Lecture notes on Elliptic Partial Differential Equations

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1.1 The Dirichlet problem and minimization

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set and $\Delta = \sum_{i=1}^n \partial_{ii}$ be the Laplacian.

Definition 1.1. A function $u \in C^2(\Omega)$ is said to be *harmonic* if $\Delta u = 0$, *subharmonic* if $\Delta u \geq 0$ and *superharmonic* if $\Delta u \leq 0$ in Ω.

We are concerned with existence of harmonic functions with prescribed boundary value, i.e. with finding a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the *Dirichlet problem*:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $g \in C^0(\partial\Omega)$ is given.

Problem (1.1) is closely related to the following variational problem: find a minimizer for

$$\mathcal{D}[u] \equiv \frac{1}{2} \int_\Omega |\mathrm{D}u|^2 \, \mathrm{d}x$$

in the class

$$\mathcal{A} \equiv \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : u = g \text{ on } \partial\Omega\}.$$

Indeed, if a minimizer exists then for any $\varphi \in C_c^\infty(\Omega)$ we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{D}[u + t\varphi]\bigg|_{t=0} = \int_\Omega \mathrm{D}u \cdot \mathrm{D}\varphi = -\int_\Omega \Delta u \, \varphi$$

and so, due to the arbitrariness of φ, u is harmonic.

Dirichlet thought that the existence of minimizers of $\mathcal{D}$ was clear, and did not require a proof. It turns out that this is not the case, and that minimizers of variational integrals do not always exist:

Example 1.2. Consider the functional

$$\mathcal{F}[u] = \int_0^1 (1 + \dot{u}^2)^{\frac{1}{4}} \, \mathrm{d}x,$$

defined on $\mathcal{A} \equiv \{u \in \mathrm{Lip}([0,1]) : u(0) = 0, u(1) = 1\}$. The sequence $u_n(x) = (1 - nx)\chi_{\{x \leq \frac{1}{n}\}}$ shows that the infimum of $\mathcal{F}$ on $\mathcal{A}$ is 1, but clearly it cannot be achieved.

This example already shows that the existence of minimizers is subtle. In fact, although any minimizer of $\mathcal{D}$ is a harmonic function, the converse is not true (see Example Sheet 1):

Example 1.3. There is $u \in C^\infty(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$ such that $\Delta u = 0$ in $\mathbb{D}$, but $\mathcal{D}[u] = \infty$. In this case, with $g = u_{\mathbb{S}^1}$, there is no extension of g to $W^{1,2}(\mathbb{D})$.

The above shows that the variational approach is in some sense unsuitable (although we will return to it later), and so we will take a different approach.

1.2 Basic properties of harmonic functions

We begin with the following key result:

Proposition 1.4 (Mean value property)**.** *Let $u \in C^2(\Omega)$ be subharmonic. Then, for every ball $B_r(x_0) \Subset \Omega$, we have*

$$u(x) \le \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u \, \mathrm{d}\sigma = ⨍_{\partial B_r(x_0)} u \, \mathrm{d}\sigma,$$
$$u(x) \le \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u \, \mathrm{d}x = ⨍_{B_r(x_0)} u \, \mathrm{d}x.$$

Here σ denotes the area measure of the sphere and $\omega_n = |B_1(0)|$, so that $\sigma(\partial B_1(0)) = n\omega_n$.

Proof. We have

$$\begin{aligned} 0 \le \int_{B_\rho(x_0)} \Delta u &= \int_{\partial B_\rho(x_0)} \mathrm{D}u(x) \cdot \frac{x - x_0}{\rho} \, \mathrm{d}\mathscr{H}^{n-1}(x) \\ &= \rho^{n-1} \int_{\mathbb{S}^{n-1}} \mathrm{D}u(x_0 + \rho\omega) \cdot \omega \, \mathrm{d}\omega \\ &= \rho^{n-1} \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial \rho} u(x_0 + \rho\omega) \, \mathrm{d}\omega. \end{aligned}$$

Since this holds for all $\rho \le r$, we see that $\rho \mapsto \int_{\mathbb{S}^{n-1}} u(x_0 + \rho\omega) \, \mathrm{d}\omega$ is increasing, so

$$\int_{\mathbb{S}^{n-1}} u(x_0 + \rho\omega) \, \mathrm{d}\omega \le \int_{\mathbb{S}^{n-1}} u(x_0 + r\omega) \, \mathrm{d}\omega = \frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u$$

and sending $\rho \to 0$, by continuity of u, we find that

$$n\omega_n r^{n-1} u(x_0) \le \int_{\partial B_r(x_0)} u.$$

This proves the first inequality, and after integrating in r we find the second one. □

Corollary 1.5 (Strong maximum principle)**.** *Let Ω be connected. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is subharmonic then it cannot achieve its maximum in Ω unless it is constant.*

Proof. By continuity the set $S = \{x \in \Omega : u(x) = \max_\Omega u\}$ is closed, and by the mean value inequality it is open. Hence it is either empty or Ω. □

The strong maximum principle holds for more general equations, although the proof is then more involved. We will return to this point later. Let us note two further consequences:

Corollary 1.6. *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then:*

(i) *Comparison principle: if u is subharmonic, v is superharmonic and $u \le v$ on $\partial\Omega$ then $u \le v$ in Ω.*

(ii) *Uniqueness: two harmonic functions on Ω that agree on $\partial\Omega$ are equal.*

Proof. (i) follows from $u - v$ being subharmonic, (ii) follows from (i). □

An important fact is that the mean value property characterizes harmonic functions:

Proposition 1.7. *If $u \in C^0(\Omega)$ satisfies the mean value equality*

$$u(x) = \fint_{B_r(x)} u(y)\,\mathrm{d}y \text{ for all } B_r(x) \subset \Omega,$$

then $u \in C^\infty(\Omega)$ and it is harmonic.

Proof. For $u \in C^2(\Omega)$ this follows from the proof of the mean value inequality. For the general case, see Example Sheet 1. $\square$

Another important consequence of the mean value formula is:

Theorem 1.8 (Harnack inequality)**.** *Let $u \in C^2(\Omega)$ be harmonic and $u \geq 0$. If $B_{4r}(x_0) \subset \Omega$ then*

$$\sup_{B_r(x_0)} u \leq 3^n \inf_{B_r(x_0)} u.$$

Proof. If $y_1, y_2 \in B_r(x_0)$, then $B_r(y_1) \subset B_{3r}(y_2) \subset B_{4r}(x_0) \subseteq \Omega$, so by the mean value formula

$$u(y_1) = \frac{1}{\omega_n r^n} \int_{B_r(y_1)} u\,\mathrm{d}x \leq \frac{1}{\omega_n r^n} \int_{B_{3r}(y_2)} u\,\mathrm{d}x = \frac{3^n}{\omega_n (3r)^n} \int_{B_{3r}(y_2)} u\,\mathrm{d}x = 3^n u(y_2),$$

as claimed. $\square$

Corollary 1.9 (Harnack's Principle)**.** *Given an increasing sequence of harmonic functions (u_j) on Ω, either $u_j(x) \to \infty$ for all $x \in \Omega$ or there is a harmonic function u such that $u_j \to u$ locally uniformly.*

Proof. Fix some $x_0 \in \Omega$. If $(u_j(x_0))$ is Cauchy, then for any $\varepsilon > 0$ we have $0 \leq u_n(x_0) - u_m(x_0) < \varepsilon$ provided $n \geq m$ are sufficiently large, so by Harnack's inequality we see that $\sup_{B_r(x)}[u_n - u_m] \leq 3^n \varepsilon$ whenever $B_{3r}(x_0) \Subset \Omega$. $\square$

Corollary 1.10 (Liouville's theorem)**.** *If $u \in C^\infty(\mathbb{R}^n)$ is harmonic and bounded from below then it is constant.*

Proof. With $m = \inf u$ we have $\sup_{B_R}(u - m) \leq 3^n \inf_{B_R}(u - m) \to 0$ as $R \to \infty$. $\square$

Proposition 1.11 (Derivative estimates)**.** *Let $u \in C^2(\Omega)$ be harmonic. If $B_r(x_0) \subset \Omega$ then*

$$|\mathrm{D}u(x_0)| \leq \frac{C(n)}{r^{n+1}} \|u\|_{L^1(B_r(x_0))}.$$

Proof. Note that, for any i, the function u_{x_i} is harmonic, therefore

$$|u_{x_i}(x_0)| = \left| \fint_{B_{r/2}(x_0)} u_{x_i}\,\mathrm{d}x \right| = \left| \frac{2^n}{\omega_n r^n} \int_{\partial B_{r/2}(x_0)} u \nu_i \,\mathrm{d}\sigma \right| \leq \frac{2^n}{r} \|u\|_{L^\infty(\partial B_{r/2}(x_0))}.$$

By the mean value formula, if $x \in \partial B_r(x_0)$ then $B_{r/2}(x) \subset B_r(x_0)$, so

$$|u(x)| \leq \frac{1}{\omega_n} \left(\frac{2}{r}\right)^n \|u\|_{L^1(B_r(x_0))}$$

as wished. $\square$

Proposition 1.12. *Let $u \in C^\infty(B_{2r}(x_0))$ be harmonic. Then*

$$|\partial^\alpha u(x_0)| \le \frac{n^k e^{k-1} k!}{r^k} \max_{B_r(x_0)} |u|, \qquad |\alpha| = k.$$

Proof. By induction. Let

$$\rho = (1-\theta) r, \qquad \theta \in (0,1).$$

Let $\alpha = \beta + e_i$, with $|\beta| = k$. Then

$$|\partial^\alpha u(x_0)| = |\partial_i \partial^\beta u(x_0)| \le \frac{n}{\rho} \max_{B_\rho(x_0)} |\partial^\beta u|.$$

Using the induction hypothesis,

$$|\partial^\alpha u(x_0)| \le \frac{n}{\rho} \cdot \frac{n^k e^{k-1} k!}{(r-\rho)^k} \max_{B_r(x_0)} |u|.$$

Since $r - \rho = \theta r$ and $\rho = (1-\theta) r$, this becomes

$$|\partial^\alpha u(x_0)| = \frac{n^{k+1} e^{k-1} k!}{r^{k+1} \theta^k (1-\theta)} \max_{B_r(x_0)} |u|.$$

Set $\theta = \frac{k}{k+1}$, then

$$\frac{1}{1-\theta} = k+1, \qquad \frac{1}{\theta^k(1-\theta)} = \left(1 + \frac{1}{k}\right)^k (k+1) < e(k+1)$$

and the conclusion follows. □

Remark 1.13. By the mean value formula, for any $x \in B_{r/2}(x_0)$,

$$u(x) = \frac{1}{\omega_n r^n} \int_{B_r(x)} u \le \frac{1}{\omega_n r^n} \int_{B_{2r}(x_0)} u = \frac{1}{\omega_n r^n} \|u\|_{L^1(B_{2r}(x_0))}.$$

Hence

$$\max_{B_{r/2}(x_0)} u \le \frac{1}{\omega_n r^n} \|u\|_{L^1(B_{2r}(x_0))}.$$

Theorem 1.14. *Let $u \in C^\infty(\Omega)$ be harmonic. Then u is analytic.*

Proof. Fix $x \in \Omega$, and let $B_{2r}(x) \subset \Omega$. Let $|h| \le r$. Then

$$u(x+h) = u(x) + \sum_{j=1}^{k-1} \frac{1}{j!} \Big[(h_1 \partial_1 + \cdots + h_n \partial_n)^j u \Big](x) + R_k(h),$$

where

$$R_k(h) = \frac{1}{k!} \Big[(h_1 \partial_1 + \cdots + h_n \partial_n)^k u \Big](x + \theta h), \qquad \text{for some } \theta \in (0,1).$$

Using the previous proposition,

$$|R_k(h)| \le \frac{|h|^k n^k}{k!} \cdot \frac{n^k e^{k-1} k!}{r^k} \max_{B_{2r}(x)} |u| \le \left(\frac{|h| n^2 e}{r} \right)^k \max_{B_{2r}(x)} |u|.$$

So if $|h|\, n^2 e < \frac{r}{2}$, then $R_k(h) \to 0$ as $k \to \infty$. Therefore the power series of u is convergent, and u is analytic. □

1.3 Perron's method and barriers

We now return to the Dirichlet problem (1.1), with the goal of constructing solutions. Let us note that, in the case of a ball, one can always solve (1.1) *explicitly*:

Proposition 1.15. *Let $g \in C^0(\partial B_r(0))$, and define*

$$u(x) = \begin{cases} \frac{r^2-|x|^2}{n\omega_n r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} \, \mathrm{d}\sigma(y) & \text{if } x \in B_r(0), \\ g(x) & \text{if } x \in \partial B_r(0). \end{cases}$$

Then $u \in C^\infty(B_r(0)) \cap C^0(\overline{B}_r(0))$ is harmonic.

Proof. The smoothness of u in the interior is clear. Now let $P(x,y) = \frac{r^2-|x|^2}{n\omega_n r|x-y|^n}$. A direct computation shows that

$$\Delta_x P(x,y) = \frac{2}{\omega_n r} \frac{r^2-|y|^2}{|x-y|^{n+2}},$$

and so u is harmonic. Moreover, we have

$$\int_{\partial B_r(0)} P(x,y) \, \mathrm{d}\sigma(y) = 1.$$

This can be seen in many ways, but follows for instance by taking $g = 1$ and showing that, in that case, also $u = 1$. Indeed, from the definition it is easy to see that u is radial, i.e. $u(x) = u(Qx)$ for all $Q \in \mathrm{O}(n)$, hence $u(x) = \varphi(|x|)$. Since u is harmonic, we have

$$\Delta u = \ddot{\varphi}(r) + \frac{n-1}{r}\dot{\varphi}(r) = 0,$$

and so $\varphi(r) = A + Br^{2-n}$ if $n \geq 3$ and $\varphi(r) = A + B \log r$ if $n = 2$. Since u is smooth we must have $B = 0$, i.e. $u = u(0)$ is constant, but

$$u(0) = \int_{\partial B_r} \frac{r^2}{n\omega_n r r^n} \, \mathrm{d}\sigma(y) = 1.$$

For the continuity, let $x_0 \in \partial B_r(0)$ and for $\varepsilon > 0$ choose $\delta > 0$ so that $|g(x) - g(x_0)| \leq \varepsilon$ if $x \in \partial B_r(0) \cap B_\delta(x_0)$. Then, for $x \in B_r(0) \cap B_{\delta/2}(x_0)$, we have

$$\begin{aligned} |u(x) - g(x_0)| &= \left| \int_{\partial B_r(0)} P(x,y)[g(y) - g(x_0)] \, \mathrm{d}\sigma(y) \right| \\ &\leq \left[\int_{\partial B_r(0) \cap B_\delta(x_0)} + \int_{\partial B_r(0) \setminus B_\delta(x_0)} \right] P(x,y)|g(y) - g(x_0)| \, \mathrm{d}\sigma(y) \\ &\leq \varepsilon + \frac{(r^2-|x|^2)r^{n-2}}{(\frac{\delta}{2})^n} 2\|g\|_{C^0(\partial\Omega)} \end{aligned}$$

and so $u(x) \to g(x_0)$ as $x \to x_0$. □

Given $g \in C^0(\partial\Omega)$, let us consider the classes

$$S_\pm \equiv \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : \pm\Delta u \leq 0 \text{ in } \Omega, \pm u \geq \pm g \text{ on } \partial\Omega\}.$$

Note that $S_\pm \neq \emptyset$, since the constant function $\max_{\partial\Omega} g \in S_+$, and $\min_{\partial\Omega} g \in S_-$. By the comparison principle, we have $v \le u$ for each $v \in S_-, u \in S_+$. Now we define:

$$u_*(x) = \sup_{u\in S_-} u(x), \qquad u^*(x) = \inf_{u\in S_+} u(x).$$

Lemma 1.16. *The functions u_* and u^* are harmonic in Ω.*

Proof. We just prove that u_* is harmonic in a ball $B \subset \Omega$. Fix $x_0 \in B$, then by definition there is a sequence $v_j \in S_-$ with $v_j(x_0) \to u_*(x_0)$. Define

$$v_j' = \max\{v_1, \ldots, v_j\} \in S_-,$$

and let v_j'' be the harmonic extension of $v_j'|_{\partial B}$ to B, given by Proposition 1.15. Observe that v_j' is an increasing sequence and that therefore, by the comparison principle, so is v_j''. By the maximum principle, the sequence v_j'' is bounded above by $\sup_{\partial\Omega} g$ and so by Harnack's Principle there is a harmonic function h with $v_j'' \to h$ locally uniformly in B.

We claim that $h = u_*$ in B. Certainly $h \le u_*$ and $h(x_0) = u_*(x_0)$. If $h(z) < u_*$ for some $z \in B$, let $w \in S_-$ be such that $w(z) > h(z)$ and define $w_j = \max\{v_j'', w\}$. Define also w_j', w_j'' similarly to before, so again $w_j'' \to \tilde{h}$ for some harmonic function $\tilde{h}$. Since $v_j'' \le w_j''$ we must have $h \le \tilde{h}$ and again $h(x_0) = \tilde{h}(x_0)$, so by the Strong Maximum Principle in fact $h = \tilde{h}$. However, this is a contradiction:

$$\tilde{h}(z) = \lim_j w_j''(z) \ge w(z) > h(z) = \tilde{h}(z).$$

This finishes the proof. □

At this point, it is not clear if u_*, u^* are equal, or whether they achieve the boundary data; in fact, this is not the case in general.

Definition 1.17. A point $x_0 \in \partial\Omega$ is *regular* if for every $g \in C^0(\partial\Omega)$ and every $\varepsilon > 0$ sufficiently small there is $v \in S_-$ and $w \in S_+$ such that $g(x_0) - v(x_0) \le \varepsilon$ and $w(x_0) - g(x_0) \le \varepsilon$.

Remark 1.18. Note that (1.1) has a solution for every $g \in C^0(\partial\Omega)$ if and only if every point of $\partial\Omega$ is regular. Indeed, if (1.1) always has a solution u we can take v, w as $u - \varepsilon$ and $u + \varepsilon$ respectively. For the other direction, note that if x_0 is regular then

$$g(x_0) - \varepsilon \le v(x_0) \le u_*(x_0) \le u^*(x_0) \le w(x_0) \le g(x_0) + \varepsilon$$

and so, sending $\varepsilon \to 0$, we conclude that u_* and u^* attain the boundary value g continuously. By uniqueness they also coincide and therefore (1.1) is solvable.

Barriers provide a useful criterion for a point to be regular:

Definition 1.19. Given $x_0 \in \partial\Omega$, an *upper barrier* at x_0 is a superharmonic function $b \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $b(x_0) = 0$ and $b > 0$ in $\overline{\Omega} \setminus \{x_0\}$.

Lemma 1.20. *Suppose that $x_0 \in \partial\Omega$ admits upper barriers. Then x_0 is regular.*

Proof. Define $M = \max_{\partial\Omega} |g|$ and, for each $M > \varepsilon > 0$, let $\delta > 0$ be such that $|g(x) - g(x_0)| < \varepsilon$ when $|x - x_0| < \delta$ for $x \in \partial\Omega$. Now let b be an upper barrier and note that by compactness $\inf_{\overline{\Omega}\setminus B_\delta(x_0)} b > 0$, hence we can find $k > 0$ such that $kb(x) \ge 3M$ if $|x - x_0| \ge \delta$. Now define

$$v(x) = g(x_0) - \varepsilon - kb(x), \qquad w(x) = g(x_0) + \varepsilon + kb(x),$$

and observe that $v \in S_-, w \in S_+$. Moreover $w(x_0) - g(x_0) = \varepsilon$ and $g(x_0) - v(x_0) = \varepsilon$, so the conclusion follows from the above remark. □

Proposition 1.21. *Let Ω satisfy the exterior sphere condition, i.e.*

$$\text{for every } x_0 \in \partial\Omega \text{ there is an open ball } B \text{ with } B \cap \Omega = \emptyset, \overline{B} \cap \overline{\Omega} = \{x_0\}.$$

Then every point of $\partial\Omega$ admits upper barriers and is therefore regular.

Proof. For $x_0 \in \partial\Omega$ let $B = B_R(y)$ be a ball as in the statement. Let $b(x) = R^{2-n} - |x-y|^{2-n}$ if $n > 2$ and $b(x) = \log \frac{|x-y|}{R}$ if $n = 2$. Then it is easy to see that $\Delta b = 0$ in $\mathbb{R}^n \setminus \{y\}$, so b is an upper barrier. □

2 Variational methods

2.1 Existence of Lipschitz minimizers

We now turn to constructing minimizers of rather general variational integrals

$$\mathcal{F}[u] = \int_\Omega F(\mathrm{D}u)\,\mathrm{d}x.$$

The Dirichlet energy is a particular example of such a functional, but we can consider other examples, such as the area functional

$$\mathcal{A}[u] = \int_\Omega \sqrt{1+|\mathrm{D}u|^2}\,\mathrm{d}x,$$

or the exponential energy

$$\mathcal{E}[u] = \int_\Omega \exp(|\mathrm{D}u|^2)\,\mathrm{d}x.$$

A basic question at this point is what is the class of functions in which we should look for minimizers. This is a surprisingly subtle question, and in the general setting different choices may lead to different minimizers (see Example Sheet).

In this course we will minimize $\mathcal{F}$ over Lipschitz functions; by Rademacher's theorem, such functions are a.e. differentiable and hence, whenever the integrand satisfies $F \in C^0(\mathbb{R}^n)$, the above integrals are perfectly well-defined. Let us give the precise definition:

Definition 2.1. Let $g \in \mathrm{Lip}(\partial\Omega)$ and $\mathrm{Lip}_g(\Omega) = \{v \in \mathrm{Lip}(\Omega) : v = g \text{ on } \partial\Omega\}$.

A function $u \in \mathrm{Lip}(\Omega)$ is a minimizer of $\mathcal{F}$ if $\mathcal{F}[u] \leq \mathcal{F}[v]$ for all $v \in \mathrm{Lip}_u(\Omega)$.

We are particularly interested, as in the above examples, in *convex* integrands $F\colon \mathbb{R}^n \to \mathbb{R}$. In this case, there is a clean relationship between minimizers and solutions of PDEs. Let us recall the notion of weak solution:

Definition 2.2. A function $u \in \mathrm{Lip}(\Omega)$ is a *weak solution* of

$$\mathrm{div}(F'(\mathrm{D}u)) = 0 \tag{2.1}$$

if, for all $\varphi \in \mathrm{Lip}_0(\Omega)$, we have

$$\int_\Omega \langle F'(\mathrm{D}u), \mathrm{D}\varphi\rangle\,\mathrm{d}x = 0.$$

Recall that F is *convex* if $F(\theta\xi_1 + (1-\theta)\xi_2) \leq \theta F(\xi_1) + (1-\theta)F(\xi_2)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$ and $\theta \in (0,1)$. It is *strictly convex* if the inequality is strict when $\xi_1 \neq \xi_2$. If $F \in C^1(\mathbb{R}^n)$, convexity is equivalent to the inequality

$$F(\xi_1) \geq F(\xi_2) + \langle F'(\xi_2), \xi_1 - \xi_2\rangle, \qquad \xi_1, \xi_2 \in \mathbb{R}^n.$$

Proposition 2.3 (Euler–Lagrange equation)**.** *Let $F \in C^1(\mathbb{R}^n)$ and let $u \in \mathrm{Lip}(\Omega)$.*

(i) *If u is a minimizer of $\mathcal{F}$ then u is a weak solution of* (2.1)*.*

(ii) *If F is convex and u is a weak solution of* (2.1) *then u is a minimizer of $\mathcal{F}$.*

Proof. For (i), take any $\varphi \in \mathrm{Lip}_0(\Omega)$ and let $f(t) = \mathcal{F}[u + t\varphi]$. We can then write

$$\frac{f(t)-f(0)}{t} = \frac{1}{t}\int_\Omega \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} F(\mathrm{D}u(x) + s\,\mathrm{D}\varphi(x))\,\mathrm{d}s\,\mathrm{d}x = \int_\Omega h(x,t)\,\mathrm{d}x,$$

where we set

$$h(x,t) \equiv \frac{1}{t}\int_0^t \langle F'(\mathrm{D}u(x) + s\,\mathrm{D}\varphi(x)), \mathrm{D}\varphi(x)\rangle\,\mathrm{d}s.$$

Since u is a minimizer, we must have

$$0 = \lim_{t\to 0}\frac{f(t)-f(0)}{t} = \lim_{t\to 0}\int_\Omega h(x,t)\,\mathrm{d}x$$

provided the limits exist. Moreover, for $|t| \le 1$, we have

$$|h(\cdot,t)| \le \|\mathrm{D}\varphi\|_{L^\infty(\Omega)}\|F'\|_{L^\infty(B_{\mathrm{Lip}(\varphi)+\mathrm{Lip}(u)}(0))},$$

thus the limit exists by the Dominated Convergence Theorem, and since for a.e. x we have

$$\lim_{t\to 0} h(x,t) = F'(\mathrm{D}u(x))\cdot\mathrm{D}\varphi(x),$$

the conclusion follows.

For (ii), by the convexity assumption, we have a.e. the inequality

$$F(\mathrm{D}v) \ge F(\mathrm{D}u) + \langle F'(\mathrm{D}u), \mathrm{D}v - \mathrm{D}u\rangle$$

whenever $v \in \mathrm{Lip}(\Omega)$ and so

$$\mathcal{F}[v] \ge \mathcal{F}[u] + \int_\Omega \langle F'(\mathrm{D}u), \mathrm{D}(v-u)\rangle\,\mathrm{d}x.$$

If moreover $v = u$ on $\partial\Omega$ then the last integral vanishes since u is a weak solution of (2.1). □

In order to construct minimizers for $\mathcal{F}$, we use the Direct Method. The key is the following lower semicontinuity result:

Lemma 2.4. *If $F \in C^1(\mathbb{R}^n)$ is convex and if $u_j \to u$ in $C^0(\Omega)$ with $\mathrm{Lip}(u_j) \le k$, then $\mathrm{Lip}(u) \le k$ and*

$$\liminf_{j\to\infty}\mathcal{F}[u_j] \ge \mathcal{F}[u].$$

Proof. The fact that $\mathrm{Lip}(u) \le k$ is clear. Approximate $F'(\mathrm{D}u)$ in $L^1(\Omega,\mathbb{R}^n)$ by a sequence $G_\varepsilon \in C_c^\infty(\Omega,\mathbb{R}^n)$. Then, by convexity,

$$\begin{aligned}\int_\Omega F'(\mathrm{D}u)\,\mathrm{d}x &\le \int_\Omega F'(\mathrm{D}u_j)\,\mathrm{d}x - \int_\Omega \langle F'(\mathrm{D}u) - G_\varepsilon, \mathrm{D}u_j - \mathrm{D}u\rangle\,\mathrm{d}x - \int_\Omega \langle G_\varepsilon, \mathrm{D}u_j - \mathrm{D}u\rangle\,\mathrm{d}x\\ &= \int_\Omega F'(\mathrm{D}u_j)\,\mathrm{d}x + o(1) + \int_\Omega (u_j - u)\,\mathrm{div}\,G_\varepsilon\,\mathrm{d}x\end{aligned}$$

as $\varepsilon \to 0$, where we integrated by parts. Hence the conclusion follows by first sending $j \to \infty$ and then $\varepsilon \to 0$. □

We say that $u \in \mathrm{Lip}(\Omega,k)$ if $u \in \mathrm{Lip}(\Omega)$ with $\mathrm{Lip}(u) \le k$.

Proposition 2.5. *Let $F \in C^1(\mathbb{R}^n)$ be convex and $g \in \mathrm{Lip}(\partial\Omega,k)$. Then F has a minimizer u in $\mathrm{Lip}_g(\Omega,k)$. If $\mathrm{Lip}(u) < k$ then u is a minimizer in $\mathrm{Lip}_g(\Omega)$.*

Proof. Take a sequence $(u_j) \subset \mathrm{Lip}_g(\Omega,k)$ such that $\mathcal{F}[u_j] \to \inf_{\mathrm{Lip}_g(\Omega,k)}\mathcal{F}$. Such a sequence is equi-bounded and equi-continuous, hence by the Ascoli–Arzelà Theorem it converges (up to a subsequence) uniformly to $u \in \mathrm{Lip}_g(\Omega, k)$. By Lemma 2.4,

$$\mathcal{F}[u] \le \liminf_{j\to\infty} \mathcal{F}[u_j] = \inf_{\mathrm{Lip}_g(\Omega,k)} \mathcal{F},$$

hence u is a minimizer in $\mathrm{Lip}_g(\Omega, k)$.

Now if $\mathrm{Lip}(u) < k$, given $w \in \mathrm{Lip}_g(\Omega)$ we can find t small enough so that $tw + (1-t)u \in \mathrm{Lip}_g(\Omega, k)$. Since F is convex and u minimizes $\mathcal{F}$ in $\mathrm{Lip}_g(\Omega, k)$, we have

$$\mathcal{F}[u] \le \mathcal{F}[tw + (1-t)u] \le t\mathcal{F}[w] + (1-t)\mathcal{F}[u],$$

i.e. $\mathcal{F}[u] \le \mathcal{F}[w]$. □

Thus, to meaningfully solve the minimization problem in $\mathrm{Lip}(\Omega)$ we need to prove a priori gradient estimates on minimizers.

2.2 Weak maximum principle

Definition 2.6. Given a variational integral $\mathcal{F}$, we say that $u \in \mathrm{Lip}(\Omega)$ is a *super-minimum* (respectively *sub-minimum*) if $\mathcal{F}[v] \ge \mathcal{F}[u]$ for all $v \in \mathrm{Lip}_u(\Omega)$ with $v \ge u$ (respectively $v \le u$).

Proposition 2.7 (Comparison principle)**.** *If F is strictly convex, u is a super-minimum and v is a sub-minimum in* $\mathrm{Lip}(\Omega)$ *with $v \le u$ on $\partial\Omega$ then $v \le u$ in Ω.*

Proof. If not, then the open set $O = \{v > u\} \Subset \Omega$ is non-empty. Consider the competitors $M(x) = \max\{u(x), v(x)\}$ and $m(x) = \min\{u(x), v(x)\}$, so that $\mathcal{F}[u] \le \mathcal{F}[M]$ and $\mathcal{F}[v] \le \mathcal{F}[m]$, or equivalently

$$\int_O F(\mathrm{D}v)\,\mathrm{d}x \le \int_O F(\mathrm{D}u)\,\mathrm{d}x \le \int_O F(\mathrm{D}v)\,\mathrm{d}x,$$

i.e. equality holds. But then, by strict convexity of F, with $w = \frac{u+v}{2}$, we have

$$\int_O F(\mathrm{D}w)\,\mathrm{d}x < \int_O \frac{1}{2}F(\mathrm{D}u) + \frac{1}{2}F(\mathrm{D}v) = \int_O F(\mathrm{D}v) \le \int_O F(\mathrm{D}w)\,\mathrm{d}x,$$

since $w \ge u$ in O and $w = u$ on ∂O. □

Corollary 2.8. *If u is a super-minimum and v is a sub-minimum of $\mathcal{F}$ in* $\mathrm{Lip}(\Omega)$ *then*

$$\sup_{\Omega}(v-u) \le \sup_{\partial\Omega}(v-u).$$

Proof. It is easy to see that $u + \sup_{\partial\Omega}(v-u)$ is a super-minimum which is not smaller than v on $\partial\Omega$, hence it is above v in Ω by the proposition. □

Lemma 2.9 (Reduction to the boundary)**.** *Let $u \in$* $\mathrm{Lip}(\Omega)$ *be a minimizer of $\mathcal{F}$ in* $\mathrm{Lip}(\Omega)$*. If F is strictly convex then*

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|} = \sup_{x\in\Omega, y\in\partial\Omega} \frac{|u(x)-u(y)|}{|x-y|}. \tag{2.2}$$

Proof. For $x_1, x_2 \in \Omega$ distinct, let $\tau = x_2 - x_1$ and define $u_\tau = u(\cdot + \tau)$, $\Omega_\tau = \tau + \Omega$. Then both u, u_τ are minimizers in $\Omega \cap \Omega_\tau$. Hence, by the corollary, there is $z \in \partial(\Omega \cup \Omega_\tau)$ such that

$$|u(x_1) - u(x_2)| = |u(x_1) - u_\tau(x_1)| \le |u(z) - u_\tau(z)| = |u(z) - u(z+\tau)|.$$

Since $\partial(\Omega \cap \Omega_\tau) \subseteq \partial\Omega \cup \partial\Omega_\tau$, either z or $z+\tau$ is in $\partial\Omega$. If we let M be the right-hand side in (2.2), and as both points are in $\overline{\Omega}$, we see that

$$|u(x_1) - u(x_2)| \le M|\tau| = M|x_1 - x_2|,$$

and the claim follows. □

2.3 Gradient estimates and barriers

Let d be the distance function to $\partial\Omega$ and let $\Omega_t = \{x \in \Omega : d(x) < t\}$.

Definition 2.10. Given a boundary datum $g \in \mathrm{Lip}(\partial\Omega)$, an *upper barrier* (resp. *lower barrier*) is a function $b \in \mathrm{Lip}(\Omega_t)$ such that $b = g$ on $\partial\Omega$, b is a super-minimum (resp. sub-minimum in Ω_t and $b \ge \sup_{\partial\Omega} g$ on $\partial\Omega_t$ (resp. $b \le \inf_{\partial\Omega} g$).

Proposition 2.11. *If F is strictly convex, $g \in \mathrm{Lip}(\partial\Omega)$ and there are upper b_+ and lower barriers b_-, then $\mathcal{F}$ has a minimizer in $\mathrm{Lip}_g(\Omega)$.*

Proof. Consider the Lipschitz functions

$$v^+ = \begin{cases} \min\{b^+, \sup_{\partial\Omega} g\} & \text{in } \Omega_t, \\ \sup_{\partial\Omega} g & \text{in } \Omega \setminus \Omega_t, \end{cases} \qquad v^- = \begin{cases} \max\{b^-, \sup_{\partial\Omega} g\} & \text{in } \Omega_t, \\ \inf_{\partial\Omega} g & \text{in } \Omega \setminus \Omega_t, \end{cases}$$

which are respectively super- and sub-minimum for $\mathcal{F}$ in Ω. Let $k > \mathrm{Lip}(v^+), \mathrm{Lip}(v^-)$ and let u be a minimizer for $\mathcal{F}$ in $\mathrm{Lip}_g(\Omega, k)$ given by Proposition 2.5. By the comparison principle we have

$$\inf_{\partial\Omega} g \le u \le \sup_{\partial\Omega} g,$$

and since also $b^- \le u \le b^+$ in $\partial\Omega_t$ by assumption, in fact we have by the comparison principle

$$v^- \le u \le v^+ \text{ in } \Omega,$$

with equality on $\partial\Omega$. Therefore, given $x \in \Omega, y \in \partial\Omega$, we have

$$v^-(x) - v^-(y) \le u(x) - u(y) \le v^+(x) - v^+(y)$$

and from Lemma 2.9 we see that $\mathrm{Lip}(u) < k$ as well. The conclusion then follows from Proposition 2.5. □

[Strictly speaking, we have used results of the previous section for (super/sub)minima in $\mathrm{Lip}(\Omega, k)$, but the same proofs go through without modifications.]

Thus our task is now to construct barriers. It turns out that this depends on the convexity properties of F. Our first observation is that, just as for Proposition 2.3, a function $b \in C^2(\Omega)$ is a super-minimum if and only if

$$\mathcal{L}(v) := \langle F''(\mathrm{D}b), \mathrm{D}^2 b\rangle = \mathrm{div}(F'(\mathrm{D}b)) \le 0.$$

For simplicity we will write $A(\xi) = F''(\xi)$. When F is strictly convex this is a positive definite matrix and so there are numbers λ, Λ such that, for every $v \in \mathbb{R}^n$, we have

$$\lambda(\xi)|v|^2 \le A(\xi)v \cdot v \le \Lambda(\xi)|v|^2.$$

It turns out that the existence of barriers depends on these values, as well as on the function $\mathcal{E}(\xi) = \langle A(\xi)\xi, \xi\rangle = A^{ij}(\xi)\xi_i\xi_j$:

Lemma 2.12. *Suppose that Ω is C^2 and that $g \in C^2(\mathbb{R}^n)$. If*

$$\limsup_{|\xi|\to\infty} \frac{|\xi|\Lambda(\xi)}{\mathcal{E}(\xi)} < +\infty, \tag{2.3}$$

then upper and lower barriers exist.

Proof. Throughout the proof, C denotes a constant depending only on Ω, g, n. We look for barriers of the form

$$b(x) = g(x) + \psi(d(x)),$$

where ψ is smooth, $\psi(0) = 0, \psi' > 0$ and $\psi'' < 0$. We have $\mathrm{D}b = \mathrm{D}g + \psi'(d)\mathrm{D}d$, and

$$\begin{aligned}
\mathcal{L}b &= \langle A, \mathrm{D}^2 g\rangle + \psi'(d)\langle A, \mathrm{D}^2 d\rangle + \psi''(d)\langle A\,\mathrm{D}d, \mathrm{D}d\rangle \\
&= \langle A, \mathrm{D}^2 g\rangle + \psi'(d)\langle A, \mathrm{D}^2 d\rangle + \frac{\psi''(d)}{\psi'(d)^2}\langle A\,\mathrm{D}(b-g), \mathrm{D}(b-g)\rangle \\
&= \langle A, \mathrm{D}^2 g\rangle + \psi'(d)\langle A, \mathrm{D}^2 d\rangle + \frac{\psi''(d)}{\psi'(d)^2}\left(\mathcal{E}(\mathrm{D}b) + \langle A\,\mathrm{D}g, \mathrm{D}g\rangle - 2\langle A\,\mathrm{D}b, \mathrm{D}g\rangle\right)
\end{aligned}$$

where we write $A = A(\mathrm{D}b)$ for simplicity. Now $\langle A, \mathrm{D}^2 g\rangle \le C\Lambda$, so by the Cauchy–Schwarz inequality for the A-inner product,

$$2|\langle A\,\mathrm{D}b, \mathrm{D}g\rangle| \le 2\sqrt{\mathcal{E}(\mathrm{D}b)}\sqrt{\langle A\,\mathrm{D}g, \mathrm{D}g\rangle} \le \frac{1}{2}\mathcal{E}(\mathrm{D}b) + 2\langle A\,\mathrm{D}g, \mathrm{D}g\rangle \le \frac{1}{2}\mathcal{E}(\mathrm{D}b) + C\Lambda,$$

where again $\Lambda = \Lambda(\mathrm{D}b)$. Thus, since $\psi'' < 0$:

$$\begin{aligned}
\mathcal{L}b &\le C\Lambda + \psi'(d)\langle A, \mathrm{D}^2 d\rangle + \frac{\psi''(d)}{\psi'(d)^2}\left(\tfrac{1}{2}\mathcal{E}(\mathrm{D}b) - C\Lambda\right) \\
&\le C_1(1 + |\mathrm{D}b|)\Lambda + \frac{\psi''(d)}{\psi'(d)^2}\left(\tfrac{1}{2}\mathcal{E}(\mathrm{D}b) - C_2\Lambda\right),
\end{aligned}$$

since

$$\psi'(d) = |\mathrm{D}b - \mathrm{D}g| \le |\mathrm{D}b| + C.$$

Now let ψ' be large enough, so that by (2.3) we have

$$C_2\Lambda \le \frac{1}{4}\mathcal{E}(\mathrm{D}b), \qquad C_1(1 + |\mathrm{D}b|)\Lambda \le \frac{1}{4}C_3\mathcal{E}(\mathrm{D}b),$$

for a new constant C_3, which depends also on the limit in (2.3). Thus

$$\mathcal{L}b \le \frac{1}{4}\mathcal{E}(\mathrm{D}b)\left(C_3 + \frac{\psi''(d)}{\psi'(d)^2}\right).$$

Now choose

$$\psi(t) = \log(1 + \sigma t)/C_3,$$

so that $\frac{\psi''(d)}{\psi'(d)^2} = -C_3$ and $\mathcal{L}b \le 0$. If we take $t_0 = 1/\sqrt{\sigma}$, on Ω_{t_0} we have

$$\psi'(d(x)) = \frac{1}{C_3}\frac{\sigma}{1 + \sigma d(x)} \ge \frac{1}{C_3}\frac{\sigma}{1 + \sigma t_0} = \frac{1}{C_3}\frac{\sigma}{1 + \sqrt{\sigma}}.$$

Since as $\sigma \to \infty$ we have that $t_0 \to 0$, $\mathrm{D}b \to \infty$, by choosing σ large enough we find an upper barrier. Finally, if b^+ is an upper barrier for $-g$ then $-b^+$ is a lower barrier for g, so similarly there are lower barriers. □

We note that (2.3) always holds if F'' is *uniformly elliptic*, i.e. if $\inf_{\xi\in\mathbb{R}^n} \frac{\lambda(\xi)}{\Lambda(\xi)} > 0$. However, this condition is not necessary, since $F(\xi) = \exp(|\xi|^2)$ also satisfies (2.3).

2.4 The area functional

Let $A(\xi) = \sqrt{1+|\xi|^2}$ be the area integrand. Then (2.3) does not hold, and so the previous proposition is not applicable. However, we will see in the example sheet that, under suitable conditions on Ω, the above construction can nevertheless be adapted to the case of the area.

We conclude this section by showing that the Dirichlet problem for the area functional is quite different from the one for the Dirichlet energy, and the following result should be contrasted with Example 1.3:

Proposition 2.13. *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of*

$$\operatorname{div}(A'(\mathrm{D}u)) = 0,$$

then $\mathcal{A}[u] < \infty$.

In fact, this result follows from the following general:

Lemma 2.14. *Let $u \in \mathrm{Lip}(\Omega)$ be a minimizer of $\mathcal{F}$ and assume that F is convex and has linear growth, so there is a constant $M > 0$ such that*

$$F(\xi_1 + \xi_2) \leq F(\xi_1) + M\xi_2.$$

For any $v \in C^1(\overline{\Omega})$ we have

$$\mathcal{F}[u] \leq \mathcal{F}[v] + M\int_{\partial\Omega} |u-v| \,\mathrm{d}\mathscr{H}^{n-1}.$$

Proof. Take $\eta_\varepsilon \in \mathrm{Lip}_0(\Omega)$ so that $\eta_\varepsilon(x) = 1$ if $d(x) > \varepsilon$ and $\eta_\varepsilon(x) = d(x)/\varepsilon$ otherwise. Take the competitor $w_\varepsilon = \eta_\varepsilon v + (1-\eta_\varepsilon)u \in \mathrm{Lip}_u(\Omega)$, so

$$\begin{aligned}\int_\Omega F(\mathrm{D}u) &\leq \int_\Omega F(\mathrm{D}w_\varepsilon) = \int_\Omega F(\eta_\varepsilon \mathrm{D}v + (1-\eta_\varepsilon)\mathrm{D}u + (v-u)\mathrm{D}\eta_\varepsilon) \\ &\leq \int_\Omega F(\eta_\varepsilon \mathrm{D}v + (1-\eta_\varepsilon)\mathrm{D}u) + M|v-u||\mathrm{D}\eta_\varepsilon| \\ &\leq \int_\Omega \eta_\varepsilon F(\mathrm{D}u) + (1-\eta_\varepsilon)F(\mathrm{D}v) + M\frac{1}{\varepsilon}\int_0^\varepsilon \int_{\{d=t\}} |v-u| \,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t \\ &\to \int_\Omega F(\mathrm{D}u) + M\int_{\partial\Omega} |u-v| \,\mathrm{d}\mathscr{H}^{n-1}\end{aligned}$$

as $\varepsilon \to 0$, by the dominated convergence theorem. □

Proof of Proposition 2.13. Apply the lemma in the domains $\{d > \varepsilon\}$ with $v = 0$, noting that u minimizes $\mathcal{A}$ in those domains. □

3 Maximum principles

In this chapter we will develop various forms of the maximum principle that will apply to subsolutions of very general scalar elliptic PDEs, and will also see some applications to linear and nonlinear problems. As opposed to what we did for the Laplacian, we will start with the weak maximum principle, and will deduce the strong one from it by passing through the Hopf boundary point lemma.

We will deal with second order differential operators in non-divergence form,

$$Lu(x) := \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x), \qquad u \in C^2(\Omega), \tag{3.1}$$

where Ω is a domain in $\mathbb{R}^n$, $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ and $a_{ij} = a_{ji}$ (this is no loss provided that $u \in C^2$). We say that L is *elliptic* in Ω if (a_{ij}) is positive-definite everywhere in Ω, meaning that for every $x \in \Omega$ there exist constants $0 < \lambda(x) \le \Lambda(x) < \infty$ such that

$$\lambda(x)|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n.$$

Here and throughout we are implicitly summing over repeated indices. Let us make a few remarks before we begin:

- For most of this section we will not assume any regularity at all for the coefficients, which makes these results very robust and applicable to nonlinear problems.
- We will prove qualitative results, as opposed to quantitative estimates, and our hypotheses will be of a qualitative nature (e.g. boundedness with an arbitrary constant). In particular, for the weak maximum principle no uniform ellipticity will be needed.
- Divergence form operators, $Lu = \partial_i(a_{ij}\partial_j u)$, can be expanded to non-divergence form operators provided that the coefficients (a_{ij}) are regular enough (say C^1). For coefficients with lower regularity (which arise, for example, as the PDE satisfied by the derivatives of minimizers of variational integrals) we need the theory of weak solutions, which we will study at the end of the course. The upshot is that a maximum principle is avaliable, but not all of the results in this chapter hold true.

3.1 The weak maximum principle

Definition 3.1. If L is elliptic in Ω and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, we will say that u is a *subsolution* (resp. *supersolution*) to $Lu = 0$ if $Lu \ge 0$ (resp. $Lu \le 0$).

Theorem 3.2 (Weak maximum principle)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set, L is elliptic in Ω and $|b_i|/\lambda$ is bounded in Ω. Given $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $Lu \ge 0$ in Ω, we have:*

(i) if $c \equiv 0$, then $\sup_\Omega u = \sup_{\partial\Omega} u$.

(ii) if $c \le 0$, then $\sup_\Omega u \le \sup_{\partial\Omega} u^+ = \max\{\sup_{\partial\Omega} u, 0\}$

Proof. We first assume that $c \equiv 0$ and show (i). Note that if we have the strict inequality $Lu > 0$, then u cannot have an interior maximum (i.e. a strong maximum principle holds): indeed, if $x_0 \in \Omega$ attains the maximum, then $\mathrm{D}u(x_0) = 0$ and $\mathrm{D}^2u(x_0) \le 0$, so that $Lu(x_0) =$

$a_{ij}(x_0)\partial_{ij}(x_0) \leq 0$ is a contradiction. Since u is continuous on $\overline{\Omega}$ and Ω is bounded, the maximum in $\overline{\Omega}$ exists and hence must attained in $\partial\Omega$, yielding the weak maximum principle.

In order to achieve the strict inequality, consider $v(x) := e^{\gamma x_1}$ and $u_\varepsilon := u + \varepsilon v$. Then

$$\partial_1 v = \gamma e^{\gamma x_1}, \qquad \partial_i v = 0 \quad \text{for } i \neq 1$$

and

$$\partial_{11} v = \gamma^2 e^{\gamma x_1}, \qquad \partial_{ij} v = 0 \quad \text{for } (i,j) \neq (1,1).$$

Thus

$$Lu_\varepsilon = Lu + \varepsilon Lv \geq (\gamma^2 a_{11} + \gamma b_1) e^{\gamma x_1} \geq \gamma(\gamma\lambda - |b_1|) e^{\gamma x_1} > 0$$

provided that we choose $\gamma > \sup_\Omega |b_1|/\lambda$. Then the weak maximum principle holds for u_ε and

$$\sup_\Omega u \leq \sup_\Omega u_\varepsilon = \sup_{\partial\Omega} u_\varepsilon = \sup_{\partial\Omega} u + \varepsilon \sup_{\partial\Omega} e^{\gamma x_1}.$$

Letting $\varepsilon \searrow 0$ the first assertion follows.

For the second assertion, let $m := \sup_{\partial\Omega} u$. We are claiming that $u \leq m$ in Ω in case $m > 0$, and $u \leq 0$ in Ω in case $m \leq 0$. Let $\Omega^+ := \{u > 0\}$ and denote $L_0 u := a_{ij}\partial_{ij}u + b_i\partial_i u = Lu - cu$. Then, assuming that $\Omega^+ \neq \varnothing$, $L_0 u = Lu - cu \geq Lu \geq 0$ in Ω^+, so that (i) applied to u in Ω^+ gives

$$0 < \sup_{\Omega^+} u = \sup_{\partial\Omega^+} u = \sup_{\partial\Omega^+ \cap \partial\Omega} u = \sup_{\partial\Omega} u.$$

Here we have used the fact that, by continuity of u, at any point $x \in \partial\Omega^+ \cap \Omega$ we must have $u(x) = 0$. Now $0 < \sup_{\partial\Omega} u = m$ leads to a contradiction if $m \leq 0$, so $\Omega^+ = \varnothing$ in this case and hence $u \leq 0$ in Ω. Finally, if $m > 0$, then also by continuity $\Omega^+ \neq \varnothing$ and thus $\sup_\Omega u = \sup_{\Omega^+} u = \sup_{\partial\Omega} u = m$. □

Remark 3.3. The weak maximum principle is in general not valid for $c > 0$, at least when it is not small. For example, given any bounded smooth domain Ω, let φ_1 be the first Dirichlet eigenfunction for the Laplacian and $\lambda_1 > 0$ its eigenvalue, which satisfy

$$-\Delta\varphi_1 = \lambda_1\varphi_1 \text{ in } \Omega, \qquad \varphi_1 = 0 \text{ on } \partial\Omega.$$

It is a well-known fact that $\varphi_1 > 0$ in Ω, but $(\Delta + c)\varphi_1 = 0$ with $c \equiv \lambda_1 > 0$. For concreteness one can take the cube $\Omega = (0,\pi)^n \subset \mathbb{R}^n$ and $\varphi_1(x_1, \ldots, x_n) = \sin(x_1)\cdots\sin(x_n)$, with $\lambda_1 = n$. In Example Sheet 2 we will see that λ_1 is in fact the sharp threshold.

Remark 3.4. If $c \leq 0$ and $u < 0$ on $\partial\Omega$, the weak maximum principle does not give any better bound that $u \leq 0$ in Ω. Indeed, let $\Omega = (-1,1) \subset \mathbb{R}$ and

$$u(x) = -\frac{\cosh(\alpha x)}{\cosh(\alpha)},$$

which satisfies $u = -1$ on $\partial\Omega$ and

$$\Delta u(x) - \alpha^2 u(x) = \frac{-\alpha^2\cosh(\alpha x) + \alpha^2\cosh(\alpha x)}{\cosh(\alpha)} = 0,$$

but $u(0) = -\cosh(\alpha)^{-1}$ can be made arbitrarily close to zero by taking α suitably large. However, we will later see that in general the strict inequality $u < 0$ holds in Ω, thanks to the strong maximum principle.

Corollary 3.5. *Suppose that* Ω *is a bounded domain in* $\mathbb{R}^n$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *and* L *is an elliptic operator whose coefficients satisfy* $\sup_\Omega |b|/\lambda < \infty$ *and* $c \le 0$. *Then:*

- *if* $Lu \le 0$ *in* Ω, *then* $\inf_\Omega u \ge \inf_{\partial\Omega}(-u^-)$, *where* $u^- = \max\{-u, 0\}$;
- *if* $Lu = 0$ *in* Ω, *then* $\sup_\Omega |u| = \sup_{\partial\Omega} |u|$.

Remark 3.6. The weak maximum principle is valid more generally assuming only that $|b|/\lambda$ is locally bounded in Ω and $u \in C^2(\Omega)$; in that case, the conclusion is that

$$\sup_\Omega u \le \limsup_{x \to \partial\Omega} u^+(x).$$

To see this, apply the above version in the set $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$, where $|b|/\lambda$ is bounded, and pass to the limit:

$$\sup_\Omega u = \limsup_{\varepsilon \searrow 0} \sup_{\Omega_\epsilon} u \le \limsup_{\varepsilon \searrow 0} \sup_{\partial\Omega_\epsilon} u^+ = \limsup_{x \to \partial\Omega} u^+(x).$$

Corollary 3.7 (Comparison principle)**.** *Let* Ω *and* L *be as above, and suppose that* $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfy* $Lu \ge Lv$ *in* Ω *and* $u \le v$ *on* $\partial\Omega$. *Then* $u \le v$ *in* $\overline{\Omega}$.

Proof. Since $u - v \le 0$ on $\partial\Omega$ and $L(u - v) \ge 0$, the weak maximum principle gives that $\sup_\Omega u - v \le \sup_{\partial\Omega}(u - v)^+ = 0$ and the conclusion follows. □

Corollary 3.8 (Uniqueness for the Dirichlet problem)**.** *Let* Ω *and* L *be as above, and suppose that* $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfy* $Lu = Lv$ *in* Ω *and* $u = v$ *on* $\partial\Omega$. *Then* $u \equiv v$ *in* $\overline{\Omega}$.

Proof. Apply the comparison principle in both directions. □

A simple comparison argument gives also the following useful estimate:

Proposition 3.9 (Maximum principle a priori estimate)**.** *Suppose that* L *is elliptic with* $c \le 0$ *and* $\beta := \sup_\Omega |b|/\lambda < \infty$. *Let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *and* $f : \Omega \to \mathbb{R}$. *Then*

$$Lu \ge f \text{ in } \Omega \quad \Longrightarrow \quad \sup_\Omega u \le \sup_{\partial\Omega} u^+ + C \sup_\Omega \frac{|f|}{\lambda},$$

where $C > 0$ *depends only on* β *and the diameter of* Ω. *In particular,*

$$Lu = f \text{ in } \Omega \quad \Longrightarrow \quad \sup_\Omega |u| \le \sup_{\partial\Omega} |u| + C \sup_\Omega \frac{|f|}{\lambda}.$$

Proof. Up to a translation, we may assume that $\Omega \subset (0, d) \times \mathbb{R}^{n-1}$, where d is its diameter. Let

$$v(x) := \sup_{\partial\Omega} u^+ + M \left(e^{\alpha d} - e^{\alpha x_1}\right) \sup_\Omega \frac{|f|}{\lambda}$$

for $M, \alpha > 0$ to be determined later. It is clear that $v \ge \sup_{\partial\Omega} u^+ \ge 0$ on $\overline{\Omega}$. We compute

$$a_{ij}\partial_{ij}e^{\alpha x_1} + b_i\partial_i e^{\alpha x_1} = (\alpha^2 a_{11} + \alpha b_1)e^{\alpha x_1} \ge (\alpha^2\lambda - \alpha\beta\lambda)e^{\alpha x_1} \ge \alpha(\alpha - \beta)\lambda.$$

Choosing $\alpha := \beta + d^{-1}$ and $M := (\alpha(\alpha - \beta))^{-1} = \frac{d^2}{1 + d\beta}$ this gives

$$Lv \le cv - \frac{1}{\alpha(\alpha - \beta)}\alpha(\alpha - \beta)\lambda \sup_\Omega \frac{|f|}{\lambda} \le -\lambda \sup_\Omega \frac{|f|}{\lambda} \le -|f| \le f \le Lu,$$

since this is just $\sup_\Omega(|f|/\lambda) \geq |f|/\lambda$. Now the comparison principle yields $u \leq v$ in Ω, so

$$\sup_\Omega u \leq \sup_\Omega v \leq \sup_{\partial\Omega} u^+ + \frac{e^{d\alpha}}{\alpha(\alpha-\beta)} \sup_\Omega \frac{|f|}{\lambda} = \sup_{\partial\Omega} u^+ + \frac{e^{1+d\beta}}{1+d\beta} d^2 \sup_\Omega \frac{|f|}{\lambda}$$

and the other inequality follows from applying the estimate to $-u$ as well. □

3.2 The Hopf boundary point lemma and the strong maximum principle

It is often useful to complement the weak maximum principle with a rigidity statement that excludes the presence of local maxima unless the solution is constant. We will derive that from a related boundary rigidity statement. In contrast to the weak maximum principle, which is global, these are local statements. Moreover we will need the stronger condition of uniform ellipticity:

Definition 3.10. An elliptic operator L with ellipticity constants $0 < \lambda(x) \leq \Lambda(x)$ is called *uniformly elliptic* in Ω if Λ/λ is bounded in Ω.

Theorem 3.11 (Boundary point lemma of E. Hopf)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose that a point $y \in \partial\Omega$ satisfies the* interior sphere condition, *that is, $\exists R > 0$ and $z \in \Omega$ such that $B_R(z) \subset \Omega$ and $y \in \partial B_R(z)$.*

Suppose that L is an elliptic operator in Ω with $(|a_{ij}| + |b_i| + |c|)/\lambda$ bounded (this implies that L is uniformly elliptic). Suppose that $u \in C^2(\Omega) \cap C^0(\Omega \cup \{y\})$ satisfies $u(x) < u(y)$ for all $x \in \Omega$ and $Lu \geq 0$. Assume one of the following:

(i) $c \equiv 0$ *in* Ω;

(ii) $c \leq 0$ *in* Ω *and* $u(y) \geq 0$;

(iii) $u(y) = 0$ *and no assumption on the sign of* c.

Then, if ν denotes the outer normal to Ω at y, it holds that

$$\liminf_{t \searrow 0} \frac{u(y) - u(y - t\nu)}{t} > 0.$$

In particular, if $\partial_\nu u(y)$ exists, then it is strictly positive.

Proof. Assume first (i) or (ii). Consider the annulus $A := B_R(z) \backslash \overline{B_{R/2}(z)}$ and the "Gaussian"

$$v(x) := e^{-\alpha|x-z|^2} - e^{-\alpha R^2},$$

which is nonnegative on A and satisfies $v(y) = 0$. We have

$$\partial_i v(x) = -2\alpha(x_i - z_i)e^{-\alpha|x-z|^2}$$

and

$$\partial_{ij} v(x) = 4\alpha^2(x_i - z_i)(x_j - z_j)e^{-\alpha|x-z|^2} - 2\alpha\delta_{ij}e^{-\alpha|x-z|^2}.$$

We compute, using the inequalities $c \leq 0$ and $2|x - z| \geq R$:

$$\begin{aligned} Lv &= \Big[a_{ij}(4\alpha^2(x_i - z_i)(x_j - z_j) - 2\alpha\delta_{ij}) + b_i(-2\alpha(x_i - z_i)) + c\Big] e^{-\alpha|x-z|^2} - ce^{-\alpha R^2} \\ &\geq \Big[4\alpha^2 a_{ij}(x_i - z_i)(x_j - z_j) - 2\alpha a_{ii} - 2b_i\alpha(x_i - z_i) + c\Big] e^{-\alpha|x-z|^2} \\ &\geq \Big[4\alpha^2\lambda|x - z|^2 - 2\alpha K\lambda - 2K'\lambda\alpha R - K''\lambda\Big] e^{-\alpha|x-z|^2} \end{aligned}$$

$$\geq \lambda \left[\alpha^2 R^2 - \alpha(2K + 2K'R) - K''\right] e^{-\alpha|x-z|^2} \geq 0$$

provided that we choose α large enough. Now consider $w := u - u(y) + \varepsilon v$ for $\varepsilon > 0$ small, which satisfies

$$Lw = Lu - cu(y) + \varepsilon Lv \geq -cu(y) \geq 0.$$

On $\partial B_R(z)$ clearly $w \leq 0$ because v vanishes there. Since $u - u(y)$ attains a negative maximum on the compact set $\partial B_{R/2}(z)$ and v is equal to a positive constant there, we can choose $\varepsilon > 0$ so that $w \leq 0$ there as well. Hence, by the weak maximum principle applied to A, $w \leq 0$ on A, i.e. $u(x) - u(y) + \varepsilon(v(x) - v(y)) \leq 0$. Setting $x = y - t\nu$ we have

$$\liminf_{t \searrow 0} \frac{u(y) - u(y - t\nu)}{t} \geq -\varepsilon \liminf_{t \searrow 0} \frac{v(y) - v(y - t\nu)}{t} = -\varepsilon \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{-\alpha(R+t)^2} > 0.$$

Finally, for case (iii) consider $\tilde{L} := L - c^+$, which has negative 0-th order coefficient. Then $\tilde{L}u(x) = Lu(x) - c^+u(x) \geq -c^+u(x) \geq -c^+u(y) = 0$ and we can apply case (ii). □

From this it is easy to deduce the strong maximum principle:

Theorem 3.12 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a (not necessarily bounded) domain, and L a (locally uniformly) elliptic operator in Ω such that $(|a_{ij}| + |b_i| + |c|)/\lambda$ is locally bounded. Suppose that $u \in C^2(\Omega)$ satisfies $Lu \geq 0$, and that u attains its maximum M inside Ω. Assume one of the following:*

(i) $c \equiv 0$ in Ω.

(ii) $c \leq 0$ in Ω and $M \geq 0$.

(iii) $M = 0$ and no assumption on the sign of c.

Then $u \equiv M$ in Ω.

Proof. Consider the set $\Sigma := \{u = M\} \subseteq \Omega$. Since Ω is connected and Σ is nonempty and relatively closed, it is enough to show that Σ is open. Let $x \in \Sigma$ and choose $R > 0$ such that $B_R(x) \subset \Omega$. We claim that $B_{R/2}(x) \subseteq \Sigma$; otherwise, choose $z \in B_{R/2}(x) \setminus \Sigma$ and let $r := \operatorname{dist}(z, \Sigma)$, so that $r \leq |x - z| < R/2$, which implies that $B_r(z) \subseteq B_R(x) \setminus \Sigma \subseteq \Omega \setminus \Sigma$.

Choose a point $y \in \Sigma$ realizing $|y - z| = r$. Since $u < M = u(y)$ in $B_r(z)$, the Hopf boundary point lemma (with the corresponding hypothesis) applied to $B_r(z) \Subset \Omega$ gives that $\mathrm{D}u(y) \cdot (y - z) > 0$ and hence $\mathrm{D}u(y) \neq 0$, contradicting the fact that y is a local maximum. □

Corollary 3.13 (Uniqueness for the Neumann problem)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain which satisfies the interior sphere condition at every point of $\partial\Omega$. Let L be a (uniformly) elliptic operator satisfying $\sup_\Omega(|a_{ij}| + |b_i| + c)/\lambda < \infty$ and $c \leq 0$, and suppose that $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solve the same Neumann problem:*

$$Lu = Lv \text{ in } \Omega, \qquad \partial_\nu u = \partial_\nu v \text{ on } \partial\Omega.$$

Then $u - v$ is constant.

Proof. We first observe that either $\sup_\Omega(u - v)$ or $\sup_\Omega(v - u)$ is nonnegative; otherwise $u \leq v$ and $v \leq u$, so $u \equiv v$ in Ω and we are done. Thus we may assume that $m := \sup_\Omega(u - v) \geq 0$.

Since $L(u - v) = 0$, by the strong maximum principle m cannot attained inside Ω unless $u - v$ is constant, hence $u - v < m$ in Ω. Finally, let $y \in \partial\Omega$ be a point attaining the maximum, so that the Hopf boundary point lemma applies and we get that $\partial_\nu(u-v)>0$, contradicting the Neumann condition. $\square$

Remark 3.14. If $c<0$ at some point in Ω, then the only constant solution is zero. Thus, this is a true uniqueness statement unless $c\equiv 0$.

3.3 The comparison principle for quasilinear equations

We finish this chapter by applying the above techniques to nonlinear operators of the form

$$Qu(x)=a_{ij}(x,\mathrm{D}u)\partial_{ij}u(x)+b(x,\mathrm{D}u),$$

which are called quasilinear in the literature (since the highest derivatives of u appear linearly). The fact that the coefficients are independent of u is a technical condition required for the next result, and can be relaxed but not too much. Note that the Euler–Lagrange equations of functionals of the form

$$\mathcal{F}[u]=\int_\Omega F(\mathrm{D}u)\,\mathrm{d}x,$$

for $F\in C^2$ satisfying $\mathrm{D}^2F>0$ everywhere, can be written as

$$\partial_{ij}F(\mathrm{D}u)\partial_{ij}u=\operatorname{div}(\mathrm{D}F(\mathrm{D}u))=0,$$

so critical points (minimizers) solve a quasilinear elliptic equation $Qu=0$ as above with $a_{ij}(p)=\partial_{ij}F(p)$ and $b\equiv 0$ above. In particular, $F\in C^3$, the next theorem is applicable. This is the case for example with the area functional $A(p)=\sqrt{1+|p|^2}$.

Theorem 3.15. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, and suppose that $a_{ij}=a_{ji}:\Omega\times\mathbb{R}^n\to\mathbb{R}$ and $b:\Omega\times\mathbb{R}^n\to\mathbb{R}$ are of class C^1. Let $u,v\in C^2(\Omega)\cap C^0(\overline{\Omega})$ and suppose either that $(a_{ij}(x,\mathrm{D}u(x)))$ or $(a_{ij}(x,\mathrm{D}v(x)))$ is elliptic in Ω. Then, with Q as above,*

$$Qu\ge Qv \text{ in } \Omega \quad\text{and}\quad u\le v \text{ on } \partial\Omega \qquad\Longrightarrow\qquad \text{either } u<v \text{ in } \Omega \text{ or } u\equiv v \text{ in } \Omega.$$

Proof. We assume that $(a_{ij}(x,\mathrm{D}u(x)))$ is elliptic (the other case is proved in the same way). Write

$$\begin{aligned}0\le Qu-Qv&=a_{ij}(x,\mathrm{D}u)\partial_{ij}u-a_{ij}(x,\mathrm{D}v)\partial_{ij}v+b(x,\mathrm{D}u)-b(x,\mathrm{D}v)\\&=a_{ij}(x,\mathrm{D}u)\partial_{ij}(u-v)+\left(a_{ij}(x,\mathrm{D}u)-a_{ij}(x,\mathrm{D}v)\right)\partial_{ij}v+b(x,\mathrm{D}u)-b(x,\mathrm{D}v)\\&=A_{ij}(x)\partial_{ij}(u-v)+B_k(x)\partial_k(u-v),\end{aligned}$$

where

$$B_k(x):=\int_0^1\left[\partial_{p_k}a_{ij}(x,\mathrm{D}v+t(\mathrm{D}u-\mathrm{D}v))\partial_{ij}v+\partial_{p_k}b(x,\mathrm{D}v+t(\mathrm{D}u-\mathrm{D}v))\right]\,\mathrm{d}t.$$

Since $A_{ij},B_k\in C^0(\Omega)$ and (A_{ij}) is locally uniformly elliptic, we may apply the (remark after the) weak maximum principle and obtain that $u-v\le 0$ on $\partial\Omega$ implies that $u-v\le 0$ inside Ω. Finally, if equality holds at some interior point, the strong maximum principle gives that $u-v\equiv 0$, proving the dichotomy. $\square$

4 Morrey–Campanato spaces and Schauder theory

In this chapter we develop the classical perturbation theory of Schauder for linear elliptic equations of the form

$$Lu(x) = a_{ij}(x)\partial_{ij}u(x) + b_i(x)\partial_i u(x) + c(x)u(x) = f(x), \tag{4.1}$$

which we have studied before. Schauder estimates tell us that, provided that the coefficients and the right hand side are Hölder continuous, u also has Hölder continuous second derivatives. Furthermore, the method developed here is very flexible and also gives optimal results for solutions to equations in divergence form or with merely continuous coefficients.

In order to obtain these estimates, we will introduce two new related families of spaces that express Hölder continuity as a suitable decay condition for L^p norms, the so-called Morrey and Campanato spaces. Besides the technical advantage of allowing us to consider solutions $u \in W^{2,2}$, the integral decay conditions defining them will hold immediately for harmonic functions, and will transfer easily to more complicated equations thanks to its robustness.

Schauder estimates are also useful as a-priori estimates, that is, even if we assume from the outset that $u \in C^{2,\alpha}$. In this case, the global version of the estimates, together with the continuity method, will allow us to show solvability of the Dirichlet boundary value problem for very general second order elliptic equations.

4.1 Hölder, Morrey and Campanato spaces

Definition 4.1. For $\alpha \in (0,1]$ and a subset $\Omega \subseteq \mathbb{R}^n$, a function $u\colon \Omega \to \mathbb{R}$ is said to be α-Hölder continuous, and we write $u \in C^{0,\alpha}(\Omega) = C^\alpha(\Omega)$, if $[u]_{C^{0,\alpha}(\Omega)} < \infty$, where

$$[u]_{C^{0,\alpha}(\Omega)} := \sup_{x,y\in\Omega: x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

When Ω is bounded, the space $C^{0,\alpha}(\Omega)$ is a Banach space with the norm

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{L^\infty(\Omega)} + [u]_{C^{0,\alpha}(\Omega)}.$$

If Ω is unbounded, the condition $[u]_{C^{0,\alpha}(\Omega)} < +\infty$ imposes also a sublinear growth condition on u at large scales, in addition to the continuity condition. Therefore, some authors restrict the supremum to pairs $x, y \in \Omega$ with $|x-y| \le 1$. This defines an equivalent space in case Ω is bounded and connected, which will be our focus, so we will not worry about this subtlety.

Note that $C^{0,\alpha}(\Omega) = C^{0,\alpha}(\overline{\Omega})$. As usual, we say that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ if any point in Ω has a neighborhood Ω' where $u \in C^{0,\alpha}(\Omega')$. We also define for $k \in \mathbb{N}$ the spaces $C^{k,\alpha}(\Omega)$ in a similar way.

In these notes, we will follow Campanato's approach to Schauder theory. For this approach, we will introduce the following spaces, which are naturally motivated by the two decay estimates of Lemma 4.11:

Definition 4.2. For $x_0 \in \Omega$ and $r > 0$, let $\Omega(x_0, r) \equiv \Omega \cap B_r(x_0)$. Given $1 \le p < \infty$ and $\mu \ge 0$ we define:

(i) the *Morrey space* $L^{p,\mu}(\Omega)$ as the space of those functions in $L^p(\Omega)$ such that

$$\|u\|^p_{L^{p,\mu}(\Omega)} := \sup_{x_0\in\Omega, 0<r<\mathrm{diam}(\Omega)} r^{-\mu} \int_{\Omega(x_0,r)} |u|^p \,\mathrm{d}x < \infty;$$

(ii) the *Campanato space* $\mathcal{L}^{p,\mu}(\Omega)$ as the space of those functions in $L^p(\Omega)$ such that

$$[u]^p_{\mathcal{L}^{p,\mu}(\Omega)} := \sup_{x_0\in\Omega, 0<r<\mathrm{diam}(\Omega)} r^{-\mu} \int_{\Omega(x_0,r)} |u-(u)_{x_0,r}|^p \,\mathrm{d}x < \infty,$$

where we exceptionally write $(u)_{x_0,r} := \frac{1}{|\Omega(x_0,r)|}\int_{\Omega(x_0,r)} u\,\mathrm{d}x$. We write

$$\|u\|_{\mathcal{L}^{p,\mu}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{\mathcal{L}^{p,\mu}}.$$

Both Morrey and Campanato spaces are Banach spaces, although we will not use this fact.

Let us first look quickly into Morrey spaces. They are only interesting when $\mu \in [0,n]$, and they serve as a scale of function spaces between L^p and L^∞:

$$L^{p,0} = L^p, \qquad L^{p,n} = L^\infty, \qquad L^{p,\mu} = \{0\} \text{ if } \mu > n,$$

as can be easily checked using Lebesgue's differentiation theorem. Hölder's inequality shows that $L^{q,\mu}$ are ordered, i.e. $L^{q,\mu} \subseteq L^{p,\mu}$ whenever $q \geq p$ and Ω is bounded.

We now investigate in more detail Campanato spaces. Here and in the rest of this section, we implicitly assume that Ω is bounded and satisfies the following mild regularity condition, which is always satisfied for smooth or even Lipschitz domains:

$$|\Omega(r,x_0)| \geq cr^n \quad \text{for all } x_0 \in \overline{\Omega}, r \in (0,\mathrm{diam}(\Omega)). \tag{4.2}$$

We will split our analysis in the cases $\mu < n$ and $\mu > n$; the critical case $\mu = n$ is more subtle, it turns out that the space it defines is independent of p and is called BMO.

Theorem 4.3 (Characterization of Campanato spaces)**.** *Let $p \in [1,\infty)$.*

(i) *$0 \leq \mu < n$: then $\mathcal{L}^{p,\mu}(\Omega)$ and $L^{p,\mu}(\Omega)$ are equivalent spaces.*

(ii) *$n < \mu \leq n+p$: then $\mathcal{L}^{p,\mu}(\Omega)$ and $C^{0,\alpha}(\Omega)$, with $\alpha = \frac{\mu-n}{p}$, are equivalent spaces.*

Proof. Let us first dispense with the easy inclusions. For each $\mu \geq 0$, we have $L^{p,\mu} \subseteq \mathcal{L}^{p,\mu}$: using $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ and Jensen's inequality, we have

$$\int_{\Omega(x_0,r)} |u-(u)_{x_0,r}|^p \,\mathrm{d}x \leq 2^{p-1}\left[\int_{\Omega(x_0,r)} |u|^p \,\mathrm{d}x + |\Omega(x_0,r)||(u)_{x_0,r}|^p\right] \leq 2^p \int_{\Omega(x_0,r)} |u|^p \,\mathrm{d}x.$$

The inclusion $C^{0,\alpha} \subset \mathcal{L}^{p,n+\alpha p}$ is also easy, because we can take averages of the pointwise estimate $|u(x)-u(y)| \leq [u]_{C^{0,\alpha}}(2r)^\alpha$, which holds for all $x, y \in B_r(x_0)$. Thus

$$|u(x)-(u)_{x_0,r}| = \left|\frac{1}{|\Omega(x_0,r)|}\int_{\Omega(x_0,r)} [u(x)-u(y)]\,\mathrm{d}y\right| \leq [u]_{C^{0,\alpha}}(2r)^\alpha$$

and so, integrating in x, we get

$$\int_{\Omega(x_0,r)} |u(x)-(u)_{x_0,r}|^p \,\mathrm{d}x \leq [u]^p_{C^{0,\alpha}}(2r)^{\alpha p}|B_r(x_0)| \leq C[u]^p_{C^{0,\alpha}} r^{n+\alpha p}.$$

Hence the rest of the proof is dedicated to prove that $\mathcal{L}^{p,\mu}$ is contained either in a Morrey space in case (i), or in a Hölder space in case (ii). Throughout the proof, we let C denote a generic constant depending on n, p, μ and Ω.

The key point in the proof, in either of the two cases, is to have good estimates on the growth of the averages $f(r) \equiv |(u)_{x_0,r}|^p$, as $r \to 0$. To do so, we want to estimate how much

f can change from one dyadic scale to the next. Given $0 < r < R$ and $x, x_0 \in \Omega$, we have

$$|(u)_{x_0,r} - (u)_{x_0,R}|^p \leq 2^{p-1} \left(|u(x) - (u)_{x_0,R}|^p + |u(x) - (u)_{x_0,r}|^p\right)$$

and so, integrating in x and using (4.2), we get

$$\begin{aligned}|(u)_{x_0,r} - (u)_{x_0,R}|^p &\leq \frac{2^{p-1}}{cr^n} \left(\int_{\Omega(x_0,R)} |u - (u)_{x_0,R}|^p \,\mathrm{d}x + \int_{\Omega(x_0,r)} |u - (u)_{x_0,r}|^p \,\mathrm{d}x\right) \\ &\leq \frac{C}{r^n} \left(R^\mu + r^\mu\right) [u]^p_{\mathcal{L}^{p,\mu}} \\ &\leq C \frac{R^\mu}{r^n} [u]^p_{\mathcal{L}^{p,\mu}},\end{aligned}$$

since $r < R$. Thus, taking the p-th rooth, we get

$$|(u)_{x_0,r} - (u)_{x_0,R}| \leq C R^{\frac{\mu}{p}} r^{-\frac{n}{p}} [u]_{\mathcal{L}^{p,\mu}} = C \left(\frac{R}{r}\right)^{\frac{n}{p}} R^{\frac{\mu-n}{p}} [u]_{\mathcal{L}^{p,\mu}}.$$

Now we assume that r, R are in adjacent dyadic scales. To be precise, fix $0 < \rho < \mathrm{diam}(\Omega)$, which we think of as a fixed constant, corresponding to the initial dyadic scale, and take

$$r \equiv 2^{-(k+1)}\rho, \qquad R \equiv 2^{-k}\rho.$$

With these choices, the previous estimate takes the form

$$|(u)_{x_0,\rho/2^{k+1}} - (u)_{x_0,\rho/2^k}| \leq C[u]_{\mathcal{L}^{p,\mu}} \left(\frac{\rho}{2^k}\right)^{\frac{\mu-n}{p}} \tag{4.3}$$

and notice how the behavior of the right-hand side will change depending on whether we are in case (i) or (ii). Let us sum (4.3) from the initial scale $k = 0$ to some large scale $k = N - 1$ to get

$$|(u)_{x_0,\rho/2^N} - (u)_{x_0,\rho}| \leq C[u]_{\mathcal{L}^{p,\mu}} \rho^{\frac{\mu-n}{p}} \left(\frac{2^{N\frac{n-\mu}{p}} - 1}{2^{\frac{n-\mu}{p}} - 1}\right). \tag{4.4}$$

At this point, we split the analysis depending on whether $\mu < n$ or $\mu > n$.

Let us first deal with the case (i), where $\mu < n$; in this case, the term in parentheses in (4.4) is comparable to $2^{N\frac{n-\mu}{p}}$, and so we obtain

$$|(u)_{x_0,\rho/2^N} - (u)_{x_0,\rho}| \leq C[u]_{\mathcal{L}^{p,\mu}} \left(\frac{\rho}{2^N}\right)^{\frac{\mu-n}{p}}. \tag{4.5}$$

Now let $r \in (0, \mathrm{diam}(\Omega))$ be arbitrary. We can thus find a unique $N \in \mathbb{N}$ and a unique $\frac{1}{2}\mathrm{diam}(\Omega) \leq \rho < \mathrm{diam}(\Omega)$ such that $r = \rho/2^N$. Thus (4.5) yields

$$|(u)_{x_0,r} - (u)_{x_0,\rho}|^p \leq C[u]^p_{\mathcal{L}^{p,\mu}} r^{\mu-n}$$

and using the triangle inequality as before gives

$$\begin{aligned}r^{-\mu} \int_{\Omega(x_0,r)} |u|^p &\leq C r^{-\mu} \left(\int_{\Omega(x_0,r)} |u - (u)_{x_0,r}|^p + r^n |(u)_{x_0,r} - (u)_{x_0,\rho}|^p + r^n |(u)_{x_0,\rho}|^p\right) \\ &\leq C[u]^p_{\mathcal{L}^{p,\mu}} + C r^{n-\mu} \left([u]^p_{\mathcal{L}^{p,\mu}} r^{\mu-n} + \rho^{-n} \int_\Omega |u|^p\right)\end{aligned}$$

$$\leq C[u]^p_{\mathcal{L}^{p,\mu}} + C\,\mathrm{diam}(\Omega)^{-\mu}\int_\Omega |u|^p \leq C\|u\|^p_{\mathcal{L}^{p,\mu}(\Omega)}.$$

Now let us deal with case (ii), so we assume that $\mu > n$. Going back to (4.3), note that since $\mu > n$ this inequality asserts that, for fixed ρ and x_0, the sequence $\left((u)_{x_0,\rho/2^k}\right)_k$ is Cauchy and so it has a limit, say $\tilde{u}(x_0)$. By Lebesgue's differentiation theorem, we must have

$$u(x_0) = \tilde{u}(x_0) \quad \text{for a.e. } x_0 \text{ in } \Omega.$$

We now claim that $\tilde{u}$ is a continuous representative for u. Sending $N \to \infty$ in (4.4), as $\mu > n$, we obtain

$$|\tilde{u}(x_0) - (u)_{x_0,\rho}| \leq C[u]_{\mathcal{L}^{p,\mu}}\rho^{\frac{\mu-n}{p}}. \tag{4.6}$$

Thus we see that $(u)_{x,\rho} \to \tilde{u}(x)$ *uniformly* as $\rho \to 0$; since $x \mapsto (u)_{x,\rho}$ is continuous, we deduce that also $\tilde{u}$ is continuous. Thus we can identify u with its continuous representative $\tilde{u}$.

We now prove that, in fact, u is Hölder continuous: given $x, y \in \Omega$ and writing $\rho = |x-y|$, using (4.6) we have

$$\begin{aligned} |u(x) - u(y)| &\leq |(u)_{x,2\rho} - u(x)| + |(u)_{x,2\rho} - (u)_{y,2\rho}| + |(u)_{y,2\rho} - u(y)| \\ &\leq C[u]_{\mathcal{L}^{p,\mu}}\rho^{\frac{\mu-n}{p}} + |(u)_{x,2\rho} - (u)_{y,2\rho}|. \end{aligned} \tag{4.7}$$

Thus it remains to deal with the last term. Since $\Omega(x,\rho) \subset \Omega(y, 2\rho)$, we have

$$\begin{aligned} |(u)_{x,2\rho} - (u)_{y,2\rho}|^p &\leq C\rho^{-n}\int_{\Omega(x,\rho)} |(u)_{x,2\rho} - (u)_{y,2\rho}|^p \,\mathrm{d}z \\ &\leq 2^{p-1}\rho^{-n}\left(\int_{\Omega(x,2\rho)} |u(z) - (u)_{x,2\rho}|^p \,\mathrm{d}z + \int_{\Omega(y,2\rho)} |u(z) - (u)_{y,2\rho}|^p \,\mathrm{d}z\right) \\ &\leq 2^p\rho^{\mu-n}[u]_{\mathcal{L}^{p,\mu}} \end{aligned}$$

This, combined with (4.7), yields the conclusion. □

Theorem 4.3(ii) is interesting in that it gives an *integral* (rather than pointwise) characterization of Hölder spaces. Note that Hölder spaces are only interesting for $\alpha \leq 1$, so the restriction $\mu \leq n + p$ is natural, although not strictly necessary for the above argument.

We now observe that Theorem 4.3 actually implies Morrey's embedding theorem (Corollary 4.5). To see this, we first state the following:

Lemma 4.4. *Let $p \in (1,\infty)$ and $\mu \geq 0$. If $|\mathrm{D}u| \in L^{p,\mu}_{\mathrm{loc}}(\Omega)$ then $u \in \mathcal{L}^{p,\mu+p}_{\mathrm{loc}}(\Omega)$.*

Proof. We have

$$\frac{1}{r^{\mu+p}}\int_{B_r(x_0)} |u - (u)_{x_0,r}|^p \,\mathrm{d}x \leq C\frac{1}{r^\mu}\int_{B_r(x_0)} |\mathrm{D}u|^p \,\mathrm{d}x \leq C\|\mathrm{D}u\|^p_{L^{p,\mu}(\Omega)},$$

by the Poincaré inequality. □

Corollary 4.5. *Let $p > n$. If $u \in W^{1,p}(\Omega)$ then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \frac{n}{p}$.*

Proof. By Lemma 4.4, if $u \in W^{1,p}(\Omega)$ then $\mathrm{D}u \in L^p(\Omega) = L^{p,0}(\Omega)$ and so $u \in \mathcal{L}^{p,p}_{\mathrm{loc}}(\Omega)$. The conclusion then follows from Theorem 4.3(ii). □

Remark 4.6. The corollary also holds globally in Ω provided that its boundary has some mild regularity. Indeed, if Ω has the extension property for $W^{1,p}$ (Lipschitz boundary suffices), then we can apply the above to an extension to $\mathbb{R}^n$ and conclude.

4.2 Decay estimates for equations with constant coefficients

We will deduce the Schauder estimates for solutions to general equations (with Hölder continuous coefficients, lower order terms and right hand side) from the corresponding estimates for equations with constant coefficients and no lower order terms/right hand side.

For scalar equations, up to a change of variables, these are simply harmonic functions, for which we have already seen a complete regularity theory. Nevertheless, as a preparation for the next section, we will phrase the relevant decay estimate in the language of Campanato spaces, and prove it using a fundamental technique which will be crucial when we study weak solutions to divergence-form equations with rough coefficients in the last chapter: the Caccioppoli inequality.

Since the proof is the same, we will already present the general Caccioppoli inequality: let $a_{ij} \in L^\infty(\Omega)$ satisfy $a_{ij} = a_{ji}$ and the strict ellipticity condition

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and almost every } x \in \Omega, \tag{4.8}$$

and consider functions $u \in W^{1,2}(\Omega)$ solving $\partial_i(a_{ij}(x)\partial_j u) = 0$ distributionally, that is,

$$\int_\Omega a_{ij}(x)\partial_i u(x)\partial_j\varphi(x)\ \mathrm{d}x = 0 \qquad \text{for every } \varphi \in C_c^\infty(\Omega). \tag{4.9}$$

Notice that by density this also holds provided that $\varphi \in W_0^{1,2}(\Omega)$.

Theorem 4.7 (Caccioppoli inequality)**.** *Suppose that $u \in W^{1,2}(\Omega)$ solves (4.9) in Ω with (a_{ij}) satisfying (4.8). Then, if $\eta \in C_c^\infty(\Omega)$,*

$$\int_\Omega |\mathrm{D}(\eta u)|^2\ \mathrm{d}x \le \frac{\Lambda}{\lambda}\int_\Omega |u|^2|\mathrm{D}\eta|^2\ \mathrm{d}x.$$

In particular, if $B_r(x_0) \subset B_R(x_0) \subset \Omega$,

$$\int_{B_r(x_0)} |\mathrm{D}u|^2\ \mathrm{d}x \le \frac{4\Lambda/\lambda}{(R-r)^2}\int_{B_R(x_0)} |u|^2\ \mathrm{d}x. \tag{4.10}$$

Proof. We will use $\eta^2 u$ as a test function; before that, we compute

$$\begin{aligned}
&a_{ij}\partial_i(\eta u)\partial_j(\eta u) - a_{ij}\partial_i u\partial_j(\eta^2 u)\\
&\quad = a_{ij}u\partial_i\eta\partial_j(\eta u) + \cancel{a_{ij}\eta\partial_i u\partial_j(\eta u)} - \cancel{a_{ij}\eta\partial_i u\partial_j(\eta u)} - a_{ij}\eta u\partial_i u\partial_j\eta\\
&\quad = a_{ij}u^2\partial_i\eta\partial_j\eta + \cancel{a_{ij}\eta u\partial_i\eta\partial_j u} - \cancel{a_{ij}\eta u\partial_i u\partial_j\eta}\\
&\quad = a_{ij}u^2\partial_i\eta\partial_j\eta,
\end{aligned}$$

using the symmetry of a_{ij}. Hence, using (4.9),

$$\begin{aligned}
\lambda\int_\Omega |\mathrm{D}(\eta u)|^2 &\le \int_\Omega a_{ij}\partial_i(\eta u)\partial_j(\eta u) = \int_\Omega a_{ij}u^2\partial_i\eta\partial_j\eta + a_{ij}\partial_i u\partial_j(\eta^2 u)\\
&= \int_\Omega a_{ij}u^2\partial_i\eta\partial_j\eta \le \Lambda\int_\Omega u^2|\mathrm{D}\eta|^2.
\end{aligned}$$

The second assertion follows from choosing $\eta \in C_c^\infty(B_R(x_0))$ with $\eta \equiv 1$ on $B_r(x_0)$ and $|\mathrm{D}\eta| \le \frac{2}{R-r}$ everywhere. □

Remark 4.8. This is actually a very general estimate:

- The inequality also holds when u is a subsolution of the corresponding equation, provided that $u \ge 0$.
- This also holds for elliptic systems, whose coefficients satisfy appropriate (Legendre or Legendre–Hadamard) ellipticity conditions.

Remark 4.9. As motivation for what follows, let us inspect Caccioppoli's inequality closer. There are two things which are yet to take advantage of:

(i) since $u - u_0$ is a solution to (4.9) whenever u is a solution and $u_0 \in \mathbb{R}$ is any constant, we can replace $|u|^2$ by $|u-u_0|^2$ on the right hand side of (4.10).

(ii) since we took η to satisfy $\mathrm{D}\eta \equiv 0$ on $B_r(x_0)$ in the proof, we can actually reduce the domain of integration on the right hand side to the annulus $B_R(x_0) \setminus B_r(x_0)$.

Thus we get an improved inequality

$$\int_{B_r(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x \le \frac{4\Lambda/\lambda}{(R-r)^2} \int_{B_R(x_0)\setminus B_r(x_0)} |u-u_0|^2 \,\mathrm{d}x, \qquad u_0 \in \mathbb{R}. \tag{4.11}$$

We will use the first point together with the Poincaré inequality below in this section, while the full strength of (4.11) will be exploited in Example Sheet 4 by means of Widman's hole-filling trick.

We will get the strongest estimate with a choice of $u_0 \in \mathbb{R}$ which minimizes the right hand side. The next exercise shows that the optimal choice is given by the average of u:

Exercise 4.10. Let $1 \le p < \infty$ and consider, for $u \in L^p(\Omega)$, the problem

$$\inf_{u_0\in\mathbb{R}} \int_\Omega |u(x) - u_0|^p \,\mathrm{d}x.$$

(i) Show that, when $p = 2$, the infimum is attained at $(u)_\Omega \equiv \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x$.

(ii) Show that, for general p, we still have for any $u_0 \in \mathbb{R}$ the inequality

$$\int_\Omega |u - (u)_\Omega|^p \,\mathrm{d}x \le 2^p \int_\Omega |u-u_0|^p \,\mathrm{d}x.$$

Taking u_0 to be an average will allow us to use Poincaré's inequality (see Example Sheet 3 or [3, Section 5.8]). Recall that, when $\Omega = B_r(x_0)$, we write $(u)_\Omega \equiv (u)_{x_0,r}$. We then have:

Lemma 4.11 (Decay estimates for homogeneous equations with constant coefficients)**.** *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of* (4.9)*, where (a_{ij}) is constant and satisfies* (4.8)*. We have*

$$\int_{B_r(x_0)} |u|^2 \,\mathrm{d}x \le C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |u|^2 \,\mathrm{d}x, \tag{4.12}$$

$$\int_{B_r(x_0)} |u - (u)_{x_0,r}|^2 \,\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |u - (u)_{x_0,R}|^2 \,\mathrm{d}x, \tag{4.13}$$

for any $B_r(x_0) \subset B_R(x_0) \Subset \Omega$, where $C = C(n,\lambda,\Lambda)$.

Proof. Let us first prove (4.12). By rescaling and translating (i.e. by considering $u_{x_0,R} \equiv u(x_0 + R\cdot)$ instead of u) we can assume that $x_0 = 0, R = 1$. Let k be an integer such that $k > \frac{n}{2}$, i.e. such that $W^{k,2} \subset C^0$. By Caccioppoli's inequality, we have

$$\int_{B_{\frac{1}{2}}} |\mathrm{D}u|^2 \,\mathrm{d}x \leq C(\lambda, \Lambda) \int_{B_1} |u|^2 \,\mathrm{d}x.$$

Since the coefficients are constant, each derivative of u also solves (4.9), and either using the fact that u is harmonic up to a linear change of variables and thus smooth, or using a version of (4.10) with difference quotients, we deduce that all derivatives of u are in $W^{1,2}_{\mathrm{loc}}$. Therefore (4.10) gives

$$\|u\|_{L^\infty(B_{2^{-k}})} \leq C\|u\|_{W^{k,2}(B_{2^{-k}})} \leq C(\lambda, \Lambda)\|u\|_{L^2(B_1)},$$

and for $r \leq 2^{-k}$, we have

$$\int_{B_r} |u|^2 \,\mathrm{d}x \leq Cr^n \|u\|^2_{L^\infty(B_{2^{-k}})} \leq Cr^n \int_{B_1} |u|^2 \,\mathrm{d}x.$$

This is the only interesting case, because if $r > 2^{-k}$ then the inequality holds trivially with $C = (2^k)^n$, since $\int_{B_r} |u|^2 \,\mathrm{d}x \leq \int_{B_1} |u|^2 \,\mathrm{d}x$.

We now prove (4.13), which follows by applying the previous inequality to the derivatives of u. Let us first assume that $r \leq \frac{R}{2}$; as before, this is the only interesting case. Applying the Poincaré inequality, (4.12) and Caccioppoli's inequality, we have

$$\begin{aligned}\int_{B_r(x_0)} |u - (u)_{x_0,r}|^2 \,\mathrm{d}x &\leq Cr^2 \int_{B_r(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x \\ &\leq Cr^2 \left(\frac{r}{R}\right)^n \int_{B_{R/2}(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x \\ &\leq C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |u - (u)_{x_0,R}|^2 \,\mathrm{d}x.\end{aligned}$$

When $r \geq \frac{R}{2}$, we can simply estimate

$$\int_{B_r(x_0)} |u - (u)_{x_0,r}|^2 \,\mathrm{d}x \leq \int_{B_r(x_0)} |u - (u)_{x_0,R}|^2 \,\mathrm{d}x \leq 2^{n+2}\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |u - (u)_{x_0,r}|^2 \,\mathrm{d}x,$$

where in the first inequality we used Exercise 4.10. □

The above decay estimate can be adapted to handle equations with a right hand side in divergence form. This will be the main tool in the proof of Schauder estimates.

Lemma 4.12. *Suppose that* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a weak solution to* $\operatorname{div}(A\mathrm{D}u) = \operatorname{div} F$, *where* $F \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, *that is,*

$$\int_\Omega a_{ij}\partial_i u \partial_j \varphi \,\mathrm{d}x = \int_\Omega F_i \partial_i \varphi \,\mathrm{d}x \qquad \forall \varphi \in C^\infty_c(\Omega), \tag{4.14}$$

with A *constant satisfying* (4.8). *Then the following hold whenever* $B_r(x_0) \subset B_R(x_0) \Subset \Omega$:

$$\int_{B_r(x_0)} |\mathrm{D}u|^2 \leq C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\mathrm{D}u|^2 + C\int_{B_R(x_0)} |F|^2, \tag{4.15}$$

$$\int_{B_r(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,r}|^2 \le C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,R}|^2 + C\int_{B_R(x_0)} |F - (F)_{x_0,R}|^2, \tag{4.16}$$

where $C = C(n, \Lambda, \lambda) > 0$.

Proof. Decompose $u = v + w$, where v and w are the unique solutions to

$$\begin{cases} \operatorname{div}(A\,\mathrm{D}v) = 0 & \text{in } B_R(x_0), \\ v = u & \text{on } \partial B_R(x_0) \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{div}(A\,\mathrm{D}w) = \operatorname{div} F & \text{in } B_R(x_0), \\ w = 0 & \text{on } \partial B_R(x_0), \end{cases}$$

respectively. Note that w (and thus also v) exists by the Riesz representation theorem on $W_0^{1,2}(B_R(x_0))$ together with the Poincaré inequality. We show (4.16), since (4.15) is analogous but easier. By Lemma 4.11, we get

$$\int_{B_r(x_0)} |\mathrm{D}v - (\mathrm{D}v)_{x_0,r}|^2 \,\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |\mathrm{D}v - (\mathrm{D}v)_{x_0,R}|^2 \,\mathrm{d}x$$

and so, using the triangle inequality and Exercise 4.10, we get

$$\begin{aligned} \int_{B_r(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,r}|^2 &\le 2\int_{B_r(x_0)} |\mathrm{D}v - (\mathrm{D}v)_{x_0,r}|^2 + 2\int_{B_r(x_0)} |\mathrm{D}w - (\mathrm{D}w)_{x_0,r}|^2 \\ &\le C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |\mathrm{D}v - (\mathrm{D}v)_{x_0,R}|^2 + 2\int_{B_r(x_0)} |\mathrm{D}w - (\mathrm{D}w)_{x_0,R}|^2 \\ &\le C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,R}|^2 + C\int_{B_R(x_0)} |\mathrm{D}w - (\mathrm{D}w)_{x_0,R}|^2. \end{aligned} \tag{4.17}$$

We now want to estimate the last term, using Exercise 4.10 to write

$$\int_{B_R(x_0)} |\mathrm{D}w - (\mathrm{D}w)_{x_0,R}|^2 \,\mathrm{d}x \le \int_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x.$$

The weak formulation of the PDE satisfied by w states that

$$\int_{B_R(x_0)} \langle A\,\mathrm{D}w, \mathrm{D}\varphi\rangle \,\mathrm{d}x = \int_{B_R(x_0)} \langle F, \mathrm{D}\varphi\rangle \,\mathrm{d}x = \int_{B_R(x_0)} \langle F - (F)_{x_0,R}, \mathrm{D}\varphi\rangle \,\mathrm{d}x$$

for any $\varphi \in W_0^{1,2}(B_R(x_0))$. Thus, taking $\varphi = w$ and using the ellipticity of A, we get

$$\begin{aligned} \lambda\int_{B_R(x_0)} |\mathrm{D}w|^2 &\le \int_{B_R(x_0)} \langle A\,\mathrm{D}w, \mathrm{D}w\rangle \,\mathrm{d}x \\ &\le \left(\int_{B_R(x_0)} |F - (F)_{x_0,R}|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_{B_R(x_0)} |\mathrm{D}w|^2 \,\mathrm{d}x\right)^{1/2}. \end{aligned}$$

Combining the previous estimates, (4.16) follows. For (4.15) one uses (4.12) instead of (4.13) and estimates without subtracting the average. □

4.3 Interior Schauder estimates

The fundamental idea in Schauder theory is to reduce the study of (4.1) to the constant coefficient case, so we begin by proving estimates in that case. We will deduce the $C^{2,\alpha}$ estimates when the right hand side is in $C^{0,\alpha}$ from $C^{1,\alpha}$ estimates when the right hand side is as in (4.14) with $F \in C^{0,\alpha}$.

We have already done all the hard work, and now we just need to put it together with the help of the following useful elementary lemma:

Lemma 4.13 (Iteration lemma)**.** *Consider a non-decreasing function $\phi\colon (0,R_0] \to [0,+\infty)$ which satisfies, for some constants $A, B, \varepsilon \geq 0$ and $0 < \beta < \alpha$,*

$$\phi(r) \leq A\Big[\Big(\frac{r}{R}\Big)^{\alpha} + \varepsilon\Big]\phi(R) + BR^{\beta} \quad \textit{for all } 0 < r \leq R \leq R_0.$$

Then there is $C = C(\alpha,\beta,A)$ and $\varepsilon_0 = \varepsilon_0(\alpha,\beta,A)$ such that, if $\varepsilon \leq \varepsilon_0$, we have

$$\phi(r) \leq C\Big[\frac{\phi(R)}{R^{\beta}} + B\Big]r^{\beta} \quad \textit{for all } 0 < r \leq R \leq R_0.$$

Proof. Since $\varepsilon \geq 0$ we may assume without loss of generality that $A > \frac{1}{2}$. Let us take $\gamma \equiv \frac{\alpha+\beta}{2}$. We choose $\tau \in (0,1)$ with $2A\tau^{\alpha} = \tau^{\gamma}$ and ε_0 so that $\varepsilon_0 \leq \tau^{\alpha}$. Thus

$$\phi(\tau R) \leq A(\tau^{\alpha} + \varepsilon_0)\phi(R) + BR^{\beta} \leq 2A\tau^{\alpha}\phi(R) + BR^{\beta} = \tau^{\gamma}\phi(R) + BR^{\beta}.$$

Thus, iterating once, we get

$$\phi(\tau^2 R) \leq \tau^{\gamma}\phi(\tau R) + B\tau^{\beta}R^{\beta} \leq \tau^{2\gamma}\phi(R) + \tau^{\gamma}BR^{\beta} + B\tau^{\beta}R^{\beta} = \tau^{2\gamma}\phi(R) + BR^{\beta}\tau^{\beta}(1+\tau^{\gamma-\beta}).$$

Hence, iterating the first estimate k times, we get

$$\begin{aligned}\phi(\tau^k R) &\leq \tau^{k\gamma}\phi(R) + BR^{\beta}\tau^{(k-1)\beta}\sum_{i=0}^{k-1}\tau^{i(\gamma-\beta)} \\ &= \tau^{k\gamma}\phi(R) + BR^{\beta}\tau^{(k-1)\beta}\frac{1-\tau^{k(\gamma-\beta)}}{1-\tau^{\gamma-\beta}} \leq C\tau^{(k+1)\beta}[\phi(R) + BR^{\beta}].\end{aligned}$$

Now for $0 < r < R$, let $k \in \mathbb{N}$ be such that $\tau^{k+1}R < r \leq \tau^k R$. Then

$$\phi(r) \leq \phi(\tau^k R) \leq C\tau^{(k+1)\beta}[\phi(R) + BR^{\beta}] \leq C\left[\phi(R) + BR^{\beta}\right]\left(\frac{r}{R}\right)^{\beta},$$

as wished. □

Theorem 4.14 (Constant coefficients)**.** *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of (4.14), where A is constant and elliptic. If $F \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ for $\mu \in [0,n+2)$ then $\mathrm{D}u \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$, with the estimate*

$$\|\mathrm{D}u\|_{\mathcal{L}^{2,\mu}(\Omega'')} \leq C(\|\mathrm{D}u\|_{L^2(\Omega')} + [F]_{\mathcal{L}^{2,\mu}(\Omega')}),$$

where $\Omega'' \Subset \Omega' \Subset \Omega$ and $C = C(n,\Omega',\Omega'',\lambda,\Lambda,\mu)$.

Proof. Choose $R_0 > 0$ such that $B_{R_0}(x_0) \Subset \Omega'$ for all $x_0 \in \Omega''$. For a fixed $x_0 \in \Omega''$ and $0 < r < R_0$, let

$$\phi(r) := \int_{B_r(x_0)} |\mathrm{D}u - (\mathrm{D}u)_{x_0,r}|^2 \,\mathrm{d}x.$$

Now the campanato decay of F together with (4.16) show that

$$\phi(r) \leq C\Big(\frac{r}{R}\Big)^{n+2}\phi(R) + C[F]^2_{\mathcal{L}^{2,\mu}}R^{\mu} \qquad \forall 0 < r < R \leq R_0,$$

so Lemma 4.13, applied with $\alpha = n+2$, $\beta = \mu < n+2$ and $\varepsilon = 0$ yields

$$\phi(r) \le C\left(\left(\frac{r}{R}\right)^{\mu}\phi(R) + C[F]^2_{\mathcal{L}^{2,\mu}} r^{\mu}\right) \le C\left(R^{-\mu}\|\mathrm{D}u\|^2_{L^2} + [F]^2_{\mathcal{L}^{2,\mu}}\right) r^{\mu},$$

which gives the conclusion. □

As a corollary we get $\mathcal{L}^{2,\mu}$ estimates for the second derivatives of solutions to equations with constant coefficients where the right hand side is in $\mathcal{L}^{2,\mu}$:

Corollary 4.15. *Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ solve*

$$a_{ij}\partial_{ij}u = f \tag{4.18}$$

for a constant elliptic matrix (a_{ij}). If $0 \le \mu < n+2$ and $f \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega)$, then $\mathrm{D}^2u \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$, with the estimate

$$\|\mathrm{D}^2u\|_{\mathcal{L}^{2,\mu}(\Omega'')} \le C\left(\|\mathrm{D}^2u\|_{L^2(\Omega')} + [f]_{\mathcal{L}^{2,\mu}(\Omega')}\right),$$

where $\Omega'' \Subset \Omega' \Subset \Omega$ and $C = C(n, \Omega', \Omega'', \lambda, \Lambda, \mu)$.

Proof. We claim that for each $1 \le k \le n$, $u_k := \partial_k u$ solves an equation of the form (4.14): $\partial_i(a_{ij}\partial_j\partial_k u) = \partial_i F_i$ where $F_i = \delta_{ik} f$: for every $\varphi \in C^\infty_c(\Omega)$,

$$\int_\Omega a_{ij}\partial_j\partial_k u\partial_i\varphi = -\int_\Omega a_{ij}\partial_j u\partial_i\partial_k\varphi = \int_\Omega a_{ij}\partial_{ij}u\partial_k\varphi = \int_\Omega f\partial_k\varphi = \int_\Omega f\delta_{ik}\partial_i\varphi$$

where the integration by parts is justified since $u \in W^{2,2}$. Now Theorem 4.14 gives the desired estimate after summing over k. □

We now treat the case of variable coefficients in nondivergence form. For simplicity we deal with equations of the form

$$a_{ij}(x)\partial_{ij}u(x) = f(x), \tag{4.19}$$

without lower order terms—those can be taken into account by the same method.

Theorem 4.16 (Variable coefficients)**.** *Fix $\sigma \in (0,1)$. Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ be a solution of (4.19), where $f \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$ and $a_{ij} \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$ satisfies the strict ellipticity condition (4.8). Then $\mathrm{D}^2u \in C^{0,\sigma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n})$, with the estimate*

$$\|\mathrm{D}^2u\|_{C^{0,\sigma}(\Omega'')} \le C(\|\mathrm{D}^2u\|_{L^2(\Omega')} + \|f\|_{C^{0,\sigma}(\Omega')}),$$

where $\Omega'' \Subset \Omega' \Subset \Omega$ and $C = C(n, \Omega', \Omega'', \lambda, \Lambda, \sigma, [A]_{C^{0,\sigma}})$.

Proof. The idea is to treat the case of variable coefficients as a perturbation of the constant coefficient case. In particular, we use Korn's trick: fix $x_0 \in \Omega$ and let $\bar{a}_{ij} := a_{ij}(x_0)$. Then

$$\bar{a}_{ij}\partial_{ij}u = g, \qquad g(x) \equiv -\left(a_{ij}(x) - \bar{a}_{ij}\right)\partial_{ij}u(x) + f(x) \in L^2_{\mathrm{loc}}.$$

Recall that $u_k := \partial_k u$ solves an equation of the form (4.14) with $F_i = \delta_{ik} g \in L^2$, so Lemma 4.12 gives (after summing over k):

$$\int_{B_r(x_0)} |\mathrm{D}^2u - (\mathrm{D}^2u)_{x_0,r}|^2 \le C\left(\frac{r}{R}\right)^{n+2}\int_{B_R(x_0)} |\mathrm{D}^2u - (\mathrm{D}^2u)_{x_0,R}|^2 + C\int_{B_R(x_0)} |g - (g)_{x_0,R}|^2$$

and

$$\int_{B_r(x_0)} |\mathrm{D}^2 u|^2 \leq C \left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\mathrm{D}^2 u|^2 + C \int_{B_R(x_0)} |g|^2. \tag{4.20}$$

Using the fact that $A \in C^{0,\sigma}$ and discarding the average for the second term, we have:

$$\begin{aligned} \int_{B_R(x_0)} |g - (g)_{x_0,R}|^2 &\leq 2 \int_{B_R(x_0)} |f - (f)_{x_0,R}|^2 + 2 \int_{B_R(x_0)} |\overline{a}_{ij} - a_{ij}|^2 |\mathrm{D}^2 u|^2 \\ &\leq C[f]^2_{\mathcal{L}^{2,n+2\sigma}} R^{n+2\sigma} + C R^{2\sigma} \int_{B_R(x_0)} |\mathrm{D}^2 u|^2. \end{aligned} \tag{4.21}$$

Thus, writing

$$\phi(r) := \int_{B_r(x_0)} |\mathrm{D}^2 u - (\mathrm{D}^2 u)_{x_0,r}|^2 \,\mathrm{d}x,$$

we arrive at

$$\phi(r) \leq C\left(\frac{r}{R}\right)^{n+2} \phi(R) + C[f]^2_{\mathcal{L}^{2,n+2\sigma}} R^{n+2\sigma} + C R^{2\sigma} \int_{B_R(x_0)} |\mathrm{D}^2 u|^2. \tag{4.22}$$

In order to be able to conclude, we need an estimate on the last term in (4.22). This will come from (4.20) and a similar computation to (4.21), using the fact that f is bounded:

$$\begin{aligned} \int_{B_r(x_0)} |\mathrm{D}^2 u|^2 \,\mathrm{d}x &\leq C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\mathrm{D}^2 u|^2 \,\mathrm{d}x + C \int_{B_R(x_0)} |g|^2 \,\mathrm{d}x \\ &\leq C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\mathrm{D}^2 u|^2 \,\mathrm{d}x + C \int_{B_R(x_0)} |f|^2 + C R^{2\sigma} \int_{B_R(x_0)} |\mathrm{D}^2 u|^2 \,\mathrm{d}x \\ &\leq C \left[\left(\frac{r}{R}\right)^n + R^{2\sigma}\right] \int_{B_R(x_0)} |\mathrm{D}^2 u|^2 \,\mathrm{d}x + C R^n \|f\|^2_{L^\infty}. \end{aligned}$$

Let $\delta > 0$ be arbitrarily small and suppose without loss of generality that $R \leq R_0 \leq 1$ is sufficiently small. Since $R^n \leq R^{n-\delta}$, if we set $\psi(r) = \int_{B_r(x_0)} |\mathrm{D}^2 u|^2$, we can apply Lemma 4.13 to ψ with $\alpha = n$, $\beta = n - \delta$ and $R^{2\sigma} \leq R_0^{2\sigma} \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is given by the lemma. This gives

$$\int_{B_r(x_0)} |\mathrm{D}^2 u|^2 \,\mathrm{d}x \leq C(\delta, R_0) \left(\|\mathrm{D}^2 u\|^2_{L^2(B_{R_0}(x_0))} + \|f\|^2_{L^\infty(B_{R_0}(x_0))} \right) r^{n-\delta}$$

and now we can complete the proof. Returning to (4.22), we see that

$$\phi(r) \leq C \left(\left(\frac{r}{R}\right)^{n+2} \phi(R) + \left(\|f\|^2_{C^{0,\sigma}} + \|\mathrm{D}^2 u\|^2_{L^2} \right) R^{2\sigma} R^{n-\delta} \right)$$

and so, by Lemma 4.13, we have that $\mathrm{D}^2 u \in \mathcal{L}^{2,n+2\sigma-\delta}_{\mathrm{loc}}(\Omega) \cong C^{0,\sigma-\delta/2}_{\mathrm{loc}}$, by Theorem 4.3(ii). Thus in particular $\mathrm{D}^2 u$ is bounded (with an estimate) and (4.22) improves to

$$\phi(r) \leq C \left(\left(\frac{r}{R}\right)^{n+2} \phi(R) + \left(\|f\|^2_{C^{0,\sigma}} + \|\mathrm{D}^2 u\|^2_{L^2} \right) R^{n+2\sigma} \right)$$

which, again by Lemma 4.13, yields $\phi(r) \lesssim r^{n+2\sigma}$. Thus $\mathrm{D}^2 u \in \mathcal{L}^{2,n+2\sigma}_{\mathrm{loc}}(\Omega) \cong C^{0,\sigma}_{\mathrm{loc}}$ with the claimed estimate. □

By iterating the above result, we obtain the following useful result:

Theorem 4.17 (Higher-order regularity)**.** *Let $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ be a solution of* (4.19)*. If $a_{ij} \in C^{k,\sigma}_{\mathrm{loc}}(\Omega)$ satisfies the strict ellipticity condition* (4.8) *and $f \in C^{k,\sigma}_{\mathrm{loc}}(\Omega)$ for some $k \geq 0$ and $\sigma \in (0,1)$, then we have $u \in C^{k+2,\sigma}_{\mathrm{loc}}(\Omega)$.*

Proof. Differentiate the system and argue inductively, see Example Sheet 3. To make it rigorous one can use, for example, difference quotients. □

4.4 Global Schauder estimates

So far we have seen Schauder estimates for a solution u in any interior subdomain of Ω. However, when u solves a boundary value problem with sufficiently regular data, then u satisfies the Schauder estimate in the whole domain $\overline{\Omega}$.

Theorem 4.18 (Global Schauder estimate)**.** *Fix $\sigma \in (0,1)$. Let $u \in W^{2,2}(\Omega)$ be a solution of* (4.19)*, where $f \in C^{0,\sigma}(\Omega)$ and $a_{ij} \in C^{0,\sigma}(\Omega)$ satisfies the strict ellipticity condition* (4.8)*. If in addition Ω is of class $C^{2,\sigma}$ and $u = \varphi$ on $\partial\Omega$ for a function $\varphi \in C^{2,\sigma}(\Omega)$, then $\mathrm{D}^2 u \in C^{0,\sigma}(\Omega, \mathbb{R}^{n\times n})$, with the estimate*

$$\|\mathrm{D}^2 u\|_{C^{0,\sigma}(\Omega)} \leq C(\|u\|_{W^{2,2}(\Omega)} + \|f\|_{C^{0,\sigma}(\Omega)} + \|\varphi\|_{C^{2,\sigma}(\Omega)}),$$

where $C = C(n, \Omega, \lambda, \Lambda, \sigma, [A]_{C^{0,\sigma}})$.

Proof (sketch). By considering $u - \varphi$, which solves $a_{ij}\partial_{ij}(u-\varphi) = f - a_{ij}\partial_{ij}\varphi \in C^{\sigma}(\Omega)$, we may assume $\varphi \equiv 0$. Let $R_0 > 0$ be such that for each $x_0 \in \overline{\Omega}$ with $\mathrm{dist}(x_0, \partial\Omega) < R_0/2$ there exists a $C^{2,\sigma}$ diffeomorphism $\Psi = \Psi_{x_0}$ mapping an open set $U \subset \mathbb{R}^n$ containing $B_{R_0}(x_0)$ onto B_1, with $\Psi(U \cap \Omega) = B_1^+ := B_1 \cap \{x_n > 0\}$, $\Psi(x_0) = y_0 = (0,t)$ for some $0 < t < 1/2$, and $\|\Psi\|_{C^{2,\sigma}(U)} + \|\Psi^{-1}\|_{C^{2,\sigma}(B_1)} \leq C$, for a constant $C = C(\Omega)$ independent of x_0. We will analyse the PDE satisfied by $u \circ \Psi^{-1}$ on B_1^+ and then transplant the estimates back into Ω, to get $C^{2,\sigma}$ estimates at distance at most $R_0/2$ from the boundary; for the rest of Ω the interior estimates will apply.

Note that in general there is a first order term in the PDE satisfied by $u \circ \Psi^{-1}$ coming from the chain rule. To deal with it, we may either assume that $u \in C^{1,\sigma}(\Omega)$ (in fact it is natural to assume $u \in C^2(\Omega)$ in the context of a priori estimates) and add the first order term to the right hand side (it is C^{σ}), or extend the Schauder theory to handle such terms. Here we will simply ignore the first order term to illustrate the main ideas.

Thus we have reduced the problem to showing that if $u \in W^{2,2}(B_1^+)$ solves

$$\begin{cases} a_{ij}\partial_{ij}u = f & \text{in } B_1^+, \\ u = 0 & \text{on } \partial_0 B_1^+ := \partial B_1^+ \cap B_1, \end{cases}$$

then

$$\int_{B_R(y_0)\cap B_1^+} |\partial_{ik}u - (\partial_{ik}u)_{y_0,R}|^2 \leq C\left(\|f\|_{C^0(B_1^+)} + \|\mathrm{D}^2 u\|_{L^2(B_1^+)}\right) R^{n+2\sigma} \tag{4.23}$$

for each $i,k \in \{1,\dots,n\}$, $y_0 = (0,t)$ and $0 < t, R < \frac{1}{2}$. Moreover, by further composing with a linear map (which together with its inverse is bounded by λ and $\|a_{ij}\|_{C^0}$ and hence distorts balls by a bounded amount), we may assume that $a_{ij}(0) = \delta_{ij}$.

We start by fixing $1 \leq k \leq n-1$, considering $u_k := \partial_k u \in W^{1,2}(B_1^+)$, which vanishes on $\partial_0 B_1^+$, and try to follow the proof of the interior Schauder estimate. Given $0 < r < R \leq \frac{1}{2}$, if $R < t$ we are in the interior case and the decay of the Campanato seminorm from $B_R(y_0)$ to $B_r(y_0)$ is exactly the same. Otherwise, if $R > t$, freeze the coefficients at $a_{ij}(0) = \delta_{ij}$, let $g = f - (a_{ij} - \delta_{ij})\partial_{ij} u$, and split $u_k = u + v$ with

$$\begin{cases} \Delta v = 0 & \text{in } B_R(y_0) \cap B_1^+, \\ v = u & \text{on } \partial(B_R(y_0) \cap B_1^+) \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta w = \operatorname{div}(g e_k) & \text{in } B_R(y_0) \cap B_1^+, \\ w = 0 & \text{on } \partial(B_R(y_0) \cap B_1^+). \end{cases}$$

The term with w behaves in exactly the same way as in the interior case thanks to $|a_{ij} - \delta_{ij}| \leq C(t+R)^\sigma \leq CR^\sigma$. In order to show decay for the harmonic function v we observe that odd reflection extends it to a (smooth) harmonic function $\tilde{v}$ on the whole ball $B_R(y_0)$, since $B_R(0,t) \cap \{x_n < 0\} \subseteq B_R(0,-t)$. Thus, as in Lemma 4.11, using the Poincaré inequality, the even symmetry of $|\mathrm{D}^2\tilde{v}|^2$, and a boundary version of the Caccioppoli inequality[1] we have for $r < R/2$:

$$\begin{aligned}
\int_{B_r(y_0)\cap B_1^+} |\mathrm{D}v - (\mathrm{D}v)_{B_r(y_0)\cap B_1^+}|^2 &\leq \int_{B_r(y_0)\cap B_1^+} |\mathrm{D}v - (\mathrm{D}\tilde{v})_{B_r(y_0)}|^2 \\
&\leq \int_{B_r(y_0)} |\mathrm{D}\tilde{v} - (\mathrm{D}\tilde{v})_{B_r(y_0)}|^2 \\
&\leq Cr^2 \int_{B_r(y_0)} |\mathrm{D}^2\tilde{v}|^2 \leq Cr^{n+2} \sup_{B_r(y_0)} |\mathrm{D}^2\tilde{v}|^2 \\
&\leq Cr^{n+2}R^{-n} \int_{B_{R/2}(y_0)} |\mathrm{D}^2\tilde{v}|^2 \\
&\leq Cr^{n+2}R^{-n} \int_{B_{R/2}(y_0)\cap B_1^+} |\mathrm{D}^2\tilde{v}|^2 \\
&\leq C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(y_0)\cap B_1^+} |\mathrm{D}v - (\mathrm{D}v)_{B_R(y_0)\cap B_1^+}|^2,
\end{aligned}$$

and similarly for the Morrey decay.

Since these are the only ingredients needed for the proof of the Campanato estimates, we get that the estimate of (4.23) holds for each $i \leq n$ and $k < n$. Finally, using the equation we get that $\partial_{nn} u = a_{nn}^{-1}(f - \sum_{(i,j)\neq(n,n)} a_{ij}\partial_{ij} u)$ and thus (4.23) holds also for $(i,k) = (n,n)$, whence $\mathrm{D}^2 u \in \mathcal{L}^{2,n+2\sigma}(\Omega) \cong C^\sigma(\Omega)$. □

Note that there are also higher order global estimates: the statement is as in Theorem 4.17 but removing all the "loc", with the additional conditions that $u = \varphi$ on $\partial\Omega$, and both φ and Ω are of class $C^{k+2,\sigma}$. The proof is by induction on k, flattening the boundary, and

[1] In the last line, we are applying the Caccioppoli inequality to the harmonic functions $\psi = \partial_i \tilde{v}$, which are even or odd in x_n, depending on i. For the even case, the standard Caccioppoli inequality in the doubled domain suffices, using that $(\psi)_{B_R(0,t)\cup B_R(0,-t)} = (\psi)_{B_R(0,t)\cap B_1^+}$. For the odd case, $(\psi)_{B_R(0,t)\cup B_R(0,-t)} = 0$, but one can prove that

$$\int_{B_R(0,t)\cap B_1^+} |\psi|^2 \leq C \int_{B_R(0,t)\cap B_1^+} |\psi - (\psi)_{B_R(0,t)\cap B_1^+}|^2$$

by an easy contradiction/compactness argument, provided that ψ is harmonic and odd.

using the global Schauder estimates for the tangential derivatives first. We skip it since it is mostly a matter of bookkeeping and all the ideas have already been introduced.

We end this section by improving the estimates of Theorem 4.18 so that no derivatives of u appear on the right hand side. Interestingly this is easier to do in the global setting—for the local estimates it is also true but requires a tricky covering argument (see Example Sheet 3). As we will see in the next section, even the L^p norm of u can be removed.

Corollary 4.19. *Let Ω, u, f, φ and (a_{ij}) be as in Theorem 4.18. Then*

$$\|u\|_{C^{2,\sigma}(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{C^{0,\sigma}(\Omega)} + \|\varphi\|_{C^{2,\sigma}(\Omega)}),$$

for any $p \leq \infty$, where C depends only on $n, \sigma, \Omega, \lambda, \|A\|_{C^0(\Omega)}$ and p.

Proof. We use an interpolation inequality which is proved the same way as Example Sheet 3, (Q3) but for a general domain: for any $u \in C^{2,\sigma}(\Omega)$ and any $\delta > 0$ it holds

$$\|u\|_{C^2(\Omega)} \leq \delta[\mathrm{D}^2 u]_{C^\sigma(\Omega)} + K_\delta \|u\|_{L^p(\Omega)},$$

with K_δ depending only on n, Ω, σ and δ. Since $\|u\|_{W^{2,2}(\Omega)} \leq C\|u\|_{C^2(\Omega)}$, we may insert this on the right hand side of the global Schauder estimate and get

$$\begin{aligned}
[\mathrm{D}^2 u]_{C^\sigma(\Omega)} &\leq C(\|u\|_{W^{2,2}(\Omega)} + \|f\|_{C^{0,\sigma}(\Omega)} + \|\varphi\|_{C^{2,\sigma}(\Omega)}) \\
&\leq C(\delta[\mathrm{D}^2 u]_{C^\sigma(\Omega)} + K_\delta\|u\|_{L^p(\Omega)} + \|f\|_{C^{0,\sigma}(\Omega)} + \|\varphi\|_{C^{2,\sigma}(\Omega)}).
\end{aligned}$$

Choosing $\delta = \frac{1}{2C}$ and absorbing gives the result. □

4.5 Schauder theory as an existence theory

Recall that our strategy in Section 2 was to first construct, using Functional Analytic methods, weak solutions of equations in divergence form, and then to prove regularity of such solutions. In Schauder theory, instead, one can construct regular solutions directly without passing through some generalized notion of solution. This does not require any variational structure and thus works for very general operators such as (4.19) (and even (4.1) with appropriate conditions). For this reason, Schauder theory is not just a regularity theory, but an *existence theory* as well, and in fact it is substantially older than the approach based on Sobolev spaces.

In this subsection we explain how to use Schauder theory to obtain existence of solutions to the problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{4.24}$$

over a smooth (say $C^{2,\alpha}$) bounded domain Ω, where $\varphi \in C^{2,\alpha}(\Omega)$, $f \in C^{0,\alpha}(\Omega)$ and L is as in (4.1) with $a_{ij}, b_i, c \in C^{0,\alpha}(\Omega)$ strictly elliptic ($a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \lambda \in \mathbb{R}^+$). Note that by changing f to $f - L\varphi$ we may assume that $\varphi \equiv 0$. There are two crucial ingredients in the method:

(i) *Solvability of the Poisson equation*: for $f \in C^{0,\alpha}(\overline{\Omega})$ there is a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ of

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(ii) *Global Schauder a priori estimates*: For $f \in C^{0,\alpha}(\overline{\Omega})$, any solution $u \in C^{2,\alpha}(\Omega)$ of (4.1) satisfies the *a priori* estimate

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\Omega, \lambda, \alpha, \|a_{ij}\|_{C^{0,\alpha}(\overline{\Omega})}, \|b_i\|_{C^{0,\alpha}(\overline{\Omega})}, \|c\|_{C^{0,\alpha}(\overline{\Omega})})\|f\|_{C^{0,\alpha}(\overline{\Omega})}. \tag{4.25}$$

One can solve (i) using the methods from Analysis of PDE: one constructs a weak solution $u \in W_0^{1,2}(\Omega)$ using the Riesz representation theorem, then shows a global estimate

$$\|u\|_{W^{2,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

(see Theorem 4.26 of APDE) and finally uses Theorem 4.18 to show that $u \in C^{2,\alpha}(\Omega)$.

To prove (ii), we first obtain the weaker estimate

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\Omega, \lambda, \alpha, \|a_{ij}\|_{C^{\alpha}(\overline{\Omega})}, \|b_i\|_{C^{\alpha}(\overline{\Omega})}, \|c\|_{C^{\alpha}(\overline{\Omega})}) \left(\|u\|_{C^0(\overline{\Omega})} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\right), \tag{4.26}$$

which comes from writing $a_{ij}\partial_{ij}u = g := f - cu - b_i\partial_i u \in C^{0,\alpha}(\Omega)$, applying Corollary 4.19, and then closing the estimate as in the proof of Corollary 4.19 by using interpolation and absorbing (see (Q5) of Example Sheet 3 for an interior version of this).

In order to upgrade (4.26) to (4.25), we need that $f \equiv 0 \Rightarrow u \equiv 0$, that is, uniqueness for the Dirichlet problem is a necessary condition—and in fact, arguing by compactness one can see it is also sufficient. Nevertheless, for concreteness, we will just assume $c \leq 0$: in this case one has the a priori estimate from Proposition 3.9 which shows that $\|u\|_{C^0(\Omega)} \leq C\|f\|_{C^0(\Omega)}$ and thus establishes (4.25).

Let us now see how (i) and (ii) combined yield an existence theorem for linear strictly elliptic equations. We consider two operators

$$L_0 = \Delta \colon \mathbb{X} \to \mathbb{Y}, \qquad L_1 = L \colon \mathbb{X} \to \mathbb{Y},$$

where

$$\mathbb{X} = C^{2,\alpha}(\overline{\Omega}) \cap \{u|_{\partial\Omega} = 0\}, \qquad \mathbb{Y} = C^{0,\alpha}(\overline{\Omega}).$$

From (i) we already know that L_0 is surjective, and our goal is now to show that L_1 is surjective as well. That this is the case follows from the a priori estimate (ii) together with the following abstract result:

Proposition 4.20 (Continuity method)**.** *Let $L_0, L_1 \colon \mathbb{X} \to \mathbb{Y}$ be bounded linear operators between Banach spaces. Set*

$$L_t \equiv (1-t)L_0 + tL_1, \quad t \in [0,1].$$

Suppose that there is a constant $C > 0$ (independent of t) such that

$$\|u\|_{\mathbb{X}} \leq C\|L_t u\|_{\mathbb{Y}} \qquad \text{for all } u \in \mathbb{X}, \text{ all } t \in [0,1]. \tag{4.27}$$

If L_0 is surjective, then so is L_1.

Proof. Suppose that L_s is surjective, for some $s \in [0,1]$. By (4.27), L_s is injective as well and hence it has a bounded inverse $L_s^{-1} \colon \mathbb{Y} \to \mathbb{X}$. We now rewrite the equation $L_t u = f$ as

$$L_s u = f + (L_s - L_t)u = f + (t-s)(L_0 u - L_1 u)$$

or, in yet another way,

$$u = L_s^{-1} f + (t-s)L_s^{-1}(L_0 u - L_1 u) \equiv Tu.$$

Thus we need to find a fixed point of $T\colon \mathbb{X} \to \mathbb{X}$. We estimate

$$\|Tu - Tv\| \leq \|L_s^{-1}\|(\|L_0\| + \|L_1\|)|t-s|\|u-v\|.$$

According to (4.27) we have $\|L_s^{-1}\| \leq C$ and so if $|t-s| \leq \frac{1}{2}(C(\|L_0\| + \|L_1\|))^{-1} \equiv c$, we can apply the contraction mapping theorem to find a fixed point of T. Thus, if L_s is surjective then so is L_t whenever $|t-s| \leq c$, and the conclusion follows by iterating finitely many times. □

As a result of this discussion and the fact that the coefficients of $L_t = (1-t)\Delta + tL$ are uniformly bounded in $C^\alpha(\Omega)$, so that (4.25) yields (4.27), we have proved the following theorem:

Theorem 4.21. *Let $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain. Suppose that $Lu = a_{ij}\partial_{ij}u + b_i\partial_i u + cu$ is a strictly elliptic operator with coefficients in $C^\alpha(\Omega)$ satisfying $c \leq 0$, and $\varphi \in C^{2,\alpha}(\Omega)$ and $f \in C^\alpha(\Omega)$ are given functions. Then there exists a unique solution $u \in C^{2,\alpha}(\Omega)$ to the Dirichlet problem* (4.24)*.*

5 Hilbert's 19th problem

In this section we study regularity properties of minimizers of the energy

$$\mathcal{F}[u] \equiv \int_\Omega F(\mathrm{D}u)\,\mathrm{d}x,$$

where we assume that $F\colon \mathbb{R}^n \to \mathbb{R}$ is smooth and satisfies

$$\lambda|\xi|^2 \leq F''(\eta)[\xi,\xi] \leq \Lambda|\xi|^2 \tag{5.1}$$

for all $\eta, \xi \in \mathbb{R}^n$, thus F is *strongly convex* and has bounded Hessian. In particular, minimizers are characterized as weak solutions of

$$\operatorname{div}(F'(\mathrm{D}u)) = 0. \tag{5.2}$$

In order to understand the key point in the regularity of solutions to (5.2), let us write

$$A(x) = (a_{ij}(x)) \equiv F''(\mathrm{D}u(x)) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{5.3}$$

We have the following proposition, whose proof we briefly postpone:

Proposition 5.1 ($W^{2,2}$-estimate)**.** *Let F satisfy* (5.1) *and let $u \in W^{1,2}(\Omega)$ be a solution of* (5.2)*. Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and it satisfies the second order non-divergence form equation*

$$\operatorname{div}(F'(\mathrm{D}u)) = \partial_{ij}F(\mathrm{D}u)\partial_{ij}u = a_{ij}\partial_{ij}u = 0.$$

Moreover, for any $\gamma \in \{1,\dots,n\}$, $w \equiv \partial_\gamma u$ is a weak solution of the divergence form equation

$$\operatorname{div}(A\,\mathrm{D}w) = 0. \tag{5.4}$$

Corollary 5.2. *In the setting of Proposition 5.1, suppose that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha > 0$. Then $u \in C^\infty(\Omega)$.*

Proof. If $u \in C^{1,\alpha}$ then the coefficients $a_{ij} := \partial_{ij}F(\mathrm{D}u)$ are Hölder continuous, so we can apply Theorem 4.16 to deduce that $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$. But then a_{ij} is $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, and so we can apply Theorem 4.17 to deduce that $u \in C^{3,\alpha}_{\mathrm{loc}}(\Omega)$. Iterating this reasoning we conclude that $u \in C^\infty(\Omega)$. □

We note that Corollary 5.2 holds even when $\alpha = 0$, by the last part of (Q2) of Example Sheet 3. Thus the key step in deducing smoothness of solutions to (5.2) is to show that $\mathrm{D}u$ is continuous: if this is the case, one can essentially linearize (5.2).

Remark 5.3 (Non-strongly convex integrands)**.** Note, from (5.3), that if $u \in \mathrm{Lip}(\Omega)$ with $\mathrm{Lip}(u) \leq L$ then it suffices to require that F be strongly convex in $B_L(0)$. So if we consider e.g. Lipschitz minimizers of the area functional, we can apply the above results: even though $F(\xi) = \sqrt{1+|\xi|^2}$ doesn't satisfy (5.1), it satisfies it in $B_L(0)$, with λ, Λ depending on L.

Therefore, in the rest of this section we concentrate on the equation (5.4) above satisfied by the directional derivatives of minimizers, that is,

$$Lu \equiv -\operatorname{div}(A\,\mathrm{D}u) = 0, \qquad \lambda|\xi|^2 \leq \langle A(x)\xi,\xi\rangle \leq \Lambda|\xi|^2, \tag{5.5}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. Here, the coefficients are *measurable*. Our goal is to show that weak solutions of (5.5) are continuous; once this is shown, by the previous discussion and, in particular, by Corollary 5.2, we have proved smoothness of minimizers.

We now prove Proposition 5.4. First, recall the following properties of difference quotients:

Proposition 5.4 (Weak derivatives vs difference quotients)**.** *Let $u \in L^p_{\mathrm{loc}}(\Omega)$ with $p \in (1,\infty)$. For $\alpha \in \{1,\dots,n\}$, we have $\partial_\alpha u \in L^p_{\mathrm{loc}}(\Omega)$ if and only if $(\partial_{h,\alpha} u)_{h\le h_0} \subset L^p_{\mathrm{loc}}$ is uniformly bounded. In the latter case, $\partial_{h,\alpha} u \rightharpoonup \partial_\alpha u$ in $L^p_{\mathrm{loc}}(\Omega)$.*

Proof of Proposition 5.1. Given $\varphi \in C^\infty(\Omega)$, for all h sufficiently small we have $\varphi(\cdot - he_\gamma) \in C^\infty(\Omega)$ and so, since u is a weak solution of (5.2), after changing variables,

$$\int_\Omega \langle F'(\mathrm{D}u(x+he_\gamma)) - F'(\mathrm{D}u(x)), \mathrm{D}\varphi\rangle \,\mathrm{d}x = 0.$$

Using the Fundamental Theorem of Calculus, we have

$$\begin{aligned} F'(\mathrm{D}u(x+he_\gamma)) - F'(\mathrm{D}u(x)) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} F'(t\mathrm{D}u(x+he_\gamma) + (1-t)\mathrm{D}u(x))\,\mathrm{d}t \\ &= \left(\int_0^1 F''\left(t\mathrm{D}u(x+he_\gamma) + (1-t)\mathrm{D}u(x)\right)\mathrm{d}t\right)\left(\mathrm{D}u(x+he_\gamma) - \mathrm{D}u(x)\right). \end{aligned}$$

Thus, writing $A_h(x)$ for this integral, and dividing by h, we get

$$\int_\Omega \langle A_h \partial_{h,\gamma}\mathrm{D}u, \mathrm{D}\varphi\rangle\,\mathrm{d}x = 0, \tag{5.6}$$

where we note that, for all $\xi \in \mathbb{R}^n$ and for a.e. $x \in \Omega$,

$$\lambda|\xi|^2 \le \langle A_h(x)\xi,\xi\rangle \le \Lambda|\xi|^2.$$

Here we use the notation $\partial_{h,\gamma} v \equiv \frac{v(\cdot+he_\gamma)-v}{h}$. Thus, by Caccioppoli's inequality, whenever $B_{2R}(x_0)$ and h is small enough, we obtain

$$\int_{B_{R/2}(x_0)} |\partial_{h,\gamma}\mathrm{D}u|^2\,\mathrm{d}x \le \frac{C}{R^2}\int_{B_R(x_0)} |\partial_{h,\gamma} u|^2 \le \frac{C}{R^2}\int_\Omega |\mathrm{D}u|^2\,\mathrm{d}x,$$

where the last inequality is standard for difference quotients. Thus, using again the same proposition, we see that $\mathrm{D}u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$. To derive the PDE for $\partial_\gamma u$, we pass to the limit as $h \to 0$ in (5.6): we claim that, up to subsequences, we have

$$\partial_{h,\gamma}\mathrm{D}u \rightharpoonup \partial_\gamma \mathrm{D}u \text{ in } L^2_{\mathrm{loc}}(\Omega), \qquad A_h \to A \text{ in } L^2_{\mathrm{loc}}(\Omega),$$

so that $A_h\partial_{h,\gamma}\mathrm{D}u \rightharpoonup A\,\mathrm{D}(\partial_\gamma u)$ in $L^2_{\mathrm{loc}}(\Omega)$. The first convergence follows from the above estimate. For the second convergence, we argue as follows: for any $\varepsilon > 0$ we have

$$\int_{\{|\mathrm{D}u|>\varepsilon^{-1}\}} |A_h(x) - A(x)|^2\,\mathrm{d}x \le 2\int_{\{|\mathrm{D}u|>\varepsilon^{-1}\}} \left(|A_h|^2 + |A|^2\right)\mathrm{d}x \le C(\Lambda)\varepsilon^2 \int_\Omega |\mathrm{D}u|^2\,\mathrm{d}x.$$

Since A is smooth, in the ball $\{|\mathrm{D}u| \le \varepsilon^{-1}\}$ there is a constant C_ε such that $|A'| \le C_\varepsilon$, hence

$$\int_{B_R(x_0)} |A_h(x) - A(x)|^2\,\mathrm{d}x \le C\varepsilon^2 + C_\varepsilon \int_0^1 \int_{\{|\mathrm{D}u|\le\varepsilon^{-1}\}\cap B_R(x_0)} t^2|\mathrm{D}u(x+he_\gamma) - \mathrm{D}u(x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, by continuity of translations in $L^2(B_R(x_0))$, we have

$$\lim_{h\to 0}\int_{B_R(x_0)} |A_h(x) - A(x)|^2\,\mathrm{d}x \le C\varepsilon^2$$

and, as $\varepsilon > 0$ is arbitrary, the claim follows. $\square$

5.1 A few remarks about weak subsolutions

Let us note the following basic examples of subsolutions:

Lemma 5.5. *If $u \in W^{1,2}(\Omega)$ is a subsolution of* (5.5)*, then:*

(i) *if $f \in C^2(\mathbb{R})$ is convex, non-decreasing with $0 \le f' \in L^\infty(\mathbb{R})$ then $f \circ u$ is also a subsolution of* (5.5)*;*

(ii) *the function $u_+ \equiv \max\{u, 0\}$ is also a subsolution of* (5.5).

Proof. To prove (i), let $0 \le \varphi \in W_0^{1,2}(\Omega)$ be arbitrary and take $\psi = (f' \circ u)\varphi \in W_0^{1,2}(\Omega)$. Then, by (5.5),

$$\begin{aligned} \langle A\,\mathrm{D}u, \mathrm{D}\psi\rangle &= \langle A\,\mathrm{D}u, (f' \circ u)\mathrm{D}\varphi\rangle + \varphi\langle A\,\mathrm{D}u, \mathrm{D}(f' \circ u)\rangle \\ &= \langle A\,\mathrm{D}(f \circ u), \mathrm{D}\varphi\rangle + \varphi f''(u)\langle A\,\mathrm{D}u, \mathrm{D}u\rangle \ge \langle A\,\mathrm{D}(f \circ u), \mathrm{D}\varphi\rangle \end{aligned}$$

for a.e. x in Ω. The conclusion follows by integrating.

To prove (ii), consider the family of functions $f_\varepsilon(t) \equiv \Big((t^4 + \varepsilon^4)^{1/4} - \varepsilon\Big)\, 1_{\{t \ge 0\}}$, which are such that $f_0(t) = t_+$. Applying (i) to $f_\varepsilon(u)$, we can use the Dominated Convergence Theorem to conclude that $u_+ = f_0(u)$ is also a subsolution of (5.5). □

We next recall Caccioppoli's inequality, which we proved in Chapter 4 (Theorem 4.7) and we noted that it applies as well to nonnegative subsolutions.

Lemma 5.6 (Caccioppoli's inequality)**.** *Let $u \in W^{1,2}(B_1)$ be such that $u \ge 0$ and $Lu \le 0$. Then*

$$\int_{B_1} |\mathrm{D}(\eta u)|^2 \,\mathrm{d}x \le C \int_{B_1} u^2 |\mathrm{D}\eta|^2 \,\mathrm{d}x$$

for all $\eta \in C_c^\infty(B_1)$.

5.2 De Giorgi–Nash Theorem

The final goal of this course is to prove the following:

Theorem 5.7 (De Giorgi–Nash)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (5.5)*. There is $\alpha = \alpha(n, \lambda, \Lambda) > 0$ such that, whenever $\Omega' \Subset \Omega$,*

$$\|u\|_{C^{0,\alpha}(\Omega')} \le C\|u\|_{L^2(\Omega)},$$

for a constant $C = C(\Omega, \Omega', n, \lambda, \Lambda)$.

We will always take $n \ge 3$. Indeed, the case $n = 2$ is much simpler, and was shown by Morrey in 1938 (20 years before), and will be addressed in Example Sheet 4.

We follow essentially the strategy of De Giorgi to prove Theorem 5.7. The proof is split into two key steps: we first show that solutions are necessarily bounded, and then we show that bounded solutions are in fact Hölder continuous.

We now begin with the first step in the proof of Theorem 5.7.

Theorem 5.8 (From L^2 to L^∞)**.** *Let $u \in W^{1,2}(B_1)$ be such that $Lu \le 0$. Then*

$$\|u_+\|_{L^\infty(B_{1/2})} \le C(n, \Lambda, \lambda)\|u_+\|_{L^2(B_1)}.$$

Proof. Let $\delta > 0$ be arbitrary, to be chosen later depending only on n, Λ, λ. By multiplying u by a suitable constant, it is enough to show that

$$\|u_+\|_{L^2(B_1)} \leq \delta \quad \Longrightarrow \quad \|u\|_{L^\infty(B_{1/2})} \leq 1. \tag{5.7}$$

Let us set $u_k \equiv (u-(1-2^{-k}))_+$, and also $B_k \equiv B_{\frac{1}{2}+2^{-k}}$, $a_k \equiv \int_{B_k} u_k^2 \, dx$. Note that, according to Lemma 5.5, $Lu_k \leq 0$. Our goal is to derive an iteration formula for a_k.

Let η_k be a cutoff function with $0 \leq \eta_k \leq 1$, $\eta_k = 1$ in B_k, $\eta_k = 0$ outside B_{k-1} and $|D\eta_k| \leq C2^k$. First, by Lemma 5.6, since $u_k \leq u_{k-1}$, we have

$$\int_{B_{k-1}} |D(\eta_k u_k)|^2 \, dx \leq C 2^{2k} a_{k-1}.$$

By Sobolev's inequality, we also have

$$\left(\int_{B_k} |u_k|^{2^*} \, dx\right)^{2/2^*} \leq \left(\int_{B_{k-1}} |\eta_k u_k|^{2^*} \, dx\right)^{2/2^*} \leq \int_{B_{k-1}} |D(\eta_k u_k)|^2 \, dx;$$

here we assume that $n \geq 3$, as we already dealt with $n = 2$ before. On the other hand, by Hölder's inequality we have

$$a_k = \int_{B_k} u_k^2 \, dx \leq \left(\int_{B_k} |u_k|^{2^*} \, dx\right)^{2/2^*} |\{u_k > 0\} \cap B_k|^{2/n}.$$

Finally, by Chebyshev's inequality, and since $B_k \subset B_{k-1}$, we have

$$|\{u_k > 0\} \cap B_k| \leq \left|\{u_{k-1} > 2^{-k}\} \cap B_{k-1}\right| \leq 2^{2k} \int_{B_{k-1}} |u_{k-1}|^2 \, dx = 2^{2k} a_{k-1}.$$

Thus, once we combine the above estimates, we get

$$a_k \leq C 2^{(2+4/n)k} a_{k-1}^{1+2/n}.$$

The crucial point in this iteration estimate is that the exponent of a_{k-1} is larger than one, which guarantees that a_k converges super-exponentially:

Lemma 5.9. *Suppose that there are constants $C, \gamma > 1$ such that*

$$a_k \leq C^k a_{k-1}^\gamma \quad \text{for all } k \in \mathbb{N}.$$

Then $\lim_{k\to\infty} a_k = 0$ provided that a_0 is small enough, depending only on C, γ.

Proof. Indeed, we have

$$a_k \leq C^k a_{k-1}^\gamma \leq C^{k+(k-1)\gamma} a_{k-2}^{\gamma^2} \leq \cdots \leq C^{\sum_{i=0}^{k-1} \gamma^i (k-i)} a_0^{\gamma^k} = a_0^{\gamma^k} C^{\gamma^k \sum_{i=0}^{k-1} \frac{k-i}{\gamma^{k-i}}} \leq (a_0 C_\gamma^M)^{\gamma^k},$$

where $M_\gamma \equiv \sum_{i=0}^\infty \frac{k-i}{\gamma^{k-i}} < \infty$ as $\gamma > 1$. The claim follows by choosing a_0 so that $a_0 C^M < 1$. □

Hence we deduce that, by choosing $\int_{B_1} u_+^2 \, dx = a_0 \leq \delta$ suitably small, we have

$$\int_{B_{1/2}} (u-1)_+^2 \, dx \leq \lim_{k\to\infty} \int_{B_k} u_k^2 \, dx = 0$$

and so $u \leq 1$ in $B_{1/2}$, proving (5.7). □

We now proceed to the second step of De Giorgi's theorem, and we show that bounded solutions are Hölder continuous.

Theorem 5.10 (From L^∞ to $C^{0,\alpha}$)**.** *Let $u \in W^{1,2}(B_1)$ be such that $Lu = 0$. Then*

$$\|u\|_{C^{0,\alpha}(B_{1/2})} \leq C(n, \lambda, \Lambda)\|u\|_{L^\infty(B_1)}$$

for some exponent $\alpha = \alpha(n, \lambda, \Lambda) > 0$.

Before proceeding with the core of the proof, let us show the following simple but important result:

Lemma 5.11 (De Giorgi's isoperimetric inequality)**.** *Let $u \in W^{1,2}(B_1)$ and consider the subsets*

$$A_0 \equiv \{u \leq 0\}, \qquad A_1 \equiv \{u \geq \tfrac{1}{2}\}, \qquad E \equiv \{0 < u < \tfrac{1}{2}\}$$

of B_1. For $p > 1$, there is a constant $C = C(n)$ such that

$$|A_0||A_1| \leq C\|\mathrm{D}u\|_{L^p}|E|^{\frac{1}{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let us take

$$\bar{u} = \begin{cases} u & \text{in } E, \\ 0 & \text{in } A_0, \\ \frac{1}{2} & \text{in } A_1, \end{cases}$$

which is still a Sobolev function, since e.g. $\bar{u} = \min\{\max\{u, 0\}, \frac{1}{2}\}$. We also have $|\mathrm{D}\bar{u}| \leq |\mathrm{D}u|$ a.e. and so, by replacing u with $\bar{u}$, we can assume that $u = 0$ on A_0 and $u = \frac{1}{2}$ in A_1. Now

$$\begin{aligned} |A_0||A_1| &= 2\int_{A_0}\int_{A_1} [u(x) - u(y)]\,\mathrm{d}x\,\mathrm{d}y \\ &\leq 2\int_{B_1} [|u(x) - (u)_{B_1}| + |u(y) - (u)_{B_1}|]\,\mathrm{d}x\,\mathrm{d}y \\ &= 4|B_1|\int_{B_1} |u(x) - (u)_{B_1}|\,\mathrm{d}x \leq C(n)\int_E |\mathrm{D}u|\,\mathrm{d}x \leq C(n)\|\mathrm{D}u\|_{L^p}|E|^{\frac{1}{p'}}, \end{aligned}$$

since $\mathrm{D}u = 0$ in $B_1 \setminus E$. □

The previous lemma shows that Sobolev functions in $W^{1,p}$ cannot have arbitrarily fast jumps, if $p > 1$, but the conclusion fails if $p = 1$ (see Example Sheet 4).

As we will see in a moment, Theorem 5.10 is a relatively straightforward consequence of the following key result:

Lemma 5.12 (Oscillation lemma)**.** *Let $u \in W^{1,2}(B_2)$ be such that*

$$u \leq 1 \text{ in } B_2, \qquad Lu \leq 0 \text{ in } B_2.$$

There is $\theta = \theta(\mu, n, \lambda, \Lambda) > 0$ such that

$$0 < \mu \leq |\{u \leq 0\} \cap B_1| \quad \Longrightarrow \quad \sup_{B_{1/2}} u \leq 1 - \theta.$$

Proof. Similarly to the proof of Theorem 5.8, we consider the sequence of subsolutions of L given by

$$u_k \equiv 2^k(u - (1 - 2^{-k}))_+.$$

Note that $0 \leq u_k \leq 1$ in B_2, since $u \leq 1$, that $u_0 = u_+$, and moreover let us register three basic properties of this sequence:

(i) the sets $\{0 < u_k < \frac{1}{2}\}$ are disjoint for all $k \in \mathbb{N}$;

(ii) $\{u_{k+1} > 0\} \subseteq \{u_k > \frac{1}{2}\}$;

(iii) $\{u \leq 0\} \subset \{u_k = 0\}$.

Let us fix $\delta > 0$, to be chosen later, and suppose that $0 < \mu \leq |\{u \leq 0\} \cap B_1|$. We claim that there is $k_0 = k_0(n, \delta, \mu)$ such that

$$\int_{B_1} u_{k_0}^2 \, \mathrm{d}x \leq \delta^2. \tag{5.8}$$

Indeed, suppose that $u_0, \ldots, u_{k+1}$ do not satisfy (5.8). Then from (ii) we obtain

$$|\{u_k \geq \tfrac{1}{2}\} \cap B_1| \geq |\{u_{k+1} > 0\} \cap B_1| \geq \int_{B_1} u_{k+1}^2 \, \mathrm{d}x \geq \delta^2$$

while from (iii) we obtain

$$|\{u_k = 0\} \cap B_1| \geq \mu,$$

and so we can use the lemma to conclude that

$$\|\mathrm{D}u_k\|_{L^2(B_1)}^2 |\{0 < u_k < \tfrac{1}{2}\} \cap B_1| \geq c(n)\delta^4\mu^2.$$

Caccioppoli's inequality gives us an upper bound on the energy of the sequence, since

$$\int_{B_1} |\mathrm{D}u_k|^2 \, \mathrm{d}x \leq C \int_{B_2} u_k^2 \, \mathrm{d}x \leq C.$$

The last two inequalities thus show that

$$|\{0 < u_k < \tfrac{1}{2}\} \cap B_1| \geq c(n, \lambda, \Lambda, \mu)\delta^4,$$

an estimate which is uniform in k. Thus from (i) we see that for k_0 sufficiently large, depending only on c and δ, (5.8) must hold.

Now let C be the constant in Theorem 5.8 and let us choose $\delta \leq \frac{1}{2C}$. Applying Theorem 5.8 and (5.8), we find that

$$\|u_{k_0}\|_{L^\infty(B_{1/2})} \leq C\|u_{k_0}\|_{L^2(B_1)} \leq C\delta \leq \frac{1}{2}.$$

Thus, recalling the definition of u_k, we have

$$u \leq \frac{1}{2}2^{-k_0} + (1 - 2^{-k_0}) = 1 - 2^{-(k_0+1)} \equiv 1 - \theta,$$

as wished. □

Remark 5.13. Note that Lemma 5.12 can be seen as a *quantitative* form of the strong maximum principle (proved for nondivergence equations in Theorem 3.12). Indeed, suppose that u is a smooth non-constant subsolution such that $\sup_{B_2} u = 1$. Then while the strong maximum principle says that $1 - \sup_{B_{1/2}} u > 0$, Lemma 5.12 asserts that one can in fact quantify the difference $1 - \sup_{B_{1/2}} u \geq \theta > 0$ just in terms of the ellipticity constants λ, Λ and in terms of $\mu = |\{u \leq 0\}|$, which is a measure of how far u is from being identically 1.

Proof of Theorem 5.10. We claim the following *oscillation decay estimate*: writing as usual $\mathrm{osc}_B u \equiv \sup_B u - \inf_B u$, there is a constant $\theta = \theta(n, \lambda, \Lambda) > 0$ with

$$\mathrm{osc}_{B_{1/2}} u \leq (1-\theta)\, \mathrm{osc}_{B_2} u. \tag{5.9}$$

Once this is proved, the conclusion follows from iteration. Indeed, by translating u it is enough to prove that it is Hölder continuous at 0, so let $k \in \mathbb{N}$ be such that $4^{-(k+1)} \leq |x| < 4^{-k}$. Then, by iterating (5.9) k-times, we have

$$\begin{aligned} |u(x) - u(0)| &\leq \mathrm{osc}_{B_{4^{-k}}} u \\ &\leq (1-\theta)^k \mathrm{osc}_{B_1} u \\ &\leq (1-\theta)^k \left(2\|u\|_{L^\infty(B_1)}\right) \\ &= 4^{-\alpha k} \left(2\|u\|_{L^\infty(B_1)}\right) \leq \frac{2}{4^\alpha} |x|^\alpha \|u\|_{L^\infty(B_1)}, \end{aligned}$$

provided we set $\alpha \equiv -\log_4(1-\theta)$.

In turn, (5.9) is a simple consequence of Lemma 5.12. Indeed, consider the function

$$v(x) \equiv \frac{2}{\mathrm{osc}_{B_2} u}\left(u(x) - \frac{\sup_{B_2} u + \inf_{B_2} u}{2}\right),$$

which is defined so that $\sup_{B_2} v = 1, \inf_{B_2} v = -1$. Hence we can assume (up to switching the sign of v) that $|\{v \leq 0\} \cap B_1| \geq \frac{1}{2}|B_1|$ and so, applying Lemma 5.12 with v instead of u, we find that

$$v \leq 1 - \theta \text{ in } B_{1/2} \qquad \Longrightarrow \qquad \mathrm{osc}_{B_{1/2}} v \leq 2 - \theta.$$

Rewriting this inequality in terms of u yields $\mathrm{osc}_{B_{1/2}} u \leq (1 - \frac{\theta}{2})\, \mathrm{osc}_{B_2} u$, which is exactly (5.9) up to redefining θ. □

5.3 Precise Hölder regularity in two dimensions

The above proof of Theorem 5.7 (and in fact, any other known proof) returns an exponentially small Hölder exponent: precisely, with $L = \frac{\Lambda}{\lambda}$, the proofs yield

$$\alpha \approx \exp(-CL^\beta), \qquad C, \beta \geq 1.$$

It is an open problem, going back to De Giorgi, to show that the best possible exponent is instead *algebraic* in L, and in fact that one can even take $\alpha \leq \frac{C(n)}{L}$. The motivation for this conjecture comes from the following simple example:

Example 5.14. Take spherical coordinates $(r, \omega) \in (0, \infty) \times \mathbb{S}^{n-1}$, and consider the operator

$$\mathcal{L} \equiv \frac{1}{r^{n-1}} \partial_r (L r^{n-1} \partial_r \cdot) + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}}.$$

We may write $\mathcal{L}$ is standard Euclidean coordinates if we denote by $J(\omega)$ the orthogonal matrix which associated to the change of coordinates, and then we set

$$A = J(\omega)PJ(\omega)^T = \mathrm{Id} + (L-1)\frac{x}{|x|} \otimes \frac{x}{|x|},$$

where $P = \mathrm{diag}(L, 1, \dots, 1)$. Then the corresponding ellipticity constant (which is kept unchanged in this change of coordinates) is L, as can also be verified directly. The case $L = 1$, in particular, is the usual laplacian.

If $u(r, \omega) = r^\alpha v(\omega)$, then

$$\begin{aligned}\mathcal{L}u &= L(u_{rr} + \tfrac{n-1}{r}u_r) + \tfrac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}u \\ &= r^{\alpha-2}\left(L\alpha(\alpha+n-2)v + \Delta_{\mathbb{S}^{n-1}}v\right).\end{aligned}$$

Now we take v to be a first spherical harmonic, i.e. $v(\omega) = \omega_1 = \frac{x_1}{|x|}$. Then we have

$$-\Delta_{\mathbb{S}^{n-1}}v = (n-1)v.$$

One way of seeing this is by noting that $x_1 = r\omega_1$ is harmonic, and so from the above calculation (with $\alpha = L = 1$) we find

$$0 = \Delta x_1 = r^{-1}\left((n-1)v + \Delta_{\mathbb{S}^{n-1}}v\right).$$

Thus, with this choice, we find

$$\mathcal{L}u = r^{\alpha-2}(L\alpha(\alpha+n-2) - (n-1))v,$$

and we find that u is a solution if

$$\alpha = \frac{1}{2}\left(-(n-2) \pm \sqrt{(n-2)^2 + \frac{4(n-1)}{L}}\right),$$

and we take the positive root to guarantee that u is in the right space. This means that we always have

$$\alpha(n, L) \le \frac{1}{2}\left(\sqrt{(n-2)^2 + \frac{4(n-1)}{L}} - (n-2)\right)$$

Note that, when $L \to \infty$, the right-hand side becomes

$$\frac{1}{2}\left(\sqrt{(n-2)^2 + \frac{4(n-1)}{L}} - (n-2)\right) \approx \begin{cases} \frac{1}{\sqrt{L}} & \text{if } n = 2, \\ \frac{n-1}{(n-2)L} & \text{if } n > 2. \end{cases}$$

We now show that the previous example is sharp when $n = 2$.

Theorem 5.15. *Let $u \in W^{1,2}(B_1)$ be a weak solution of (5.5) when $n = 2$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$, with $\alpha = \frac{1}{\sqrt{L}}$, where $L = \frac{\Lambda}{\lambda}$.*

Remark 5.16. One can think of the previous theorem, and in fact also of Theorem 5.7, as an a priori estimate. Indeed, suppose that $A_\varepsilon \to A$ a.e. and in $L^1(\Omega)$, and consider the solutions of the regularized problems

$$\mathrm{div}(A_\varepsilon \mathrm{D}u_\varepsilon) = 0, \qquad u_\varepsilon \in u + W^{1,2}_0(\Omega).$$

Then $u_\varepsilon \in C^\infty(\Omega)$ and $u_\varepsilon \to u$ in $W^{1,2}(\Omega)$: indeed, with $w_\varepsilon = u_\varepsilon - u \in W^{1,2}(\Omega)$ then

$\operatorname{div}(A_\varepsilon \mathrm{D}w_\varepsilon) = \operatorname{div}((A - A_\varepsilon)\mathrm{D}u))$ and so we have

$$\lambda \int_\Omega |\mathrm{D}w_\varepsilon|^2 \,\mathrm{d}x \le \int_\Omega \langle A_\varepsilon \mathrm{D}w_\varepsilon, \mathrm{D}w_\varepsilon\rangle \,\mathrm{d}x$$
$$= \int_\Omega \langle (A - A_\varepsilon)\mathrm{D}u, \mathrm{D}w_\varepsilon\rangle \,\mathrm{d}x \le \|\mathrm{D}w_\varepsilon\|_{L^2(\Omega)} \|(A_\varepsilon - A)\mathrm{D}u\|_{L^2(\Omega)}.$$

Hence $\|\mathrm{D}w_\varepsilon\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$.

Proof of Theorem 5.15. In fact, we will prove that with

$$\phi(r) \equiv \int_{B_r} \langle A\,\mathrm{D}u, \mathrm{D}u\rangle \,\mathrm{d}x,$$

the function $r^{-2/\sqrt{L}}\phi(r)$ is non-decreasing in $(0,1)$. Once this is shown, it follows in particular that $\mathrm{D}u \in L^{2,2/\sqrt{L}}$, and hence $u \in C^{0,\alpha}_{\mathrm{loc}}$ for the desired α. Equivalently, we will show the differential inequality

$$0 \le \frac{\mathrm{d}}{\mathrm{d}r} \log(r^{-2/\sqrt{L}}\phi(r)) = -\frac{2}{\sqrt{L}}\frac{1}{r} + \frac{\phi'(r)}{\phi(r)} \qquad \iff \qquad \phi(r) \le \frac{\sqrt{L}}{2} r\phi'(r).$$

The RHS is easy to calculate, since $\phi'(r) = \int_{S_r} \langle A\,\mathrm{D}u, \mathrm{D}u\rangle \,\mathrm{d}x$. Through the approximation argument mentioned in the previous remark, we can further assume that both u and the coefficients A are smooth. Note that, for any $u_0 \in \mathbb{R}$, $\operatorname{div}((u - u_0)A\,\mathrm{D}u) = \langle A\,\mathrm{D}u, \mathrm{D}u\rangle$ and so, by integration by parts, we have

$$\phi(r) = \int_{B_r} \operatorname{div}((u - u_0)A\,\mathrm{D}u) \,\mathrm{d}x = \int_{S_r} (u - u_0)\langle A\,\mathrm{D}u, \nu\rangle \,\mathrm{d}\sigma.$$

It is now convenient to use polar coordinates: let $R(\theta)$ be the rotation matrix by angle θ, and note that

$$\mathrm{D}u(x) = R(\theta)\widetilde{\mathrm{D}}u, \qquad \widetilde{\mathrm{D}}u = (\partial_r u, \tfrac{1}{r}\partial_\theta u).$$

Also $\nu = R(\theta)e_1$. So, if we write $P = R(-\theta)AR(\theta)$, which is a symmetric matrix with the same eigenvalues as A, we have

$$\phi(r) = \int_{S_r} (u - u_0)\langle P\widetilde{\mathrm{D}}u, e_1\rangle = \int_{S_r} (u - u_0)(p_{11}\partial_r u + p_{12}\tfrac{1}{r}\partial_\theta u) \,\mathrm{d}\sigma.$$

The ellipticity bounds on (5.5) imply that

$$\lambda \le p_{11} \le \Lambda, \qquad \lambda \le \frac{1}{(P^{-1})_{22}} = p_{22} - \frac{p_{12}^2}{p_{11}} \le \Lambda$$

and so, by Cauchy–Schwarz,

$$\phi(r) \le \left(\int_{S_r} p_{11}(u - u_0)^2 \,\mathrm{d}\sigma\right)^{1/2} \left(\int_{S_r} \left(\sqrt{p_{11}}\partial_r u + \frac{p_{12}}{\sqrt{p_{11}}}\frac{1}{r}\partial_\theta u\right)^2 \mathrm{d}\sigma\right)^{1/2}.$$

We now choose u_0 to be the average of u over S_r, and apply Wirtinger's inequality:

$$\int_0^{2\pi} w^2 \,\mathrm{d}t \le \int_0^{2\pi} \dot{w}^2 \,\mathrm{d}t$$

whenever w is 2π periodic and has zero mean. Thus

$$\int_{S_r} p_{11}(u-u_0)^2\,\mathrm{d}\sigma \le \Lambda \int_{S_r} (\partial_\theta u)^2\,\mathrm{d}\sigma \le \frac{\Lambda}{\lambda} r^2 \int_{S_r} (p_{22} - \tfrac{p_{12}^2}{p_{11}})\frac{(\partial_\theta u)^2}{r^2}.$$

Hence we find that

$$\begin{aligned}\phi(r) &\le \frac{\sqrt{L}}{2} r \int_{S_r} \left(p_{22} - \frac{p_{12}^2}{p_{11}}\right)\frac{(\partial_\theta u)^2}{r^2} + \left(\sqrt{p_{11}}\partial_r u + \frac{p_{12}}{\sqrt{p_{11}}}\frac{1}{r}\partial_\theta u\right)^2 \\ &= \frac{\sqrt{L}}{2} r \int_{S_r} \langle P\widetilde{\mathrm{D}}u, \widetilde{\mathrm{D}}u\rangle\,\mathrm{d}\sigma \\ &= \frac{\sqrt{L}}{2} r \int_{S_r} \langle A\mathrm{D}u, \mathrm{D}u\rangle\,\mathrm{d}\sigma,\end{aligned}$$

from which the conclusion follows. □

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