinate domain on a manifold, application of conditions (ii), then (iii) and (i), yields

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} = \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}, \tag{1.23}$$

for any smooth function f. Extend this to arbitrary differential forms by linearity. The conditions (i),(ii),(iii) then follow by direct calculation, in particular the last of these holds by independence of the order of differentiation in second partial derivatives. This proves the *uniqueness*, i.e. that if d exists then it must be expressed by (1.23) in local coordinates.

Observe another important consequence of (1.23): the operator d is necessarily **local**, which means that the value $(d\omega)_p$ at a point p is determined by the values of differential form ω on a neighbourhood of p.

To establish the *existence* of d one now needs to show that the defining formula (1.23) is consistent, i.e. the result of calculation does not depend on the system of local coordinates in which it is performed. So let d' denote the exterior differentiation constructed as in (1.23), but using *different* choice of local coordinates. Then, by (ii), we must have

$$d'(f dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = d'f \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} + \sum_{j=1}^{r} (-1)^{j+1} f dx_{i_1} \wedge \ldots d'(dx_{i_j}) \ldots \wedge dx_{i_r}.$$

But $d'f = df$ and $d'(dx_k) = d'(d'x_k) = 0$, by (i) and (iii) and because we know that the differential of a smooth function (0-form) is independent of the coordinates. Hence the right-hand side of the above equality becomes $\frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r} = d(f x_{i_1} \wedge \ldots \wedge dx_{i_r})$, and so the exterior differentiation is well-defined. □

De Rham cohomology

A differential form α is said to be **closed** when $d\alpha = 0$ and **exact** when $\alpha = d\beta$ for some differential form β. Thus exact forms are necessarily closed (but the converse is not in general true, e.g. Example Sheet 2, Q4).

Definition. The quotient space

$$H^k_{\mathrm{dR}}(M) = \frac{\text{closed } k\text{-forms on } M}{\text{exact } k\text{-forms on } M}$$

is called the k-th **de Rham cohomology group** of the manifold M.

Any smooth map between manifolds, say $f : M \to N$, induces a **pull-back** map between exterior powers of cotangent spaces $f^* : \Lambda^r T^*_{f(p)} N \to \Lambda^r T^*_p M$ $(r = 0, 1, 2, \ldots)$, which is a linear map defined, for any differential r-form α on N, by

$$(f^*\alpha)_p(v_1, \ldots, v_r) = \alpha_{f(p)}((df)_p v_1, \ldots, (df)_p v_r),$$

using the differential of f. The chain rule for differentials of smooth maps immediately gives

$$(f \circ g)^* = g^* f^*. \tag{1.24}$$

It is also straightforward to check that f^* preserves the $\wedge$-product $f^*(\alpha\wedge\beta) = (f^*\alpha)\wedge(f^*\beta)$ and f^* commutes with the exterior differentiation, $f^*(d\alpha) = d(f^*\alpha)$, hence f^* preserves the subspaces of closed and exact differential forms. Therefore, every smooth map $f : M \to N$ induces a linear map on the de Rham cohomology

$$f^* : H^r(N) \to H^r(M)$$

A consequence of the chain rule (1.24) is that if f is a diffeomorphism then f^* is a linear isomorphism. Thus the de Rham cohomology is a *diffeomorphism invariant*, i.e. diffeomorphic manifolds have isomorphic de Rham cohomology.[9]

Poincaré lemma. *$H^k(D) = 0$ for any $k > 0$, where D denotes the open unit ball in $\mathbb{R}^n$.*

The proof goes by working out a way to invert the exterior derivative. More precisely, one constructs linear maps $h_k : \Omega^k(U) \to \Omega^{k-1}(U)$ such that

$$h_{k+1} \circ d + d \circ h_k = \mathrm{id}_{\Omega^k(U)}\,.$$

Remark. In the degree 0, one has $H^0(M) = \mathbb{R}$ for any connected manifold.

Basic integration on manifolds

Throughout this subsection M is an *oriented* n-dimensional manifold. Let $\omega \in \Omega^n(M)$ and, as before, denote $\operatorname{supp}\omega = \{p \in M : \omega_p \neq 0\}$ (the support of ω). Suppose that the closure of $\operatorname{supp}\omega$ is *compact*.

Consider first the special case when the closure of $\operatorname{supp}\omega$ is contained in the domain of just one coordinate chart, (U, φ) say. If $f(x)dx_1 \wedge \ldots \wedge dx_n$ is the local expression for ω then the integral $\int_{\varphi(U_\alpha)} f(x)dx_1 \ldots dx_n$ makes sense as in the multivariate calculus. The value of this integral is independent of the choice of local coordinates, provided only that the change is orientation-preserving, i.e. the Jacobian is *positive*. This is because the local expression for ω changes precisely as required by the change of variables formula for integrals of functions of n variables (which involves the absolute value of the Jacobian). Thus $\int_M \omega = \int_U \omega$ is well defined when ω supported in just one coordinate chart.

Now let ω be any n-form with *compact support*. Consider an oriented system of charts $(U_\alpha, \varphi_\alpha)$ covering M (as in Theorem 1.20(b)). Let (ρ_i) be a partition of unity subordinate to $\{U_\alpha\}$.

[9] In fact, more is true. It can be shown, using topology, that the de Rham cohomology depends only on the topological space underlying a smooth manifold and that homeomorphic manifolds have isomorphic de Rham cohomology. The converse in not true: there are manifolds with isomorphic de Rham cohomology but e.g. with different fundamental groups.

Definition. In the above situation, the integral of ω over M is given by

$$\int_M \omega = \sum_i \int_{U_{\alpha_i}} \rho_i \omega.$$

Note that the sum in the right-hand side may be assumed finite. It can be checked that $\int_M \omega$ does not depend on the choice of a partition of unity on M, and is therefore well-defined.

Stokes' Theorem (for manifolds without boundary)[10]**.** *Suppose that $\eta \in \Omega^{n-1}(M)$ has a compact support. Then $\int_M d\eta = 0$.*

Let ρ_α be a partition of unity subordinate to some oriented system of coordinate neighbourhoods covering M (as above). Then $d\eta$ is a *finite* sum of the forms $\rho_\alpha \eta$. Now the proof of Stokes' Theorem can be completed by considering compactly supported exact forms on $\mathbb{R}^n$ and using calculus.

Corollary 1.25 (Integration by parts)**.** *Suppose that α and β are compactly supported differential forms on M and* $\deg\alpha + \deg\beta = \dim M - 1$.
Then $\int_M \alpha \wedge d\beta = (-1)^{1+\deg\alpha} \int_M (d\alpha) \wedge \beta$.

One rather elegant application of the results discussed above is the following.

Theorem 1.26. *Every (smooth) vector field on S^{2m} vanishes at some point.*

Proof of Theorem 1.26. Notation: recall that we write $S^n \subset \mathbb{R}^{n+1}$ for the unit sphere about the origin. For $r > 0$, let $\iota(r) : x \in S^n \hookrightarrow rx \in \mathbb{R}^{n+1}$ denote the embedding of S^n in the Euclidean space as the sphere of radius r about the origin and write $S^n(r) = \iota(r)(S^n)$ (thus, in particular, $S^n(1) = S^n$).

Suppose, for a contradiction, that $X(x)$ is a nowhere-zero vector field on S^n. We may assume, without loss of generality, that $|X| = 1$, identically on S^n. For any real parameter ε, define a map

$$f : x \in \mathbb{R}^{n+1} \setminus \{0\} \to x + \varepsilon|x|X(x/|x|) \in \mathbb{R}^{n+1} \setminus \{0\}.$$

Here we used the inclusion $S^n \subset \mathbb{R}^{n+1}$ (and hence $T_pS^n \subset T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$) to define a (smooth) map $x \in \mathbb{R}^{n+1} \setminus \{0\} \to X(x/|x|) \in \mathbb{R}^{n+1} \setminus \{0\}$.

Step 1.
We claim that f is a diffeomorphism, whenever $|\varepsilon|$ is sufficiently small. Firstly, for any $x_0 \neq 0$,

$$(df)_{x_0} = \mathrm{id}_{\mathbb{R}^{n+1}} + \varepsilon\big[d(|x|X(x/|x|))\big]_{x_0}$$

and straightforward calculus shows that the norm of the linear map defined by the Jacobi matrix $\big[d(|x|X(x/|x|))\big]_{x_0}$ is bounded independent of $x_0 \neq 0$. Hence there is $\varepsilon_0 > 0$, such that $(df)_{x_0}$ is a linear isomorphism of $\mathbb{R}^{n+1}$ onto itself for any $x_0 \neq 0$ and any $|\varepsilon| < \varepsilon_0$. But

[10] Manifolds with boundary are *not* considered in these lectures.

then, by the Inverse Mapping Theorem (page 10), for any $x \neq 0$, f maps some open ball $B(x, \delta_x)$ of radius $\delta_x > 0$ about x *diffeomorphically* onto its image.

Furthermore, it can be checked, by inspection of the proof of the Inverse Mapping Theorem, that (1) δ_x can be taken to be *continuous* in x and (2) δ_x can be chosen independent of ε if $|\varepsilon| < \varepsilon_0$. We shall assume these two latter claims without proof. Consequently, δ_x can be taken to depend only on $|x|$ (as $S^n(|x|)$ is compact).

Taking a smaller $\varepsilon_0 > 0$ if necessary, we ensure that f is one-to-one if $|\varepsilon| < \varepsilon_0$. For the latter, note that $|f(x)| = \sqrt{1+\varepsilon^2}|x|$ and so it suffices to check that that f is one-to-one on each $S^n(|x|)$. But two points on $S^n(|x|)$ far away from each other cannot be mapped to one because $|f(x) - x| < \varepsilon|x|$ and two distinct points at a distance less than say $\frac{1}{2}\delta_{|x|}$ cannot be mapped to one because f restricts to a diffeomorphism (hence a bijection) on a $\delta_{|x|}$-ball about each point.

A similar reasoning shows that f is surjective (onto) if ε in sufficiently small. Indeed, $f(B(x, \delta_x))$ is an open set homeomorphic to a ball and the boundary of $f(B(x, \delta_x))$ is within small distance $\varepsilon(1 + \delta_x)|x|$ from the boundary of $B(x, \delta_x)$. Therefore, x must be inside the boundary of $f(B(x, \delta_x))$ and thus in the image of f. In all of the above, 'small ε_0' can be chosen independent of x because δ_x depends only on $|x|$ and f is homogeneous of degree 1, $f(\lambda x) = \lambda f(x)$ for each positive λ.

Step 2. Now, as f is a diffeomorphism f maps the embedded submanifold $S^n(1)$ diffeomorphically onto the embedded submanifold $f(S^n(1)) = S^n(\sqrt{1+\varepsilon^2})$. Consider a differential n-form on $\mathbb{R}^{n+1}$

$$\omega = \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge \ldots (\text{omit } dx_i) \ldots \wedge dx_n,$$

We have $\int_{f(S^n(1))} \omega = \int_{S^n(1)} f^*\omega$ (change of variable formula) and this integral depends *polynomially* on the parameter ε because $f^*\omega$ does so (as f is linear in ε).

But on the other hand, for any $r > 0$,

$$\begin{aligned}
\int_{S^n(r)} \omega - \int_{S^n(1)} \omega &= \int_{S^n} \iota(r)^*\omega - \int_{S^n} \omega \\
&= \int_1^r \frac{d}{ds}\Big(\int_{S^n} \iota(s)^*\omega\Big) ds \\
&= \int_1^r \Big(\int_{S^n} \frac{d}{ds}\big(\iota(s)^*\omega\big)\Big) ds
\end{aligned}$$

applying, on each coordinate patch, a theorem on differentiation of an integral depending on a parameter s, from calculus

$$= \int_{1 \leq |x| \leq r} d\omega$$

replacing, again on each coordinate patch, a repeated integration with an $(n+1)$-dimensional integral

$$= \int_{1\le|x|\le r} (n+1)dx_0 \wedge \ldots \wedge dx_n = c_{n+1}(r^{n+1}-1),$$

where a constant c_{n+1} is $n+1$ times the volume of $(n+1)$-dimensional ball (the value of c_{n+1} does not matter here). Put $r = \sqrt{1+\varepsilon^2}$ and then the right-hand side is *not* a polynomial in ε if $n+1$ is odd (i.e. when n is even). A contradiction. □

Some page references

to Warner, and Guillemin–Pollack, and Gallot–Hulin–Lafontaine.
N.B. The material in these books is sometimes covered differently from the Lectures and may contain additional topics, thus the references are not quite 'one-to-one'.

smooth manifolds [W] 1.2–1.6
tangent and cotangent bundles [W] 1.25
exponential map on a matrix Lie group [W] 3.35
left invariant vector fields [W] 3.6–3.7
submanifolds [W] 1.27–1.31, 1.38
differential forms [GP] 153–165,174–178
de Rham cohomology [GP] pp.178–182
Poincaré Lemma [W] 4.18
partition of unity [GP] p.52 or [W] 1.8–1.11
integration and Stokes' Theorem [GP] 165–168, 183–185
non-existence of vector fields without zeros on S^{2n} (Milnor's proof) [GHL] 1.41