Symplectic topology, Lent 2020: question sheet 1

Part (a): Differential topology/forms revision.

(1) Explain how to write the de Rham complex for an open subset $U \subset \mathbb{R}^3$ as

$$0 \to C^\infty(U;\mathbb{R}) \xrightarrow{grad} C^\infty(U;\mathbb{R}^3) \xrightarrow{curl} C^\infty(U;\mathbb{R}^3) \xrightarrow{div} C^\infty(U;\mathbb{R}) \to 0$$

(2) Give a proof of Stokes' theorem (for manifolds with boundary).

(3) Show that $\omega(x,y) = \frac{xdy-ydx}{x^2+y^2}$ defines a 2-form on $\mathbb{R}^2\backslash\{0\}$ and hence on S^1 such that (i) on $\{x>0\}$, $\omega = df$ for a suitable function f but (ii) ω is not globally exact. Deduce that $H^1_{dR}(S^1) \cong \mathbb{R}$ is generated by ω.

(4) If $\theta_1, \ldots, \theta_4$ are co-ordinates on the torus T^4, show $d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4$ is not "decomposable" (not a wedge product of one-forms).

(5) Let M be a closed oriented n-manifold.

(i) If $\theta \in \Omega^{n-1}(M)$ show $d\theta$ vanishes somewhere.

(ii) If M is connected and $\omega \in \Omega^1(X)$ has $\int_\gamma \omega = 0$ for every closed curve $\gamma \subset X$, show that $\omega = df$ is globally exact.

(iii) If $M = \partial N$ with N compact of dimension $n+1$ and $f : M \to Y$ is a smooth map which extends to N, show $\int_M f^*\omega = 0$ for every closed $\omega \in \Omega^n(Y)$.

(6) To revise orientation:

(i) Exhibit non-orientable closed manifolds of every dimension strictly greater than one.

(ii) Show there is no orientation-preserving diffeomorphism from $\mathbb{C}\mathbb{P}^2$ to $\overline{\mathbb{C}\mathbb{P}}^2$.

(7) Show $\mathbb{C}\mathbb{P}^2$ is not the boundary of a compact smooth 5-manifold.

Part (b): Symplectic topology questions

(1) Let V be a symplectic vector space. A Lagrangian subspace is a half-dimensional subspace on which the symplectic form vanishes. Prove the following:

(i) If $S \subset V$ is *isotropic*, meaning $S \subset S^{\perp}$, there is a Lagrangian subspace L with $S \subset L$.

(ii) If $W \subset V$ is *coisotropic*, meaning $W \supset W^{\perp}$, there is a canonical symplectic structure on $W/W^{\perp}$.

(iii) If L, L' are Lagrangian subspaces, every linear isomorphism $L \to L'$ can be extended to a linear symplectomorphism $A \in Sp(V)$ of V.

(iv) $Sp(V)$ acts transitively on pairs of transverse Lagrangian subspaces, but in general not on pairs of transverse symplectic subspaces (even of fixed dimensions).

(2) Prove $Sp_{2n}(\mathbb{R}) \cap O_{2n} = Sp_{2n}(\mathbb{R}) \cap GL_n(\mathbb{C}) = O_{2n} \cap GL_n(\mathbb{C}) = U_n$. In fact, $Sp_{2n}(\mathbb{R})/U_n$ is contractible: why is this significant?

(3) Let (M, ω) be a symplectic manifold. The *Poisson bracket* of two smooth functions $f, g \in C^{\infty}(M, \mathbb{R})$ is defined by $\{f, g\} = \omega(X_f, X_g)$, where X_f and X_g are the Hamiltonian vector fields defined by f and g.

(i) Show that $L_{X_g} f = \{f, g\}$, and that $[X_g, X_f] = X_{\{f,g\}}$. [If you have problems with signs, ignore them.]

Hint: to prove the second identity, you may use without proof the identity $i_{[X,Y]}\alpha = di_X i_Y \alpha + i_X d i_Y \alpha - i_Y d i_X \alpha - i_Y i_X d\alpha$.

(ii) Deduce that $(C^{\infty}(M, \mathbb{R}), \{\cdot, \cdot\})$ is a Lie algebra, i.e. $\{f, g\} = -\{g, f\}$ (skew-symmetry) and $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity).

(Note: by the first identity proved in (a) and skew-symmetry of the bracket, $\{f, g\} = 0 \Leftrightarrow$ the flow of X_f preserves the level sets of $g \Leftrightarrow$ the flow of X_g preserves the level sets of f)

(iii)* Assume that $f_1, \ldots, f_k$ satisfy $\{f_i, f_j\} = 0$ $\forall i, j$, and let $F = (f_1, \ldots, f_k) : M \to \mathbb{R}^k$. Show that any regular level set of F is a coisotropic submanifold of M, and that the vector fields X_{f_i} are all tangent to this submanifold and span the tangent space to its isotropic foliation.

(For example, if $k = \frac{1}{2} \dim M$ then the regular levels of F are Lagrangian; this situation is called an *integrable system*).

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