ALGEBRAIC SURFACES, SHEET III: LENT 2021

Throughout this sheet, the symbol k will denote an algebraically closed field.

1. Calculate the plurigenera (i.e. the numbers $h^0(\omega_X^{\otimes n})$ for $n \geq 1$) of a ruled surface X.
2. Give examples of surfaces whose Albanese varieties have dimensions $0, 1$, and 2.
3. Exhibit elliptically fibered surfaces of Kodaira dimensions $-\infty, 0$, and 1 that are not isomorphic to products of curves. Prove that an elliptically fibered surface cannot have Kodaira dimension 2.
4. Let D be a divisor on a surface X. Prove that if D^2 is strictly positive, then for all n sufficiently large, either nD or $-nD$ has sections. Hint: Use the fact that Riemann–Roch is a quadratic function in D.
5. Let X be a smooth surface in $\mathbb{P}^n$ given as the complete intersection of $n-2$ hypersurfaces of degrees $d_1, \ldots, d_{n-2}$. Describe the complete intersections with non-positive Kodaira dimension.
6. (Kummer Construction I) Let A be a projective abelian surface over the complex numbers and let τ be the involution given by $a \mapsto -a$. Show that there are 16 fixed points of this involution.
7. (Kummer Construction II) In the question above, show that the quotient A/τ is a projective variety. Show that it is singular and give local equations for the singularities. Blowup the singularities to obtain a smooth surface.
8. Let X be a non-ruled surface. By using Noether's formula, prove that K_X^2 is no larger than 12 times the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$.
9. Construct a surface with intinitely many (-1)-curves. In order to do this, blowup the 9 intersection points of two smooth cubics in $\mathbb{P}^2$. Examine the automorphism group of the resulting surface and remember that an elliptic curve has a group law.
10. Let $f : X \to Y$ be a proper morphism of varieties. Stein factorization states that f factors as a finite morphism $X \to Z$ followed by a morphism $Z \to Y$ that has connected fibers. Using this, prove that if X is a surface whose image in its Albanese is a curve Y, then $X \to Y$ has connected fibers.

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