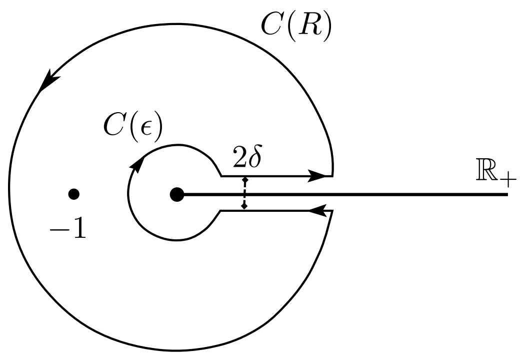
 A.G.Kovalev@dpmms.cam.ac.uk

Evaluation of a definite integral using a keyhole contour

In this example, we evaluate, for $0 < \alpha < 1$, the integral

$$I(\alpha) = \int_0^\infty \frac{x^\alpha}{x(x+1)}\,dx$$

We use a 'keyhole' contour Γ depending on three fixed parameters $0 < \delta < \varepsilon < R$. The contour consists of the straight line segment $\{x + i\delta : \sqrt{\varepsilon^2 - \delta^2} < x < \sqrt{R^2 - \delta^2}\}$ in the first quadrant going parallel to the real axis in the positive direction, then going counter-clockwise along the circle $C(R) = \{|z| = R\}$ to $\sqrt{R^2 - \delta^2} - i\delta$, then horizontally in the fourth quadrant the straight line segment $\{x - i\delta : \sqrt{\varepsilon^2 - \delta^2} < x < \sqrt{R^2 - \delta^2}\}$ in the negative direction and finally closing with arc of the circle $C(\varepsilon) = \{|z| = \varepsilon\}$ clockwise.



Extend x^α to $z^\alpha = \exp(\alpha \log z)$ understood as a holomorphic function on the cut-plane $\mathbb{C} \setminus [0, \infty)$ by choosing the holomorphic branch of $\log z = \ln|z| + i \arg z$ with $0 < \arg z < 2\pi$. Consider $\displaystyle\int_\Gamma \frac{z^\alpha}{z(z+1)}\,dz$ and let $R \to \infty$ and $\varepsilon, \delta \to 0$.

We find that each of the integrals over arcs of circles $C(R)$ and $C(\varepsilon)$ tends to zero, estimated by

$$C'R\frac{R^\alpha}{R^2} \text{ for large } R \text{ and } C''\varepsilon\frac{\varepsilon^\alpha}{\varepsilon} \text{ for small } \varepsilon,$$

for some positive constants C', C''. The integral along the upper horizontal straight line segment tends to $I(\alpha)$ and the limit of the integral over the lower horizontal straight line segment may be expressed as $-e^{2\pi i\alpha} I(\alpha)$ because the values of the chosen branch of z^α tend to $e^{2\pi i\alpha} x^\alpha$ as we approach the cut $[0, \infty)$ from below. Thus the limit of the contour integral is $(1 - e^{2\pi i\alpha}) I(\alpha)$, taking account of the directions of integration.

On the other hand, by the Residue Theorem (3.4),

$$\int_\Gamma \frac{z^\alpha}{z(z+1)}\,dz = 2\pi i \operatorname*{Res}_{z=-1} \frac{z^\alpha}{z(z+1)} = -2\pi i\, e^{i\pi\alpha},$$

noting that the only singularity inside Γ is a simple pole at $z = -1 = e^{i\pi}$. Therefore,

$$I(\alpha) = -2\pi i \frac{e^{i\pi\alpha}}{(1 - e^{2\pi i\alpha})} = \frac{-\pi i}{e^{-i\pi\alpha} - e^{\pi i\alpha}} = \frac{\pi}{\sin \pi\alpha}$$

Notice that $I(\alpha) \to \infty$ as $\alpha \to 0$ and as $\alpha \to 1$.