ALGEBRAIC SURFACES, SHEET I: LENT 2021

Throughout this sheet, the symbol k will denote an algebraically closed field.

1. Let X and Y be separated schemes over k, and let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on X and Y respectively. By using Čech cohomology, prove the sheaf Künneth formula:
$$H^m(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \bigoplus_{m=p+q} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}).$$
2. Calculate the genus of a smooth curve that is the complete intersection of 3 quadrics in $\mathbb{P}^4$. More generally, prove that a smooth curve arising as an intersection of hypersurfaces $X_{d_1}, \dots, X_{d_{n-1}}$ is
$$g = 1 + \frac{1}{2}\left(\prod_{i=1}^{n-1} d_i\right)\left(\sum_{i=1}^{n-1} d_i - n - 1\right).$$
3. Prove that there exist smooth projective varieties X of any dimension larger than 1 with trivial canonical sheaf and $h^1(\mathcal{O}_X)$ equal to 0. These are examples of *algebraic Calabi–Yau varieties.*
4. Prove that a smooth quartic hypersurface X in $\mathbb{P}^3$ has vanishing irregularity, i.e. $h^1(X, \mathcal{O}_X)$ vanishes.
5. Construct an example of smooth quartic surface in $\mathbb{P}^3$ that contains a line. For your chosen line, give a parameterisation of the family of plaines in $\mathbb{P}^3$ that contain this chosen line. The intersections of these planes with the quartic determine a family of elliptic curves. This is an example of a *elliptically fibered K3 surface.*
6. Let X and Y be two birational irreducible projective varieties. Construct a third variety Z, not necessarily smooth, that has a birational morphism to both X and Y.
7. Let X be an smooth projective surface and let C and C' be two distinct irreducible curves on C. Construct an exact[1] sequence of the form:
$$0 \to \mathcal{O}_X(-C - C') \to \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \to \mathcal{O}_X \to \mathcal{O}_{C \cap C'} \to 0.$$
The intersection $C \cap C'$ is scheme theoretic. By passing to Euler characteristics, prove that if $C \cap C'$ is a finite collection of reduced points, its cardinality can be computed via ranks of cohomology groups of line bundles associated to C, C', and $\mathcal{O}_X$.
8. Look up the statement of the Lefschetz hyperplane section theorem. Use it to prove that, at least over the complex numbers, a product of curves $C \times D$ with C and D curves of genus at least 1 is never a smooth hypersurface in $\mathbb{P}^3$.

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[1]Remember that you can check exactness locally!

9. **Weighted projective space** $\mathbb{P}(a_0, \dots, a_n)$ over k, for a_i positive integers, is the scheme $\mathrm{Proj} k[X_1, \dots, X_n]$, where the grading on the polynomial ring assigns degree a_i to X_i.

   Choose appropriate weights and construct a surface X in a weighted projective space together with a morphism
$$\pi : X \to \mathbb{P}^2,$$
such $\pi^{-1}(p)$ consists of two points for all $p \in \mathbb{P}^2 \setminus C$, where C is a smooth curve of degree d. That is *Construct a double cover of* $\mathbb{P}^2$ *branched over smooth curve of degree* d.