Part IB COMPLEX ANALYSIS (Lent 2019): Example Sheet 2

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1. (i) Use the Cauchy integral formula to compute

$$\int_{|z|=1} \frac{e^{\alpha z}}{3z^2 - 7z + 2} dz \,,$$

where $\alpha \in \mathbb{C}$.

(ii) By considering suitable complex integrals, show that if $r \in (0, 1)$,

$$\int_0^{\pi} \frac{cos(n\theta)}{1 - 2r\cos\theta + r^2} d\theta = \frac{\pi r^n}{1 - r^2}. \quad \text{and} \quad \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta) d\theta = 2\pi$$

2. Let f be an entire function. Prove that if any of the following conditions hold, then f is constant:

(i) $f(z)/z \to 0$ as $|z| \to \infty$;

(ii) for some $a \in \mathbb{C}$ and $\varepsilon > 0$, f never takes values in the disc $D(a, \varepsilon)$;

(iii) $f = u + iv$ and $|u(z)| > |v(z)|$ for all $z \in \mathbb{C}$.

3. Let $f : D(a, r) \to \mathbb{C}$ be holomorphic, and suppose that $\mathrm{Re}(f)$ attains a maximum at $z = a$. Show that f is constant.

4. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z.

(ii) Show that an entire function f is a (non-constant) polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

(iii) Let f be a function which is holomorphic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists $k \in \mathbb{Z}$ such that $|f(z)| < |z|^k$, for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

5. Prove Schwartz's lemma: if $f : D(0, 1) \to \mathbb{C}$ is a holomorphic function such that $|f(z)| \leq 1$ and $f(0) = 0$, then **either** $|f(z)| < |z|$ whenever $0 < |z| < 1$ **or** $f(z) = e^{i\theta} z$ for some real constant θ. [Hint: consider the function $g(z) = f(z)/z$ on the closed discs $\{|z| \leq 1 - \varepsilon\}$, $\varepsilon > 0$, and apply the maximum modulus principle.]

(ii) Deduce from Schwartz's lemma that any conformal equivalence from $D(0, 1)$ onto itself is given by a Möbius transformation.

6. (i) Let f be an entire function such that for every positive integer n one has $f(1/n) = 1/n$. Show that $f(z) = z$.

(ii) Let g be an entire function. If $g(n) = n^2$ for every $n \in \mathbb{Z}$, must $g(z) = z^2$?

(iii) Let h be a holomorphic function on $D(0, 2)$. Show that there exists a positive integer n such that $h(1/n) \neq 1/(n + 1)$.

7. Find the Laurent expansion, in powers of z, of $1/(z^2 - 3z + 2)$ in each of the domains:

$$\{z \in \mathbb{C} : |z| < 1\}, \qquad \{z \in \mathbb{C} : 1 < |z| < 2\}, \qquad \{z \in \mathbb{C} : |z| > 2\}.$$

8. Classify the singularities of each of the following functions:

$$\frac{z}{\sin z}, \qquad \frac{1}{z^4+z^2}, \qquad \cos\frac{\pi}{z^2}, \qquad \frac{1}{z^2}\cos\frac{\pi z}{z+1}.$$

9. (i) Let $w \in \mathbb{C}$ and let $\gamma, \delta : [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t)-\delta(t)| < |\gamma(t)-w|$. By computing the winding number $n(\sigma, 0)$ of the closed curve $\sigma(t) = \dfrac{\delta(t)-w}{\gamma(t)-w}$ about the origin, show that $n(\gamma, w) = n(\delta, w)$.

(ii) If $w \in \mathbb{C}$, $r > 0$ and γ is a closed curve which does not meet $D(w,r)$, show that $n(\gamma, w) = n(\gamma, z)$ for every $z \in D(w,r)$.

(iii) Deduce that if γ is a closed curve and U is the complement of (the image of) γ then the function $w \mapsto n(\gamma, w)$ is a locally constant function on U.

10. Show that

$$\varphi : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C} \setminus [-1,1], \quad z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$$

is a conformal equivalence between the two domains. If an entire function f never takes values in the line segment $[-1,1] \subset \mathbb{R}$, show that $\varphi^{-1} \circ f$ is holomorphic and deduce that f is constant.

11. (Casorati–Weierstrass theorem) Let f be holomorphic on a punctured disc $D^*(a,r)$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a,r)$, with $z_n \neq a$, such that $z_n \to a$ and $f(z_n) \to b$, as $n \to \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard asserts that in any neighbourhood of an essential singularity a holomorphic function takes *every* complex value except possibly one.]

12. Let f be a holomorphic function on a punctured disc $D^*(a,R)$. Show that if f has a non-removable singularity at $z = a$ then the function $\exp(f(z))$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\operatorname{Re} f(z) < M$ for $z \in D^*(a,R)$, then f has a removable singularity at $z = a$.