Abstract. We prove that for any integer \( n \in \mathbb{N} \), \( d \in \{1, \ldots, n\} \) and any \( \varepsilon, \delta \in (0,1) \), a bounded function \( f : [-1,1]^n \to [-1,1] \) of degree at most \( d \) can be learned with probability at least \( 1 - \delta \) and \( L_2 \)-error \( \varepsilon \) using \( \log( \frac{1}{\delta})\varepsilon^{-d-1}C^d\sqrt{d} \) random queries for a universal finite constant \( C > 1 \).

2020 Mathematics Subject Classification. Primary: 06E30; Secondary: 42C10, 68Q32.

Key words. Discrete hypercube, learning theory, Bohnenblust–Hille inequality.

1. Introduction

Every function \( f : [-1,1]^n \to \mathbb{R} \) admits a unique Fourier–Walsh expansion of the form

\[
\forall x \in [-1,1]^n, \quad f(x) = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S)w_S(x),
\]

where \( w_S(x) = \prod_{i \in S} x_i \) and the Fourier coefficients \( \hat{f}(S) \) are given by

\[
\forall S \subseteq \{1, \ldots, n\}, \quad \hat{f}(S) = \frac{1}{2^n} \sum_{y \in \{-1,1\}^n} f(y)w_S(y).
\]

We say that \( f \) has degree at most \( d \in \{1, \ldots, n\} \) if \( \hat{f}(S) = 0 \) for every subset \( S \) with \(|S| > d\).

1.1. Learning functions on the hypercube. Let \( \mathcal{C} \) be a class of functions \( f : [-1,1]^n \to \mathbb{R} \) on the \( n \)-dimensional discrete hypercube. The problem of learning the class \( \mathcal{C} \) can be described as follows: given a source of examples \((x, f(x))\), where \( x \in [-1,1]^n \), for an unknown function \( f \in \mathcal{C} \), compute a hypothesis function \( h : [-1,1]^n \to \mathbb{R} \) which is a good approximation of \( f \) up to a given error in some prescribed metric. In this paper we will be interested in the random query model with \( L_2 \)-error, in which we are given \( N \) independent examples \((x, f(x))\), each chosen uniformly at random from the discrete hypercube \([-1,1]^n\), and we want to efficiently construct a (random) function \( h : [-1,1]^n \to \mathbb{R} \) such that \( ||h - f||_{L_2}^2 < \varepsilon \) with probability at least \( 1 - \delta \), where \( \varepsilon, \delta \in (0,1) \) are given accuracy and confidence parameters. The goal is to construct a randomized algorithm which produces the hypothesis function \( h \) from a minimal number \( N \) of examples.

The above very general problem has been studied for decades in computational learning theory and several results are known, primarily for various classes \( \mathcal{C} \) of structured Boolean functions \( f : [-1,1]^n \to [-1,1] \). Already since the late 1980s, researchers used the Fourier–Walsh expansion (1) to design such learning algorithms (see the survey [14]). Perhaps the most classical of these is the Low-Degree Algorithm of Linial, Mansour and Nisan [12] who showed that for the class \( \mathcal{C}_b^d \) of all bounded functions \( f : [-1,1]^n \to [-1,1] \) of degree at most \( d \) there exists an algorithm which produces an \( \varepsilon \)-approximation of \( f \) with probability at least \( 1 - \delta \) using \( N = \frac{2d}{\varepsilon} \log(\frac{2d}{\varepsilon}) \) samples. In this generality, the \( O_{\varepsilon,\delta,d}(n^d \log n) \) estimate of [12] was the state of the art until the recent work [11] of Iyer, Rao, Reis, Rothvoss and Yehudayoff who employed analytic techniques to derive new bounds on the \( \ell_1 \)-size of the Fourier spectrum of bounded functions.

ALEXANDROS ESKENAZIS AND PAATA IVANISVILI

A. E. was supported by a Junior Research Fellowship from Trinity College, Cambridge. P. I. was partially supported by the NSF grants DMS-2152346 and CAREER-DMS-2152401.

\( ^1 \)We will by no means attempt to survey this (vast) field, so we refer the interested reader to the relevant chapters of O’Donnell’s book [15] and the references therein.
functions (see also Section 3) and used these estimates to show that \( N = O_{\varepsilon, \delta, d}(n^{d-1} \log n) \) examples suffice to learn \( e^C_\theta \). The goal of the present paper is to further improve this result and show that in fact \( N = O_{\varepsilon, \delta, d}(\log n) \) samples suffice for this purpose.

**Theorem 1.** Fix \( \varepsilon, \delta \in (0,1) \), \( n \in \mathbb{N} \), \( d \in \{1, \ldots, n\} \) and a bounded function \( f : [-1,1]^n \to [-1,1] \) of degree at most \( d \). If \( N \in \mathbb{N} \) satisfies

\[
N \geq \min \left\{ 4d n^d \left( \frac{\log d}{\varepsilon} \right), \log \left( \frac{n}{\delta} \right) \right\}
\]

where \( C \in (0, \infty) \) is a large numerical constant, then \( N \) uniformly random independent queries of pairs \((x, f(x))\), where \( x \in [-1,1]^n \), suffice for the construction of a random function \( h : [-1,1]^n \to \mathbb{R} \) satisfying the condition \( \|h - f\|_{L_2} < \varepsilon \) with probability at least \( 1 - \delta \).

The proof of Theorem 1 relies on some important approximation theoretic estimates going back to the 1930s which we shall now describe (see also [9]). To the best of our knowledge, these tools had not yet been exploited in the computational learning theory literature.

1.2. **The Fourier growth of Walsh polynomials in \( \ell^\infty_{2^d} \).** Estimates for the growth of coefficients of polynomials as a function of their degree and their maximum on compact sets go back to the early days of approximation theory (see [5]). A seminal result of this nature is Littlewood’s celebrated \( \frac{1}{4} \)-inequality [13] for bilinear forms which was later generalized by Bohnenblust and Hille [4] for multilinear forms on the torus \( \mathbb{T}^n \) or the unit square \([-1,1]^n\). By means of polarization, one can use this multilinear estimate to derive an inequality for polynomials which reads as follows\(^2\). For every \( K \in \{\mathbb{R}, \mathbb{C}\} \) and \( d \in \mathbb{N} \), there exists \( B^K_d \in (0, \infty) \) such that for every \( n \in \mathbb{N} \) and every coefficients \( c_n \in K \), \( n \in (\mathbb{N} \cup \{0\})^n \) with \(|\alpha| \leq d\), we have

\[
\left( \sum_{|\alpha| \leq d} |c_\alpha| \right)^{\frac{d+1}{d+1}} \leq B^K_d \max \left\{ \left\| \sum_{|\alpha| \leq d} c_\alpha x^\alpha : x \in \mathbb{K}^n \text{ with } \|x\|_{\ell_\infty(K)} \leq 1 \right\}.
\]

Moreover, \( \frac{d+1}{d+1} \) is the smallest exponent for which the optimal constant in (4) is independent of the number of variables \( n \) of the polynomial. The exact asymptotics of the constants \( B^\mathbb{R}_d \) and \( B^\mathbb{C}_d \) remain unknown, however it is known that there is a significant gap between \( B^\mathbb{R}_d \) and \( B^\mathbb{C}_d \), namely that \( \limsup_{d \to \infty} (B^\mathbb{R}_d)^{1/d} = 1 + \sqrt{2} \) whereas \( B^\mathbb{C}_d \leq C\sqrt{\min d} \) for a finite constant \( C > 1 \) (see [7, 1, 9, 6, 8] for these and other important advances of the last decade). Restricting inequality (4) to real **multilinear** polynomials, convexity shows that the maximum on the right-hand side is attained at a point \( x = (\pm 1, \ldots, \pm 1) \) which, in view of (1), makes (4) an estimate for the Fourier-Walsh growth of functions on the discrete hypercube. We shall denote by \( B_{d}^{[\pm 1]} \) the corresponding optimal constant (first explicitly investigated by Blei in [3, p. 175]), that is, the least constant such that for every \( n \in \mathbb{N} \) and every function \( f : [-1,1]^n \to \mathbb{R} \) of degree at most \( d \),

\[
\left( \sum_{S \subseteq \{1, \ldots, n\}} \|\hat{f}(S)\|_{\ell_2}^\frac{d+1}{d} \right)^{\frac{d+1}{d+1}} \leq B_{d}^{[\pm 1]} \|f\|_{\ell_\infty}.
\]

The best known quantitative result in this setting is due to Defant, Mastylo and Pérez [8] who showed that \( B_{d}^{[\pm 1]} \leq \exp(\kappa \sqrt{d} \log d) \) for a universal constant \( \kappa \in (0, \infty) \). The main contribution of this work is the following theorem relating the growth of the constant \( B_{d}^{[\pm 1]} \) and learning.

**Theorem 2.** Fix \( \varepsilon, \delta \in (0,1) \), \( n \in \mathbb{N} \), \( d \in \{1, \ldots, n\} \) and a bounded function \( f : [-1,1]^n \to [-1,1] \) of degree at most \( d \). If \( N \in \mathbb{N} \) satisfies

\[
N \geq \frac{e^{8d^2}}{\varepsilon^{d+1}} \left( B_{d}^{[\pm 1]} \right)^{2d+2} \log \left( \frac{n}{\delta} \right),
\]

\(^2\)For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \), we use the standard notations \(|\alpha| = \alpha_1 + \cdots + \alpha_n\) and \( x^n = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).
then given $N$ uniformly random independent queries of pairs $(x, f(x))$, where $x \in \{-1, 1\}^n$, one can construct a random function $h : \{-1, 1\}^n \to \mathbb{R}$ satisfying $\|h - f\|_{L_2}^2 < \varepsilon$ with probability at least $1 - \delta$.

In Section 2 we will prove Theorem 2 and use it to derive Theorem 1. In Section 3 we will present some additional remarks on Boolean analysis and learning, in particular improving the recent bounds of [11] on the $\ell_1$-Fourier growth of bounded functions of low degree.

Acknowledgements. We are very grateful to Assaf Naor for constructive feedback.

2. Proofs

Proof of Theorem 2. Fix a parameter $b \in (0, \infty)$ and denote by

$$N_b \overset{\text{def}}{=} \left\lceil \frac{2}{b^2} \log \left( \frac{2 \sum_{k=0}^{d} \binom{n}{k}}{\delta} \right) \right\rceil. \quad (7)$$

Let $X_1, \ldots, X_{N_b}$ be independent random vectors, each uniformly distributed on $\{-1, 1\}^n$. For a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq d$ consider the empirical Walsh coefficient of $f$, given by

$$\alpha_S = \frac{1}{N_b} \sum_{j=1}^{N_b} f(X_j) w_S(X_j). \quad (8)$$

As $\alpha_S$ is a sum of bounded i.i.d. random variables and $\mathbb{E}[\alpha_S] = \hat{f}(S)$, the Chernoff bound gives

$$\mathbb{P}[|\alpha_S - \hat{f}(S)| > b] \leq 2 \exp(-N_b b^2/2). \quad (9)$$

Therefore, using the union bound and taking into account that $f$ has degree at most $d$, we get

$$\mathbb{P}[|\alpha_S - \hat{f}(S)| \leq b \text{ for every } S \subseteq \{1, \ldots, n\}] \geq 1 - 2 \sum_{k=0}^{d} \binom{n}{k} \exp(-N_b b^2/2) \geq 1 - \delta. \quad (10)$$

Fix an additional parameter $a \in (b, \infty)$ and consider the random collection of sets given by

$$S_a \overset{\text{def}}{=} \{S \subseteq \{1, \ldots, n\} : |\alpha_S| \geq a\}. \quad (11)$$

Observe that if the event $G_b$ of equation (10) holds, then

$$\forall S \not\in S_a, \quad |\hat{f}(S)| \leq |\alpha_S - \hat{f}(S)| + |\alpha_S| < a + b \quad (12)$$

and

$$\forall S \in S_a, \quad |\hat{f}(S)| \geq |\alpha_S| - |\alpha_S - \hat{f}(S)| \geq a - b. \quad (13)$$

Finally, consider the random function $h_{a,b} : [1, n] \to \mathbb{R}$ given by

$$\forall x \in \{-1, 1\}^n, \quad h_{a,b}(x) \overset{\text{def}}{=} \sum_{S \in S_a} \alpha_S w_S(x). \quad (14)$$

Combining (13) with inequality (5), we deduce that

$$|S_a| \overset{(13)}{=} (a - b)^{-\frac{2d}{\pi^2}} \sum_{S \subseteq [1, \ldots, n]} |\hat{f}(S)|^2 \overset{(5)}{=} (a - b)^{-\frac{2d}{\pi^2}} \sum_{S \subseteq [1, \ldots, n]} |\hat{f}(S)|^2 \overset{(5)}{=} (a - b)^{-\frac{2d}{\pi^2}} (B_d^{[\pm 1]})^2. \quad (15)$$

Therefore, on the event $G_b$ we have

$$\|h_{a,b} - f\|_{L_2}^2 \overset{(12)}{=} \sum_{S \subseteq [1, \ldots, n]} |\hat{h}_{a,b}(S) - \hat{f}(S)|^2 = \sum_{S \subseteq S_a} |\alpha_S - \hat{f}(S)|^2 + \sum_{S \not\in S_a} |\hat{f}(S)|^2 \overset{(5)}{=} (a - b)^{-\frac{2d}{\pi^2}} (B_d^{[\pm 1]})^2 \overset{(15)}{=} (a - b)^{-\frac{2d}{\pi^2}} (a + b)^{2\pi} \overset{(14)}{=} |S_a| b^2 + (a + b)^{2\pi} \sum_{S \in S_a} \hat{f}(S)|^2 \overset{(5)}{=} (a - b)^{-\frac{2d}{\pi^2}} (B_d^{[\pm 1]})^2 \overset{(15)}{=} (a - b)^{-\frac{2d}{\pi^2}} (a + b)^{2\pi}. \quad (16)$$
Choosing $a = b(1 + \sqrt{d + 1})$, we deduce that
\[
\|h_{b(1+\sqrt{d+1})}\|_2 < (B_d^{[\pm 1]}b^2)^{\frac{d}{2\pi}} ((d+1)^{-\frac{d}{2\pi}} + (2 + \sqrt{d + 1})^{\frac{d}{2\pi}}).
\] (17)

Next, we need the technical inequality
\[
(d+1)^{-\frac{d}{2\pi}} + (2 + \sqrt{d + 1})^{\frac{d}{2\pi}} \leq (e^4(d+1))^{\frac{d}{2\pi}} \quad \text{for all} \quad d \geq 1.
\] (18)

Rearranging the terms, it suffices to show that $(2 + \sqrt{d + 1})^{\frac{d}{2\pi}} \leq (d+1)^{\frac{d}{2\pi}}(e^{\frac{d}{2\pi}} - \frac{1}{d+1})$, which is equivalent to \( (\frac{2}{\sqrt{d+1}} + 1)^{\frac{d}{2\pi}} \leq e^{\frac{d}{2\pi}} - \frac{1}{d+1} \). We have
\[
\left(\frac{2}{\sqrt{d+1}} + 1\right)^{\frac{d}{2\pi}} \leq \left(\sqrt{2} + 1\right)^{\frac{d}{2\pi}} \leq 1 + 3 \left(\frac{d}{d+1}\right) \leq e^{\frac{d}{2\pi}} - \frac{1}{d+1},
\] (19)

where inequality (*) holds because the left hand side is convex in the variable $\lambda \overset{\text{def}}{=} \frac{2}{\sqrt{d+1}}$ whereas the right hand side is linear and since (*) holds at the endpoints $\lambda = 0, 1$.

Combining (17) and (18) we see that $\|h_{b(1+\sqrt{d+1})}\|_2 < \varepsilon$ holds for $b^2 \leq e^{-5d}e^{d+1}(\sqrt{d^d} - 1)$. Plugging this choice of $b$ in (7) shows that given $N$ random queries, where
\[
N = \left\lceil \frac{e^6 d(B_d^{[\pm 1]} )^{2d+2}}{\varepsilon^{d+1}} \log \left(\frac{2\varepsilon}{\delta} \sum_{k=0}^{d} \binom{n}{k} \right) \right\rceil,
\] (20)

the random function $h_{b(1+\sqrt{d+1})}$ satisfies $\|h_{b(1+\sqrt{d+1})}\|_2 < \varepsilon$ with probability at least $1 - \delta$ and the conclusion of the theorem follows from elementary estimates, such as
\[
\sum_{k=0}^{d} \binom{n}{k} \leq \sum_{k=0}^{d} \frac{n^k}{k!} = \sum_{k=0}^{d} \frac{d^k}{k!} \left( \frac{n}{d} \right)^k \leq \left( \frac{en}{d} \right)^d.
\]

Theorem 1 is a straightforward consequence of Theorem 2.

**Proof of Theorem 1.** Theorem 2 combined with the bound $b_d^{[\pm 1]} \leq \exp(\kappa \sqrt{d \log d})$ of [8] imply the conclusion of Theorem 1 for $\varepsilon \geq \exp(\kappa \sqrt{d \log d})$, where $C \in (0, \infty)$ is a large universal constant.

The case $\varepsilon < \frac{\exp(C \sqrt{d \log d})}{n}$ follows from the Low-Degree Algorithm of [12].

### 3. Concluding remarks

We conclude with a few additional remarks on the spectrum of bounded functions defined on the hypercube and corresponding learning algorithms. For a function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$, its Rademacher projection on level $\ell \in \{1, \ldots, n\}$ is defined as
\[
\forall x \in \{-1, 1\}^n, \quad \text{Rad}_\ell f(x) = \sum_{S \subseteq \{1, \ldots, n\} \atop |S| = \ell} \hat{f}(S)w_S(x).
\] (21)

1. The first main theorem of [11] asserts that if $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ is a function of degree $d$, then
\[
\forall \ell \in \{1, \ldots, d\}, \quad \|\text{Rad}_\ell f\|_{L_\infty} \leq \left\{ \begin{array}{ll}
\frac{T_d^{[\pm 1]}(0)}{\ell!} \cdot \|f\|_{L_\infty}, & \text{if } (d-\ell) \text{ is even} \\
\frac{T_d^{[\pm 1]}(0)}{\ell!} \cdot \|f\|_{L_\infty}, & \text{if } (d-\ell) \text{ is odd}
\end{array} \right.,
\] (22)

where $T_d(t)$ is the $d$-th Chebyshev polynomial of the first kind, that is, the unique real polynomial of degree $d$ such that $\cos(d\theta) = T_d(\cos\theta)$ for every $\theta \in \mathbb{R}$. Moreover, Iyer, Rao, Reis, Rothvoss and Yehudayoff observed in [11, Proposition 2] that this estimate is asymptotically sharp. We present a simple proof of their inequality (22) (see also [10] for related arguments).
Proof of (22). For any \( f : [-1, 1]^n \to \mathbb{R} \) consider its harmonic extension on \([-1, 1]^n\),

\[
\forall (x_1, \ldots, x_n) \in [-1, 1]^n, \quad f(x_1, \ldots, x_n) = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) \prod_{j \in S} x_j.
\]

(23)

By convexity \( \|\hat{f}\|_{L^\infty([-1, 1]^n)} = \|f\|_{L^\infty([-1, 1]^n)} \). In particular, the restriction of \( \hat{f} \) on the ray \( t(x_1, \ldots, x_n), t \in [-1, 1] \), i.e.

\[
\forall t \in \mathbb{R}, \quad h_x(t) \overset{\text{def}}{=} \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) w_S(x) t^{|S|}
\]

satisfies \( \max_{t \in [-1, 1]} |h_x(t)| \leq \|f\|_{L^\infty} \) for all \( (x_1, \ldots, x_n) \in [-1, 1]^n \). Therefore, since \( \deg h_x \leq d \), a classical inequality of Markov (see e.g. [5, p. 248]) gives

\[
\left| \text{Rad}_\ell f(x) \right| = \frac{|\mu_x^{(\ell)}(0)|}{\ell!} \leq \left\{ \begin{array}{ll}
\frac{|T^{(\ell)}_x(0)|}{|T^{(\ell)}_x|} \cdot \|f\|_{L^\infty}, & \text{if } (d - \ell) \text{ is even} \\
\frac{|T^{(\ell)}_x|}{|T^{(\ell)}_x|} \cdot \|f\|_{L^\infty}, & \text{if } (d - \ell) \text{ is odd}
\end{array} \right.
\]

(25)

and (22) follows by taking a maximum over all \( x \in [-1, 1]^n \). \( \square \)

In particular, as observed in [11], inequality (22) implies that if \( f \) has degree at most \( d \) then

\[
\forall \ell \in \{1, \ldots, d\}, \quad \left\| \text{Rad}_\ell f \right\|_{L^\infty} \leq \frac{d^\ell}{\ell!} \cdot \|f\|_{L^\infty}.
\]

(26)

2. The second main theorem of [11] asserts that if \( f : [-1, 1]^n \to [-1, 1] \) is a bounded function of degree at most \( d \), then for every \( \ell \in \{1, \ldots, d\} \) we have

\[
\sum_{S \subseteq \{1, \ldots, n\}} |\text{Rad}_\ell f(S)| = \sum_{S \subseteq \{1, \ldots, n\}} |\hat{f}(S)| \leq n^{\frac{\ell}{2\ell - 1}} d^{\ell - \frac{\ell}{2}}.
\]

(27)

The Bocheński–Hille-type inequality of [8] implies the following improved bound.

Corollary 3. Let \( n \in \mathbb{N} \) and \( d \in \{1, \ldots, n\} \). Then, every bounded function \( f : [-1, 1]^n \to [-1, 1] \) of degree at most \( d \) satisfies

\[
\forall \ell \in \{1, \ldots, d\}, \quad \sum_{S \subseteq \{1, \ldots, n\}} |\hat{f}(S)| \leq \left( \frac{n}{\ell} \right)^{\frac{\ell}{2\ell - 1}} e^{\kappa \sqrt{\ell} \log \ell} d^{\ell - \frac{\ell}{2}} \leq n^{\frac{\ell}{2\ell - 1}} d^{\ell - \frac{\ell}{2}},
\]

(28)

for some universal constant \( c \in (0, 1) \).

Proof. Combining Hölder’s inequality with the estimate of [8] and (26) we get

\[
\sum_{S \subseteq \{1, \ldots, n\}} |\hat{f}(S)| \leq \left( \frac{n}{\ell} \right)^{\frac{\ell}{2\ell - 1}} \left( \sum_{S \subseteq \{1, \ldots, n\}} |\text{Rad}_\ell f(S)|^{\frac{2\ell}{\ell - 1}} \right)^{\frac{\ell - 1}{\ell}} \leq \left( \frac{n}{\ell} \right)^{\frac{\ell}{2\ell - 1}} \exp(\kappa \sqrt{\ell} \log \ell) \left\| \text{Rad}_\ell f \right\|_{L^\infty}^{(26)} \leq \left( \frac{n}{\ell} \right)^{\frac{\ell}{2\ell - 1}} \exp(\kappa \sqrt{\ell} \log \ell) d^{\ell - \frac{\ell}{2}}.
\]

(29)

The last inequality of (28) follows from (22) and the elementary bound \( \binom{n}{\ell} \leq \left( \frac{ne}{\ell} \right)^\ell \). \( \square \)

We refer to the recent work [2] for a systematic study of inequalities relating the Fourier growth with various well-studied properties of Boolean functions.

3. It is straightforward to observe (see also [15, Proposition 3.31]) that if \( f : [-1, 1]^n \to [-1, 1] \) is a Boolean function and \( h : [-1, 1]^n \to \mathbb{R} \) is an arbitrary function, then

\[
\left\| \text{sign}(h) - f \right\|_{L^2}^2 = 4 \mathbb{P} \{ \text{sign}(h) \neq f \} \leq 4 \mathbb{P} \{ |h - f| \geq 1 \} \leq 4 \|h - f\|_{L^2}^2,
\]

(30)

where we define \( \text{sign}(0) \) as \( \pm 1 \) arbitrarily. Therefore, applying Theorem 1 to a Boolean function, the above algorithm produces a Boolean function \( \hat{h} = \text{sign}(h) \) which is a \( 4\varepsilon \)-approximation of \( f \).
References


(A. E.) Trinity College and Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK.

Email address: ae466@cam.ac.uk

(P. I.) Department of Mathematics, University of California, Irvine, Irvine, CA 92617, USA.

Email address: pivanisv@uci.edu