Symplectic topology: Example Sheet 4 of 4

1. Show that a smooth J-holomorphic submanifold in a symplectic manifold (M, ω) is a symplectic submanifold.

2. (a) Find a connected complex algebraic surface X and a homology class $A \in H_2(X; \mathbb{Z})$ so that there are holomorphic curves in the class A, but none of these curves are connected. (Hint: Use adjunction.)
(b) Find a compact complex algebraic surface X and a homology class $A \in H_2(X; \mathbb{Z})$ with $A \cdot A < 0$ and such that there are singular (nodal) holomorphic curves in the class A. Can such an A have a smooth holomorphic representative?

3. Find an example of a symplectic manifold X and $A \in H_2(X; \mathbb{Z})$ so that the moduli space of all J-spheres in the class A necessarily has "excess" dimension, i.e. larger than that predicted by the index.

4. Give an informal argument to say there should be a finite positive number of degree d holomorphic curves in $\mathbb{CP}^2$ through a generic set of $3d-1$ points. Compute the number for $d \leq 3$. Show that if the set of points is not generic, the number of actual curves may be different.

5. Let $\wp$ denote the Weierstrass function, a meromorphic degree 2 map $T^2 \to \mathbb{P}^1$; if $T^2 = \mathbb{C}/\Lambda$ then $\wp(z) = \sum_{\lambda \in \Lambda} (z - \lambda)^{-2}$. Consider the maps $u_n : z \mapsto \frac{\wp(z) - \wp(w)}{\wp(z) - \alpha_n \wp(w)}$, where $w \in T^2$ is a fixed non-critical point of $\wp$ and $\alpha_n \in \mathbb{C} \backslash \{1\}$. Show that as $\alpha_n \to 1$ two degree 1 bubbles develop, at $\pm w$, and elsewhere the u_n converge on compact subsets of $T^2 \backslash \{\pm w\}$ to a constant map.

6. Suppose $r \leq s$ and $r' \leq s'$ are positive reals. With the obvious product symplectic forms, show there is a symplectomorphism $B^2(r) \times B^2(s) \cong B^2(r') \times B^2(s')$ if and only if $r = r'$ and $s = s'$.

7. (Extra.) Let $\Gamma \leq \mathbb{C}^3$ be the group of upper triangular matrices

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \oplus \mathbb{Z}[i] \right\}$$

Let X be the complex manifold $\mathbb{C}^3/\Gamma$. Show
(a) The map $y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ defines a holomorphic torus $T^2 \subset X$.
(b) Fix a metric on X. Let ϕ_x denote multiplication by the element $\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Show that the area of $\phi_x(T^2)$ is $o(|x|)$ and in particular unbounded as $|x| \to \infty$.
(c) Deduce that the complex structure on X is not compatible with any symplectic form.