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A lemma about holomorphic functions on an annulus

Let A be an annulus, a domain of the form $A = \{z : r < |z-a| < R\}$, for some $a \in \mathbb{C}$ and $0 \le r < R$.

Lemma 2.22. *Let $g : A \to \mathbb{C}$ be a holomorphic function on A except possibly at a finite subset of points $P \subset A$ where g is continuous.*

Then for $r < \rho < R$ the integral $\int_{|z-a|=\rho} g(z)dz$ does not depend on the choice of ρ.

Our convention is that the path $\{|z-a| = \rho\}$ is traversed counterclockwise. We may parameterize this path as $\gamma(t) = a + \rho\exp(2\pi i t)$ for $0 \le t \le 1$.

Proof. Consider a vertical strip $S = \{\zeta = x + iy \in \mathbb{C} : \ln r < x < \ln R, y \in \mathbb{R}\}$, where we used ln to denote the real logarithm. S is a star domain and $\zeta \mapsto z = a + \exp(\zeta)$ maps S onto A.

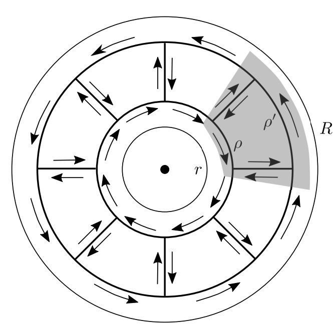
Define $\varphi(\zeta) = g(a + \exp(\zeta))\exp(\zeta)$. Then for each $M > 0$, φ is a holomorphic function on a rectangular domain $S_M = \{x + iy \in S : |y| < M\}$ except possibly at a finite subset of points in S_M, where φ is continuous.

We can therefore apply Cauchy's theorem for star domains (Corollary 2.11) to φ on S_M to obtain that $\int_{\hat\gamma} \varphi(z)dz = 0$ for every closed curve $\hat\gamma$ in S (noting $\hat\gamma$ is in S_M for some M). Then by the inverse to Fundamental Theorem of Calculus (Theorem 2.7) $\varphi(z) = \Phi'(z)$ for all $z \in S$, for some holomorphic $\Phi : S \to \mathbb{C}$ (formally we only obtain Φ on S_M but it is clear from the construction of Φ in the argument of Theorem 2.7 that by taking M larger and larger we get a well-defined function Φ on S).

Noting that $\varphi(\zeta+2\pi i) - \varphi(\zeta) = 0$ for all $\zeta \in S$, we find that the function $\Phi(\zeta+2\pi i) - \Phi(\zeta) = K$ is constant on S.

Now $\int_{|z-a|=\rho} g(z)dz = \int_0^1 g(a + \rho\exp(2\pi i t))2\pi i\rho\exp(2\pi i t)dt = 2\pi i \int_0^1 \varphi(\ln\rho + 2\pi i t)dt = 2\pi i(\Phi(\ln\rho + 2\pi i) - \Phi(\ln\rho)) = 2\pi i K$ independent of ρ as we had to prove. □

Remarks. There are alternative ways of proving the above lemma. We may e.g. consider the difference $\int_{|z-a|=\rho} g(z)dz - \int_{|z-a|=\rho'} g(z)dz$ for $R > \rho > \rho' > r$ and reduce the proof to Cauchy's theorem for star domains, by finding suitable curves which lie in a star domains of the shape of an annulus sector. For this, we can connect $\{|z-a| = \rho\}$ and $\{|z-a| = \rho'\}$ with strait line segments that will be passed in opposite directions, so that $\int_{|z-a|=\rho} g(z)dz - \int_{|z-a|=\rho'} g(z)dz$ is computed as a finite sum of vanishing integrals over closed curves, as shown on the picture below.



What really matters in the result of Lemma 2.22 is that, informally speaking, the contour $\{|z-a| = \rho\}$ 'goes exactly once 'round the hole'. This can be made rigorous by using some concepts from algebraic topology, but that's another story.