ANALYSIS II (Michaelmas 2009): EXAMPLES 4

The questions are not equally difficult. The questions marked with * may be harder, but merit some attention even if you do not write out solutions. Comments, corrections are welcome at any time and may be sent to `a.g.kovalev@dpmms.cam.ac.uk`.

1. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

2. At which points of $\mathbb{R}^2$ are the following functions $\mathbb{R}^2 \to \mathbb{R}$ differentiable?

(i) $f(x,y) = \begin{cases} x/y & y \neq 0, \\ 0 & y = 0. \end{cases}$

(ii) $f(x,y) = |x||y|$.

(iii) $f(x,y) = xy\,|x-y|$.

(iv) $f(x,y) = \begin{cases} xy\,/\sqrt{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$

(v) $f(x,y) = \begin{cases} xy\,\sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$

3. Let $f(x,y) = x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$. Show that f is continuous at $(0,0)$ and that it has directional derivatives in all directions there (i.e. for any fixed α, the function $t \mapsto f(t\cos\alpha, t\sin\alpha)$ is differentiable at $t = 0$). Is f differentiable at $(0,0)$?

4. We work in $\mathbb{R}^3$ with the usual inner product. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable except at 0 and

$$Df(x)\,h = \frac{h}{\|x\|} - \frac{x(x\cdot h)}{\|x\|^3}.$$

Verify that $Df(x)\,h$ is orthogonal to x and explain geometrically why this is the case.

5. (i) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_1 f = \partial f/\partial x$ is continuous in some open ball around (a,b), and $D_2 f = \partial f/\partial y$ exists at (a,b). Show that f is differentiable at (a,b).

(ii) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_1 f = \partial f/\partial x$ exists and is bounded near (a,b), and that for a fixed, $f(a,y)$ is continuous as a function of y. Show that f is continuous at (a,b).

6. Let $M_n = M_n(\mathbb{R})$ be the space of $n \times n$ real matrices (it can be identified with $\mathbb{R}^{n^2}$). Show that the function $f : M_n \to M_n$ defined by $f(A) = A^2$ is differentiable everywhere in M_n. Is it true that $Df(A) = 2A$? If not, what is the derivative of f at A?

7. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that

$$\sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| \le 1\} = \inf\{k \in \mathbb{R} : k \text{ is a Lipschitz constant for } A\}.$$

Show that the function which assigns to A the common value of these two expressions is a norm on the vector space $L(\mathbb{R}^n, \mathbb{R}^m)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$. [This is the *operator norm* on $L(\mathbb{R}^n, \mathbb{R}^m)$.]

Now assume $m = n$ and identify $L(\mathbb{R}^n, \mathbb{R}^n)$ with M_n using the standard basis of $\mathbb{R}^n$. Show that if the operator norm of $A \in M_n$ satisfies $\|A\| < 1$, then the sequence $B_k = I + A + A^2 + \ldots + A^{k-1}$ converges (here I is the identity matrix), and deduce that $I - A$ is then invertible. Deduce that the set $GL_n(\mathbb{R})$ of all invertible $n \times n$ real matrices is an open subset of M_n.

8. Define $g : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ by $g(A) = A^{-1}$. Show that g is differentiable at the identity matrix $I \in GL_n(\mathbb{R})$, and that $Dg(I)H = -H$.

Let $A \in GL_n(\mathbb{R})$. By writing $(A + H)^{-1} = A^{-1}(I + HA^{-1})^{-1}$, or otherwise, show that g is differentiable at A. What is $Dg(A)$?

Show further that g is twice differentiable at I, and find $D^2g(I)$ as a bilinear map.

9. (i) Define $f : M_n \to M_n$ by $f(A) = A^3$. Find the Taylor series of $f(A + H)$ about A.

(ii)* (This assumes that you did the previous question!) Let again $g : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be defined by $g(A) = A^{-1}$. Find the Taylor series of $g(I + H)$ about I.

10.* Show that $\det : M_n \to \mathbb{R}$ is differentiable at the identity matrix I with $(D\det)(I)H = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $(D\det)(A)H = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2 \det(I)$ as a bilinear map.

11. Show that there is a continuous square-root function on some neighbourhood of I in M_n; that is, show that there is an open ball $B_\varepsilon(I) \subset M_n$ for some $\varepsilon > 0$ and a continuous function $g : B_\varepsilon(I) \to M_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$.

Is it possible to define a square-root function on all of M_n? What about a cube-root function?

12. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x, x^3 + y^3 - 3xy)$ and let $C = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Show that f is locally invertible around each point of C except $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0, y_0) \in C \setminus \{(0, 0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $f(x_0, y_0)$ such that f maps U bijectively to V. What is the derivative of the local inverse function? Deduce that for each point $(x_0, y_0) \in C$ other than $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$ there exist open intervals I containing x_0 and J containing y_0 such that for each $x \in I$ there is a unique $y \in J$ with $(x, y) \in C$.

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g(x) = f(x, c - x)$ where c is a constant. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable and find its derivative

(i) directly from the definition of differentiability

and also

(ii) by using the chain rule.

Deduce that if $D_1 f = D_2 f$ holds throughout $\mathbb{R}^2$, then $f(x, y) = h(x + y)$ for some differentiable function h.

14.* Let $U \subset \mathbb{R}^2$ be an open set that contains a rectangle $[a, b] \times [c, d]$. Suppose that $g : U \to \mathbb{R}$ is continuous and that the partial derivative $D_2 g$ exists and is continuous. Set $G(y) = \int_a^b g(x, y)dx$. Show that G is differentiable on (c, d) with derivative $G'(y) = \int_a^b D_2 g(x, y)dx$. Show further that $H(y) = \int_a^y g(x, y)dx$ is differentiable. What is its derivative $H'(y)$?

[Hint: consider a function $F(y, z) = \int_a^z g(x, y)dx$ before dealing with H.]