Part IID DIFFERENTIAL GEOMETRY (Mich. 2011): Example Sheet 3

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1. Let $\alpha : I \to S$ be a geodesic. Show that if α is a plane curve and $\ddot{\alpha}(t) \neq 0$ for some $t \in I$, then $\dot{\alpha}(t)$ is an eigenvector of the differential of the Gauss map at $\alpha(t)$.
[Hint: without loss of generality suppose that α is parametrized by arc-length and observe that the normal to α and the normal to the surface have to be colinear around t.]

2. Show that if all geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.
[Hint: use the previous problem and Problem 14 of Example sheet 2].

3. Let $f : S_1 \to S_2$ be an isometry between two surfaces.
(i) Let $\alpha : I \to S_1$ be a curve and V a parallel vector field along α. Show that $df_{\alpha(t)}(V(t))$ is a parallel vector field along $f \circ \alpha$.
(ii) Show that f preserves geodesics.

4. Consider the surface of revolution from Problem 9, Example sheet 2.
(i) Find the differential equations of the geodesics;
(ii) Establish *Clairaut's relation*: $f^2\dot{u}$ is constant along geodesics. Show that if θ is the angle that a geodesic makes with a parallel and r is the radius of the parallel at the intersection point, then Clairaut's relation says that $r\cos\theta$ is constant along geodesics.
(iii) Show that meridians are geodesics; when is a parallel a geodesic?

5. Show that there are no compact minimal surfaces in $\mathbb{R}^3$.

6. Let S_1 and S_2 be surfaces with Gaussian curvatures K_{S_1} and K_{S_2}, respectively, and $f : S_1 \to S_2$ a diffeomorphism.
(i) Suppose that $K_{S_2}(f(x)) = K_{S_1}(x)$. Must f be an isometry?
(ii) Suppose instead that f maps geodesics of S_1 to geodesics of S_2. Must f be an isometry?

7. The existence of isothermal coordinates is a hard theorem. However for the case of minimal surfaces without planar points it is possible to give an easy proof along the following lines.
(i) Let S be a regular surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N : S \to S^2$ satisfies

$$\langle dN_p(v_1), dN_p(v_2)\rangle = \lambda(p)\langle v_1, v_2\rangle$$

for all $p \in S$ and all $v_1, v_2 \in T_pS$, where $\lambda(p) \neq 0$ is a number which depends only on p.
(ii) By considering stereographic projection and (i) show that isothermal coordinates exist around a non planar point in a minimal surface.

8. The intrinsic distance of a surface S is defined as follows. Given p and q in S let $d(p,q) = \inf_{\alpha\in\Omega(p,q)} \ell(\alpha)$. It can be shown that d is a distance (can you see which property requires some care?) which is compatible with the topology of S. If (S,d) is complete (and S has no boundary) the Hopf–Rinow theorem asserts that given two points p and q there exists a geodesic γ joining the points such that $d(p,q) = \ell(\gamma)$ and geodesics are defined for all $t \in \mathbb{R}$.
(i) Show that if $f : S_1 \to S_2$ is an isometry, then $d_2(f(p), f(q)) = d_1(p,q)$ for all p and q in S_1.
(ii) A geodesic $\gamma : [0,\infty) \to S$ is called a *ray leaving from p* if it realizes the distance between $\gamma(0)$ and $\gamma(s)$ for all $s \in [0,\infty)$. Let p be a point in a complete (as a metric space), noncompact surface S. Prove that S contains a ray leaving from p. [You may assume that geodesics vary smoothly (hence continuously) with their initial conditions.]

For the next five questions we consider the Weierstrass representation of a minimal surface determined by functions f and g on a simply connected domain $D \subseteq \mathbb{C}$ as we saw in lectures.

9. Show that if φ is the parametrization defined by the Weierstrass representation, then φ is an immersion if and only f vanishes only at the poles of g and the order of its zero at such a point is *exactly* twice the order of the pole of g.

10. Find D, f and g representing the catenoid $\varphi = (a \cosh v \cos u,\ a \cosh v \sin u,\ av)$ and the helicoid $\varphi = (a \sinh v \cos u,\ a \sinh v \sin u,\ au)$.

11. Show that the Gaussian curvature of the minimal surface determined by the Weierstrass representation is given by

$$K = -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2$$

Show that either $K \equiv 0$ or its zeros are isolated. [There is a way of doing this problem almost without calculations. Think about the relation between g and the Gauss map and the fact that stereographic projection is conformal.]

12. The Weierstrass representation is not unique: if $\varphi_{(f,g)} : D \to \mathbb{R}^3$ is the associated parametrization and $\alpha : W \to D$ is a bijective holomorphic map, then $\varphi_{(f,g)} \circ \alpha$ is another representation of the same minimal surface and it must have the same form with different f and g. Show that by choosing $\alpha(z) = g^{-1}(z)$ we can assume that our pair (f, g) is of the form (F, id) (g can be inverted near a point which is not a pole and for which g' is not zero). We denote such a representation by φ_F.

13. Show that the minimal surfaces given by $\varphi_{e^{-i\theta}F}$ for θ real are all locally isometric. With an appropriate choice of F, show that the catenoid and the helicoid are locally isometric.

14. Show that any geodesic of the paraboloid of revolution $z = x^2 + y^2$ which is not a meridian intersects itself an infinite number of times.
[Hint: use Clairaut's relation. You may assume that no geodesic of a surface of revolution can be asymptotic to a parallel which is not itself a geodesic. You will need to show that for a geodesic which is not a meridian, $u(t)$ does not approach some u_0 as $t \to \infty$.]

15. Suppose that on a connected surface S there is a point p such that all geodesics through p are closed, i.e. all geodesics through p induce smooth maps $S^1 \to S$. Show that S is compact.