Part IB COMPLEX ANALYSIS (Lent 2019): Example Sheet 1

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Comments and/or corrections concerning these questions are welcome at any time and can be emailed to me at a.g.kovalev@dpmms.cam.ac.uk.

1. Show that any real linear map $T : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C} \cong \mathbb{R}^2$ can be written as $T(z) = Az + B\bar{z}$, for two complex numbers A and B. Considering T as a complex-valued function on $\mathbb{C}$, deduce that T is complex differentiable on $\mathbb{C}$ if and only if $B = 0$.

2. (i) Let $f : D(a, r) \to \mathbb{C}$ be a holomorphic function on a disc. Show that f is constant if either of its real part, imaginary part, modulus or argument is constant.

(ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x+iy) = u(x) + iv(y)$, where u and v are both real valued.

(iii) Find all the functions which are holomorphic on $\mathbb{C}$ and which have the real part $x^3 - 3xy^2$. (The final answer should be in terms of the complex variable $z = x + iy$.)

3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$ and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f satisfies Cauchy–Riemann equations at 0 but is not differentiable there.

4. (i) Find the set of complex numbers z for which $|e^z| < 1$ and the set of those for which $|e^z| \leq e^{|z|}$.

(ii) Find the zeros of $1 + e^z$ and $\cosh z$.

5. (i) Define the differential operators $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\big)$ and $\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\big)$. Prove that a C^1 function f is holomorphic if and only if $\partial f/\partial \bar{z} = 0$. Show that

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\ ,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the usual Laplacian on $\mathbb{R}^2$.

(ii) Let $f : U \to V$ be holomorphic and $g : V \to \mathbb{C}$ be harmonic. Show that the composition $g \circ f$ is harmonic.

6. (i) Denote by Log the principal branch of the logarithm. If $z \in \mathbb{C}$, show that $n\,\mathrm{Log}(1 + z/n)$ is defined if n is sufficiently large, and it tends to z as n tends to ∞. Deduce that for any $z \in \mathbb{C}$,

$$\lim_{n\to\infty}\big(1 + \frac{z}{n}\big)^n = e^z.$$

(ii) Defining $z^\alpha = \exp(\alpha\,\mathrm{Log}\,z)$, where Log is the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz}(z^\alpha) = \alpha z^{\alpha-1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold?

7. Prove that each of the following series converges uniformly on the corresponding subset of $\mathbb{C}$:

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}\, e^{-nz}$, on $\{z : 0 < r \leq \mathrm{Re}\, z\}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}}$, on $\{z : |z| \leq r < \frac{1}{2}\}$.

8. Find conformal equivalences between the following pairs of domains:

(a) the sector $\{z \in \mathbb{C} : -\pi/3 < \arg z < \pi/3\}$ and the open unit disc $D(0, 1)$;

(b) the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and the half-disc $\{z \in D(0,1) : \operatorname{Re} z > 0\}$.

(c) the horizontal strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$;

By considering a suitable bounded solution of Laplace's equation $u_{xx} + u_{yy} = 0$, find a non-constant harmonic function on Q which is constant on each of the two boundaries of the quadrant (it need not be continuous at the origin).

9. (i) Show that the most general Möbius transformation which maps the unit disk onto itself has the form $z \mapsto \lambda \frac{z-a}{\bar{a}z-1}$, with $|a| < 1$ and $|\lambda| = 1$. [*Hint: first show that these maps form a group.*]

(ii) Find a Möbius transformation taking the region between the circles $\{|z| = 1\}$ and $\{|z-1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: a circle can be described by an equation of the form* $|z-a|/|z-b| = \ell$.]

(iii) Find a conformal map from an infinite strip onto an annulus. Can such a map be a Möbius transformation?

10. Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

11. Show that the following functions do not have antiderivatives (i.e. functions of which they are derivatives) on the domains indicated:

(a) $\dfrac{1}{z}$, $\quad (0 < |z| < \infty)$; $\qquad$ (b) $\dfrac{z}{1+z^2}$, $\quad (1 < |z| < \infty)$.