

Part IB COMPLEX ANALYSIS (Lent 2019): Example Sheet 3

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1. Let f be a holomorphic function on a punctured disc $D^*(a, R) = \{0 < |z - a| < R\}$ and let γ be a closed curve in $D^*(a, R)$. Show that

$$\int_{\gamma} f(z) dz = 2\pi i n(\gamma, a) \operatorname{Res}_a f.$$

2. Let $g(z) = p(z)/q(z)$ be a rational function, such that $\deg q \geq 2 + \deg p$. Show that the sum of residues of g at all its singularities is zero.

3. Evaluate the following integrals:

$$\begin{array}{ll} \text{(a)} \int_0^{\pi} \frac{d\theta}{4 + \sin^2 \theta}; & \text{(c)} \int_{-\infty}^{\infty} \frac{\sin \mu x}{x(a^2 + x^2)} dx, \text{ where } a > 0, \mu > 0; \\ \text{(b)} \int_0^{\infty} \frac{x^2}{(x^2 + 4)^2(x^2 + 9)} dx; & \text{(d)} \int_0^{\infty} \frac{\ln(x^2 + 1)}{x^2 + 1} dx. \end{array}$$

4. For $\alpha \in (-1, 1)$ with $\alpha \neq 0$ compute

$$\int_0^{\infty} \frac{x^{\alpha}}{x^2 + x + 1} dx.$$

5. Use Rouché's Theorem to prove the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ be a polynomial of degree n , and let $A = \max\{|a_i| : 0 \leq i \leq n-1\}$. Then p has n roots (counted with multiplicity) in the disc $\{|z| < A + 1\}$.

6. Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for such z . Hence sketch the curve $p \circ \gamma$, where $\gamma(t) = e^{2\pi it}$, and use your sketch to determine the number of z (counted with multiplicity), such that $|z| < 1$ and $p(z) = x$ for each real value x .

7. (i) Show that $z^4 + 12z + 1$ has exactly three zeros in the annulus $\{z \in \mathbb{C} : 1 < |z| < 4\}$. Show that these zeros are distinct.

(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{z : |z| = 3/2\}$.

8. Suppose that f is holomorphic on some open disc containing $\{|z| \leq 1\}$ and satisfies $|f(z)| < 1$ when $|z| = 1$. Show that there is exactly one complex number w , such that $|w| < 1$ and $f(w) = w$.

9. (Inverse function formula for holomorphic functions.) Let f be an analytic function on a disc $D(a, R)$, such that $f'(a) \neq 0$. Show that for sufficiently small $r > 0$ the formula

$$g(w) = \frac{1}{2\pi i} \int_{|z-a|=r} z \frac{f'(z)}{f(z) - w} dz$$

defines a holomorphic function on some neighbourhood of $f(a)$ which is inverse to f .

10. Prove that the equation $z \sin z = 1$ has only real roots.

[Hint: find the number of real roots in the interval $[-(n + 1/2)\pi, (n + 1/2)\pi]$ and compare with the number of zeros of $z \sin z - 1$ in the disc $\{|z| < (n + 1/2)\pi\}$.]

11. (i) For a positive integer N , let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N , $|\cot \pi z| < C$ on γ_N .

(ii) By integrating $\frac{\pi \cot \pi z}{z^2 + 1}$ around γ_N , show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = \frac{1 + \pi \coth \pi}{2}.$$

(iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n / (n^2 + 1)$.

The following is *not* part of the example sheet, but may be added to revision or used as optional extras for enthusiasts.

(A1) Evaluate:

(i) $\int_0^{\infty} \sin x^2 dx$;

(ii) $\int_0^{\infty} \frac{x^\alpha dx}{(x+a)(x+2a)}$, for $-1 < \alpha < 1$, $a > 0$;

(iii) $\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx} dx$, where $t \in \mathbb{R}$;

(iv) $\int_{-1}^1 \frac{\sqrt{1-x^2}}{1+x^2} dx$ ('dog-bone' contour).

(A2) By integrating $\frac{z}{a - e^{-iz}}$ around the rectangle with vertices $\pm\pi, \pm\pi + iR$, prove that

$$\int_0^{\pi} \frac{x \sin x}{1 - 2a \cos x + a^2} dx = \frac{\pi}{a} \log(1+a), \quad \text{for } 0 < a < 1.$$

(A3) (i) Show that the Taylor expansion of $z/(e^z - 1)$ near the origin has the form

$$1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_k}{(2k)!} z^{2k},$$

where the numbers B_k (the *Bernoulli numbers*) are rational.

(ii) If k is a positive integer show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1} \pi^{2k} B_k}{(2k)!}.$$