ALGEBRAIC SURFACES, SHEET II: LENT 2021

Throughout this sheet, the symbol k will denote an algebraically closed field.

1. Let X be the blowup of $\mathbb{P}^2$ at a single point. Let H and E be the divisor classes of a line in $\mathbb{P}^2$ pulled back to X and an exceptional divisor. Prove that the line bundle $\mathcal{O}_X(H - E)$ is base point free, and describe the resulting map to projective space.
2. Prove that the line bundle $2H - E$ on the blowup of $\mathbb{P}^2$ at a point is very ample.
3. Prove that $\mathbb{P}^2$ blown up at 3 non-collinear points is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at 2 points.
4. Observe that quartic plane curves are given by an open subset of $\mathbb{P}^{14}$. Using this observation, for every smooth curve C, construct a surface X equipped with a morphism $X \to C$, whose generic fiber is a genus 3 curve, but X is not a product.
5. Prove that a generic surface in $\mathbb{P}^3$ of degree at least 4 contains no lines. To begin, consider the Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$. Form the incidence correspondence of hypersurfaces of degree d and examine the dimensions of the two projections.
6. Prove that a smooth curve C on a surface S is a (-1)-curve if and only if $C \cdot C$ and $C \cdot K_S$ are both negative.
7. By examining the Picard rank[1] show that every surface is birational to a surface that does not have (-1)-curves.
8. Using the universal property of blowups, prove that if $F : X' \to X$ is a birational morphism of surfaces, then F can be factoried into a sequence of blowups.
9. Prove that on the surface obtained by blowing up $\mathbb{P}^2$ at 9 general points, neither the canonical divisor nor the anticanonical divisor is ample.
10. Prove that on the blowup of $\mathbb{P}^2$ at 7 points, the line bundle $2H - E_1 - \ldots - E_7$ is basepoint free but not ample. Find all its (-1) curves. Describe the resulting map to projective space.
11. Prove that the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ contains a curve[2] of self-intersection -2. Construct a birational morphism $X \to X_0$ such that C is contracted. By using reasoning that is independent from Castelnuovo's theorem, prove that X_0 is singular.
12. Prove that a bidegree $(3,1)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$. Describe this surface as a sequence of blowups of $\mathbb{P}^2$.

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[1]Recall that the Picard rank is the rank of the group of divisors up to numerical equivalence. You may take as given that this number is finite for any surface.

[2]A hint: this is a family of $\mathbb{P}^1$'s over $\mathbb{P}^1$. Use the structure of this bundle to find the curve